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BRANCHED SINGULARITIES OF HARMONIC MAPS

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Abstract. In this paper we give an example of energy minimizing harmonic maps for which the set of singular points are two or more lines intersecting at a point.

1. Introduction

It is well known that a harmonic map between Riemannian manifolds are not necessarily continuous. In general, even energy minimizing harmonic maps can have points of discontinuity which we will call the singularity. The Hausdorff dimension of the singularity of a minimizing harmonic map is known to be less than or equal to $m-3$ where m is the dimension of the domain [10].

The shape of the singularity and the behavior of a minimizing harmonic map near the singularity have been studied by many mathematicians. For example, when the dimension of the domain $m=3$, the singularity of minimizing harmonic map is consist of isolated points [10]. But when $m \geq 4$ the singular set can have more complicated structure. For the structure of singularity, it has been proved by Leon Simon that the singular set of a minimizing harmonic map into a real analytic manifold is a union of a pairwise disjoint locally $(n-3)$ rectifiable locally compact subsets [9]. But it is not known whether there is a branch points in singular sets even for simplest case as maps from B^4 to S^2. In fact, we do have only few explicit examples of minimizing harmonic maps with singularity.

In this note, we introduce a minimizing harmonic map for which the singular set is made of finite lines crossing at a point. So far this is the only known example of branched singular set of minimizing harmonic maps.

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2. Minimizing harmonic map with branched singularity

On a C^∞ manifold N, we define a degenerate metric h to be a positive semi-definite $(0,2)$-tensor field which is not necessarily continuous. So, for any vector field V on N, $h(V,V)$ is a nonnegative function on N.

Now consider a weakly differentiable map $u : M \to N$ from a Riemannian manifold M into a manifold N, and let h be a degenerate metric on N. We define the energy $E_h(u)$ of u with respect to the degenerate metric h as

$$E_h(u) = \int_{x \in M} tr_M h(\nabla u, \nabla u)\ (x)\ dV(x),$$

where $tr_M h(\nabla u, \nabla u) = \sum g^{ij}(x) h(u_*(\frac{\partial}{\partial x_i}),\ u_*(\frac{\partial}{\partial x_j}))$ and $g_{ij} = (g^{ij})^{-1}$ is a Riemannian metric on M. Note that $tr_M h(\nabla u, \nabla u)$ is a nonnegative function on M. Also note that if h is the Riemannian metric on N, then $E_h(u)$ is same as the usual energy $E_2(u)$. We say a map u_0 is an E_h-minimizing map if $E_h(u_0)$ has minimum energy among the weakly differentiable maps with $v|_{\partial M} = u_0|_{\partial M}$.

The next lemma states that when we accumulate degenerate metrics on the target manifold, the energy minimizing property of a given map is preserved.

Lemma 1. *Let $\{h_i | i = 1, \cdots, k\}$ is a family of degenerate metric on a C^∞-manifold N, and let M be a Riemannian manifold. If $u_0 : M \to N$ is E_{h_i}-minimizing for each i, then u_0 is an E_h-minimizing map where the degenerate metric h is given by*

$$h(V,W) = \sum_i h_i(V,W) \qquad \text{for any} \quad V, W \in T_x N \text{ for all } x \in N$$

proof. For any weakly differentiable map v, $tr_M h_i(\nabla v, \nabla v)$ is non-negative. So,

$$\begin{aligned} E_h(v) &= \int_{x \in M} \sum_i tr_M h_i(\nabla v, \nabla v) dV(x) \\ &= \sum_i \int_{x \in M} \int_M tr_M h_i(\nabla v, \nabla v) dV(x) = \sum_i E_{h_i}(v). \end{aligned}$$

Since u_0 is an E_{h_i}-minimizing map for each i,

$$E_h(v) = \sum_i E_{h_i}(v) \geq \sum_i E_{h_i}(u_0) = E_h(u_0)$$

for any v such that $v|_{\partial M} = u_0|_{\partial M}$. $\square$

Using the above lemma, we may construct examples of minimizing harmonic maps by accumulating simpler degenerate metrics on target manifold for which we know that the given map is an energy minimizer. By this method we construct a minimizing harmonic map from B^4 into (R^6, g) which is not Euclidean with branched singularities where g is some Riemannian metric.

Theorem 2. *Let B^4 be the unit ball in the Euclidean 4-space. There exists a Riemannian metric on R^6 such that the map $u_0 : B^4 \to R^6$ given by*

$$u_0(x_1, x_2, x_3, x_4) \; = \; \left(\frac{(x_1, x_2, x_3)}{|(x_1, x_2, x_3)|}, \; \frac{(x_2, x_3, x_4)}{|(x_2, x_3, x_4)|} \right) .$$

is an energy minimizing harmonic map with singularities along x_1-axis and x_4-axis.

proof. First, we construct a Riemannian structure on R^3 so that $u(X) = \frac{X}{|X|} : B^3 \to R^3$ becomes a minimizing map. Let $N_a^+ = \{ \, (X, \; x) \in R^3 \times R \mid |X|^2 + \frac{x^2}{a^2} = 1, \; x \geq 0\}$ be the upper half of a 3-ellipsoid with Riemannian metric induced from the Euclidean metric on R^4. When a is sufficiently large, $\tilde{u}(X) = (\frac{X}{|X|}, \; 0) : B^3 \to N_a^+$ is a minimizing map. In fact, Helein[5] showed that $\tilde{u}$ is minimizing when $a > 8$.

For such an a, let $P : R^3 \to N_a^+$ be a Lipschitz map defined as follows.

$$P(X) = \begin{cases} \left(\frac{\sqrt{2|X|^2 - |X|^4}}{|X|} X, \; a(1 - |X|^2) \right) & \text{if } \; |X| \leq 1 \\ \left(\frac{X}{|X|}, \; 0 \right) & \text{if } \; |X| > 1 \; . \end{cases}$$

Then we may construct a smooth metric h on R^3 so that , $h = P^*(ds^2_{N_a^+})$ for $|X| \leq 1$, and $h - P^*(ds^2_{N_a^+})$ is positive definite if $|X| > 1$.

For any weakly differentiable map $v : B^3 \to (R^3, \; h)$ such that $v|_{\partial B^3} = u|_{\partial B^3}$, we have

$$\begin{aligned} E_g(v) \; &= \int_{B^3} tr_{B^3} g(\nabla v, \nabla v) dVol \\ &\geq E_2(P_* v) \qquad\qquad (\text{ by (i) and (ii) }) \\ &\geq E_2(\tilde{u}) \qquad\qquad (\text{Since } \tilde{u} \text{ is an energy minimizing map}) \\ &= E_g(u) \end{aligned}$$

So, $u(X) = \frac{X}{|X|} : B^3 \to R^3$ is an E_h-minimizing map.

Now let h_1 and h_2 be the two degenerate metrics on R^6 given by $h_i = \Pi_i^*(h)$, $i = 1, 2,$ where $\Pi_1, \Pi_2 : R^3 \times R^3 \to R^3$ are the projections such that $\Pi_1(X, Y) = X \in$

R^3, $\Pi_2(X,Y) = Y \in R^3$. Then from the above computation, it follows immediately that $u_0 : B^4 \to R^6$ is a E_{h_i}-minimizing map for each i. By applying Lemma 1, $u_0 : B^4 \to (R^6, g)$ is an energy minimizing map where the Riemannian structure on the target manifold is $g = h_1 + h_2$. The singular set of this energy minimizing map is $\mathcal{S}_{u_0} = B^4 \cap \{ x_1 = x_2 = x_3 = 0 \text{ or } x_2 = x_3 = x_4 = 0 \}$ which is a union of two line segments crossing at the origin. $\square$

For the above minimizing map u_0, we can notice that the image lies in the product of spheres $T = S^2 \times S^2 \subset S^5_{\sqrt{2}} \subset R^6$, and the induced metric on the torus T is the standard product metric. Of course, the map u_0 is also a energy minimizing map considered as a map into the target manifold T or as a map into $S^5_{\sqrt{2}}$ with the induced metric.

In the above proof we can notice that by introducing more and more complicated structure on the target manifold one can produce minimizing tangent map in B^m whose singular set is any finite union of linear subspaces with codimension bigger than 3. Hence, the singularity of minimizing harmonic map can cross finite times with any positive angle at the branch point.

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