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AN APPROXIMATION OF THE HANKEL TRANSFORM FOR ABSOLUTELY CONTINUOUS MAPPINGS

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Abstract. Using some techniques developed by Dragomir and Wang in the recent paper [2] in connection to Ostrowski integral inequality, we point out some approximation results for the Henkel's transform of absolutely continuous mapping.

1. Introduction

Two-dimensional systems may often show circular symmetry, for example optical systems are often constructed from components that, in themselves, are circulary symmetrical.

When circular symmetry exists, that is, when $f(x,y) = f(r)$, $r^2 = x^2 + y^2$, then the bidimensional Fourier transform can be represented in the following way [8, p. 244 - p. 250]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(xu+yv)}dxdy \tag{1.1}$$

$$= \int_0^{\infty}\int_0^{2\pi} f(r)e^{-i2\pi qr\cos(\theta-\varphi)}rdrd\theta = \int_0^{\infty} f(r)\left[\int_0^{2\pi} e^{-i2\pi qr\cos(\theta-\varphi)}d\theta\right]rdr$$

$$= 2\pi\int_0^{\infty} f(r)J_0(2\pi qr)rdr$$

where $x + iy = re^{i\theta}$, $u + iv = qe^{i\varphi}$, $q^2 = u^2 + v^2$ and we have used the relation

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iz\cos\beta}d\beta \tag{1.2}$$

We refer to $G(f)(q)$ given by

$$G(f)(q) = 2\pi\int_0^{\infty} f(r)J_0(2\pi qr)rdr \tag{1.3}$$

as the Hankel transform (of zero order) of $f(r)$.

The main aim of the present article is to point out some estimates of the Hankel transform for absolutely continuous mappings defined on an finite interval $[a,b]$ by the use of some techniques developed by Dragomir and Wang in the recent paper [2] in connection to Ostrowski integral inequality. Some adaptive quadrature formulae which will allowe another approach than the classical one will be also derived.

2. An Integral Representation

Let $J_1(\cdot)$ represent the *first-order Bessel functions of the first kind*, that is,

$$J_1(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\varphi - i\varphi}d\varphi, x \in \mathbb{R}. \tag{2.1}$$

Define the corresponding *Bessel's* mean as*:*

$$B_1(z,w) = \begin{cases} z \text{ if } w = z \\ \frac{zJ_1(z)-wJ_1(w)}{z-w} \text{ if } w \neq z \end{cases} ; w, z \in \mathbb{R}. \tag{2.2}$$

The following representation of Hankel transform holds.

Theorem 1. *Let $g : [a,b] \to \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ be an absolutely continuous mapping on $[a,b]$. Then we have the representation*

$$\begin{aligned} G(g)(\rho) &= \frac{1}{\rho}B_1(2\pi b\rho, 2\pi a\rho) \times \int_a^b g(s)\,ds \\ &\quad + \frac{2\pi}{b-a}\int_a^b\int_a^b k(r,s)\,g'(s)\,rJ_0(2\pi r\rho)\,drds \end{aligned} \tag{2.3}$$

for all $\rho \in [a,b], \rho \neq 0$, where $k(\cdot,\cdot) : [a,b]^2 \to \mathbb{R}$ is given by

$$k(u,v) := \begin{cases} v-a \text{ if } v \in [a,u] \\ v-b \text{ if } v \in (u,b] \end{cases} ; (u,v) \in [a,b]^2 \tag{2.4}$$

and $J_0(\cdot)$ is the zeroth-order Bessel function of the first kind, that is,

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ix\cos\beta}d\beta, x \in \mathbb{R}. \tag{2.5}$$

Proof. Using the integration by parts formula for absolutely continuous mappings on $[a,b]$, we can write

$$\int_a^x (s-a)\,g'(s)\,ds = (x-a)\,g(x) - \int_a^x g(s)\,ds \tag{2.6}$$

and

$$\int_x^b (s-b)\,g'(s)\,ds = (b-x)\,g(x) - \int_x^b g(s)\,ds \tag{2.7}$$

for all $x \in [a,b]$.

Adding (2.6) and (2.7), we end up with (see also [2])

$$g(x) = \frac{1}{b-a}\int_a^b g(s)\,ds + \frac{1}{b-a}\int_a^b k(r,s)\,g'(s)\,ds \tag{2.8}$$

for all $x \in [a,b]$, which is of importance in itself, too.
Now, consider the Hankel transform of g on the interval $[a,b]$, that is:

$$G(g)(\rho) := 2\pi \int_a^b g(r)\, r J_0(2\pi r\rho)\, dr \tag{2.9}$$

and use the representation (2.8) to get

$$\begin{aligned} &G(g)(\rho) \\ &= 2\pi \int_a^b \left[\frac{1}{b-a}\int_a^b g(s)\,ds + \frac{1}{b-a}\int_a^b k(r,s)\, g'(s)\,ds\right] r J_0(2\pi r\rho)\, dr \\ &= \frac{1}{b-a}\int_a^b g(s)\,ds \cdot 2\pi \int_a^b r J_0(2\pi r\rho)\,dr \\ &\quad + \frac{2\pi}{b-a}\int_a^b\int_a^b k(r,s)\, g'(s)\, r J_0(2\pi r\rho)\, ds dr. \end{aligned} \tag{2.10}$$

Consider the change of variable $r' = 2\pi r\rho$. Then $r = \frac{r'}{2\pi\rho}$, $dr = \frac{dr'}{2\pi\rho}$ and

$$\begin{aligned} \int_a^b r J_0(2\pi r\rho)\,dr &= \int_{2\pi a\rho}^{2\pi b\rho} \frac{r'}{2\pi\rho} J_0(r') \frac{1}{2\pi\rho} dr' \\ &= \frac{1}{(2\pi\rho)^2}\int_{2\pi a\rho}^{2\pi b\rho} r' J_0(r')\,dr'. \end{aligned}$$

It is a well-known property of Bessel functions that

$$\int_0^x \xi J_0(\xi)\,d\xi = x J_1(x),\ x \in \mathbb{R}. \tag{2.11}$$

Consequently,

$$\begin{aligned} \int_{2\pi a\rho}^{2\pi b\rho} r' J_0(r')\,dr' &= \int_0^{2\pi b\rho} r' J_0(r')\,dr' - \int_0^{2\pi a\rho} r' J_0(r')\,dr' \\ &= 2\pi b\rho J_1(2\pi b\rho) - 2\pi a\rho J_1(2\pi a\rho) \end{aligned}$$

and

$$\begin{aligned} \frac{2\pi}{b-a}\int_a^b r J_0(2\pi r\rho)\,dr &= \frac{2\pi}{b-a}\cdot\frac{1}{(2\pi\rho)^2}\left[2\pi b\rho J_1(2\pi b\rho) - 2\pi a\rho J_1(2\pi a\rho)\right] \\ &= \frac{1}{\rho}\frac{2\pi b\rho J_1(2\pi b\rho) - 2\pi a\rho J_1(2\pi a\rho)}{2\pi b\rho - 2\pi a\rho} \\ &= \frac{1}{\rho} B_1(2\pi b\rho, 2\pi a\rho). \end{aligned}$$

Using (2.10), we deduce the desired representation (2.1). ∎

In practical applications we have $a=0$ and $b=1$. Consequently, we can state the following corollary.

Corollary 2. *Let $g:[0,1]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ be an absolutely continuous mapping on $[0,1]$. Then we have the representation*

$$G(g)(\rho) = \frac{J_1(2\pi\rho)}{\rho}\times\int_0^1 g(s)\,ds+2\pi\int_0^1\int_0^1\tilde{k}(r,s)\,g'(s)\,rJ_0(2\pi r\rho)\,drds,\ \rho\in(0,1] \tag{2.12}$$

where $\tilde{k}:[0,1]\to\mathbb{R}$ is given by

$$\tilde{k}(u,v)=\begin{cases} v & \text{if } v\in[0,u] \\ v-1 & \text{if } v\in(u,1]\end{cases}.$$

Now let us define the mapping with real values $I(g):[a,b]\to[0,\infty)$ given by

$$I(g)(\rho):=|G(g)(\rho)|^2,\ \rho\in[a,b]. \tag{2.13}$$

Using the well known property of complex numbers

$$|x+y|^2=|x|^2+2\operatorname{Re}(z\bar{y})+|y|^2 \text{ for any } x,y\in\mathbb{C} \tag{2.14}$$

we can state the following corollary:

Corollary 3. *With the assumptions from Theorem 1, we have*

$$\begin{aligned} I(g)(\rho) &= \frac{1}{\rho^2}|B_1(2\pi b\rho,2\pi a\rho)|^2\left|\int_a^b g(s)\,ds\right|^2 \\ &+\frac{4\pi}{\rho(b-a)}\operatorname{Re}\left[B_1(2\pi b\rho,2\pi a\rho)\int_a^b g(s)\,ds\right. \\ &\left.\times\int_a^b\int_a^b k(r,s)\,r\overline{g'(s)J_0(2\pi r\rho)}drds\right] \\ &+\frac{4\pi^2}{(b-a)^2}\left|\int_a^b\int_a^b k(r,s)\,g'(s)\,rJ_0(2\pi r\rho)\,drds\right|^2 \end{aligned} \tag{2.15}$$

for all $\rho \in [a,b], \rho \neq 0.$
If $a = 0,\ b = 1,$ *then*

$$I(g)(\rho) = \frac{1}{\rho^2}|J_1(2\pi\rho)|^2 \left|\int_0^1 g(s)\,ds\right|^2 + \frac{4\pi}{\rho}\operatorname{Re}\left[J_1(2\pi\rho)\int_0^1 g(s)\,ds \times \int_0^1\int_0^1 \tilde{k}(r,s)\, r\overline{g'(s)J_0(2\pi r\rho)}\,drds\right] + 4\pi^2\left|\int_0^1\int_0^1 \tilde{k}(r,s)\,g'(s)\,rJ_0(2\pi r\rho)\,drds\right|^2 \tag{2.16}$$

for all $\rho \in (0,1]$.

3. Integral Inequalities

The main aim of this section is to point out an estimate for the remainder

$$R[g](\rho) := \frac{2\pi}{b-a}\int_a^b\int_a^b k(r,s)\,g'(s)\,rJ_0(2\pi r\rho)\,drds \tag{3.1}$$

in formula (2.3).
We can state the following integral inequalities.

Theorem 4. *Let g be as in Theorem 1. Then*

$$\left|G(g)(\rho) - \frac{1}{\rho}B_1(2\pi b\rho, 2\pi a\rho) \times \int_a^b g(s)\,ds\right| \tag{3.2}$$
$$\leq \begin{cases} \frac{2\pi}{b-a}\|g'\|_\infty \int_a^b \left[\frac{(r-a)^2+(b-r)^2}{2}\right]|r|\,dr \ \text{if } g' \in L_\infty[a,b]; \\ \frac{2\pi}{b-a}\|g'\|_p \left[\int_a^b \left[\frac{(r-a)^{q+1}+(b-r)^{q+1}}{q+1}\right]|r|^q\,dr\right]^{\frac{1}{q}} \\ \text{if } g' \in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1; \\ 2\pi\|g'\|_1 \int_a^b |r|\,dr \end{cases}$$

for all $\rho \in [a,b], \rho \neq 0.$

Proof. Using the representation (2.3), we get

$$\left|G(g)(\rho) - \frac{1}{\rho}B_1(2\pi b\rho, 2\pi a\rho)\int_a^b g(s)\,ds\right| \leq \frac{2\pi}{b-a}\int_a^b\int_a^b |k(r,s)|\,|r|\,\left|g'(s)\right|\,|J_0(2\pi r\rho)|\,drds =: A(\rho)$$

for all $\rho \in [a,b] \setminus \{0\}$.
It is obvious that

$$|J_0(2\pi r\rho)| = \left|\frac{1}{2\pi}\int_0^{2\pi} e^{-x\cos\beta} d\beta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|e^{-x\cos\beta}\right| d\beta = \frac{1}{2\pi}\int_0^{2\pi} d\beta = 1.$$

In this way we can state the following inequality

$$A(\rho) \leq \frac{2\pi}{b-a}\int_a^b\int_a^b |k(r,s)|\,|r|\,\left|g'(s)\right| drds := B(\rho).$$

It is obvious that

$$\begin{aligned} B(\rho) &\leq \left\|g'\right\|_\infty \frac{2\pi}{b-a}\int_a^b\int_a^b |k(r,s)|\,|r|\,drds \\ &= \left\|g'\right\|_\infty \frac{2\pi}{b-a}\int_a^b \left(\int_a^b |k(r,s)|\,ds\right)|r|\,dr \\ &= \left\|g'\right\|_\infty \frac{2\pi}{b-a}\int_a^b \left[\int_a^r (s-a)\,ds + \int_r^b (b-s)\,ds\right]|r|\,dr \\ &= \left\|g'\right\|_\infty \frac{2\pi}{b-a}\int_a^b \left[\frac{(r-a)^2+(b-r)^2}{2}\right]|r|\,dr, \end{aligned} \tag{3.3}$$

and the first inequality in (3.2) is obtained.
For the second inequality, we use Hölder's integral inequality for double integrals to get:

$$\begin{aligned} &B(\rho) \\ &\leq \frac{2\pi}{b-a}\left(\int_a^b\int_a^b |k(r,s)|^q |r|^q drds\right)^{\frac{1}{q}}\left(\int_a^b\int_a^b \left|g'(s)\right|^p dsdr\right)^{\frac{1}{p}} \\ &= \frac{2\pi}{b-a}(b-a)^{\frac{1}{p}}\left\|g'\right\|_p \left(\int_a^b\left(\int_a^r (s-a)^q ds + \int_r^b (b-s)^q ds\right)|r|^q dr\right)^{\frac{1}{q}} \\ &= \frac{2\pi}{b-a}(b-a)^{\frac{1}{p}}\left\|g'\right\|_p \left(\int_a^b \left[\frac{(r-a)^{q+1}+(b-r)^{q+1}}{q+1}\right]|r|^q dr\right)^{\frac{1}{q}} \end{aligned} \tag{3.4}$$

and the second inequality is also proved.
Finally, we observe that

$$\begin{aligned} B(\rho) &\leq \frac{2\pi}{b-a}\sup_{(r,s)\in[a,b]^2}|k(r,s)|\int_a^b |r|\,dr \cdot \int_a^b \left|g'(s)\right| ds \\ &= \frac{2\pi}{b-a}(b-a)\left\|g'\right\|_1 \int_a^b |r|\,dr = 2\pi\left\|g'\right\|_1 \int_a^b |r|\,dr \end{aligned} \tag{3.5}$$

and the theorem is proved. ∎

Remark 1. *In practical applications* $a \geq 0$, *so the first bound in* (3.2) *becomes*

$$\begin{aligned}
&\frac{2\pi}{b-a}\left\|g'\right\|_\infty \int_a^b \left[\frac{(r-a)^2+(b-r)^2}{2}\right] rdr \\
&= \frac{2\pi}{b-a}\left\|g'\right\|_\infty \left[\frac{1}{2}\int_a^b (r-a)^2\, rdr + \frac{1}{2}\int_a^b (b-r)^2\, rdr\right] \\
&= \frac{2\pi}{b-a}\left\|g'\right\|_\infty \left[\frac{1}{24}(b-a)^3(3b+a) + \frac{1}{24}(b-a)^3(3a+b)\right] \\
&= \frac{2\pi}{b-a}\left\|g'\right\|_\infty \frac{1}{24}(b-a)^3(4b+4a) \\
&= \frac{\pi}{3}\left\|g'\right\|_\infty (b-a)^2(a+b).
\end{aligned}$$

The second term will be:

$$\frac{2\pi}{(b-a)(q+1)^{\frac{1}{q}}}\left\|g'\right\|_p \left[\int_a^b (r-a)^{q+1} r^q dr + \int_a^b (b-r)^{q+1} r^q dr\right]^{\frac{1}{q}},$$

and the third term is:

$$\pi\left\|g'\right\|_1 (b+a)(b-a).$$

Consequently, we can state that

$$\left|G(g)(\rho) - \frac{1}{\rho}B_1(2\pi b\rho, 2\pi a\rho)\int_a^b g(s)\,ds\right| \tag{3.6}$$
$$\leq \begin{cases} \frac{\pi}{3}\left\|g'\right\|_\infty (a+b)(b-a)^2 \ \ \textit{if } g' \in L_\infty[a,b]; \\ \\ \frac{2\pi}{(b-a)(q+1)^{\frac{1}{q}}}\left\|g'\right\|_p \left[\int_a^b (r-a)^{q+1} r^q dr + \int_a^b (b-r)^{q+1} r^q dr\right]^{\frac{1}{q}} \\ \textit{if } g' \in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \pi\left\|g'\right\|_1 (b+a)(b-a) \end{cases}$$

for all $\rho \in [a,b]$.

Remark 2. *If we assume that* $a = 0$ *and* $b > 0$ *in* (3.6), *then we get*

$$\left|G(g)(\rho) - \frac{J_1(2\pi b\rho)}{\rho}\times\int_0^b g(s)\,ds\right| \tag{3.7}$$
$$\leq \begin{cases} \frac{\pi}{3}\left\|g'\right\|_\infty b^3 \ \ \textit{if } g' \in L_\infty[a,b] \\ \pi\left\|g'\right\|_1 b^2 \end{cases}.$$

The above inequality shows that

$$G(g)(\rho) \approx \frac{J_1(2\pi b\rho)}{\rho} \times \int_0^b g(s)\,ds$$

when $b \to 0+$ and the precision of approximation is 3.

Remark 3. *Let us observe that the integral*

$$I = \int_a^b (b-r)^2 (r-a)^2\,dr$$

can be written in a different form. By using the change of variable $r = a(1-t) + bt$, $t \in [0,1]$, we obtain

$$\begin{aligned} I &= (b-a) \int_0^1 [b-(1-t)a-tb]^q [(1-t)a+tb-a]^s\,dt \\ &= (b-a)^{q+s+1} \int_0^1 (1-t)^q t^s dt = (b-a)^{q+s+1} \beta(q+1,s+1) \end{aligned}$$

where $\beta(\cdot,\cdot)$ is a Beta function, that is,

$$\beta(q,s) = \int_0^1 (1-t)^{q-1} t^{s-1} dt; \quad q,s>0.$$

Now, coming back to the second bound in (3.6) for $a=0$, we can observe that

$$\int_0^b r^{q+1} r^q dr = \int_0^b r^{2q+1} dr = \frac{b^{2q+2}}{2q+2}$$

and

$$\int_0^b (b-r)^{q+1} r^q dr = b^{2q+2} \beta(q+2,q+1).$$

Consequently, we have the inequality

$$\begin{aligned} &\left| G(g)(\rho) - \frac{J_1(2\pi b\rho)}{\rho} \int_0^b g(s)\,ds \right| \\ &\leq \frac{2\pi}{b(q+1)^{\frac{1}{q}}} \left\| g' \right\|_p \left[\frac{b^{2q+2}}{2q+2} + b^{2q+2}\beta(q+2,q+1) \right]^{\frac{1}{q}} \\ &= \frac{2\pi b^{1+\frac{2}{q}}}{(q+1)^{\frac{1}{q}}} \left[\frac{1}{2q+2} + \beta(q+2,q+1) \right]^{\frac{1}{q}} \end{aligned} \tag{3.8}$$

for all $\rho \in (0,b]$.

In some practical applications the upper limit of integration is $b=1$, therefore the following corollary is required.

Corollary 5. *Let* $g : [0,1] \to \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *be an absolutely continuous mapping on* $[0,1]$. *Then we have the inequality*

$$\left| G(g)(\rho) - \frac{J_1(2\pi\rho)}{\rho} \times \int_0^1 g(s)\,ds \right| \tag{3.9}$$

$$\leq \begin{cases} \frac{\pi}{3}\left\|g'\right\|_\infty \ \textit{if } g' \in L_\infty[a,b] \\ \\ \frac{2\pi}{(q+1)^{\frac{1}{q}}}\left[\frac{1}{2q+2} + \beta(q+2, q+1)\right]^{\frac{1}{q}} \left\|g'\right\|_p \\ \textit{if } g' \in L_p[0,1], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \pi\left\|g'\right\|_1 \end{cases}$$

for all $\rho \in (0,1]$.

Using the inequality (3.2) which provides upper bounds for the remainder $R[g](\rho)$, we will point out the following inequality which approximates the mapping $I(g)(\rho)$ (c.f.(2.13)).

Corollary 6. *Let* $g : [a,b] \to \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *be an absolutely continuous mapping on* $[a,b]$. *Then we have the inequality*

$$\left| I(g)(\rho) - \frac{1}{\rho^2}\left|B_1(2\pi b\rho, 2\pi a\rho)\right|^2 \left|\int_a^b g(s)\,ds\right|^2 \right|$$

$$\leq \left[\frac{2}{\rho}\left|B_1(2\pi b\rho, 2\pi a\rho)\right| \left|\int_a^b g(s)\,ds\right| + E(g)(\rho)\right] E(g)(\rho)$$

where

$$E(g)(\rho) := \begin{cases} \frac{2\pi}{b-a}\left\|g'\right\|_\infty \int_a^b \left[\frac{(r-a)^2+(b-r)^2}{2}\right]|r|\,dr \ \textit{if } g' \in L_\infty[a,b]; \\ \\ \frac{2\pi}{b-a}\left\|g'\right\|_p \left[\int_a^b \left[\frac{(r-a)^{q+1}+(b-r)^{q+1}}{q+1}\right]|r|^q\,dr\right]^{\frac{1}{q}} \\ \textit{if } g' \in L_p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ 2\pi\left\|g'\right\|_1 \int_a^b |r|\,dr \end{cases}$$

for all $\rho \in [a,b)$, $\rho \neq 0$.

The proof is obvious by Corollary 3 and Theorem 4.

4. A Quadrature Formula

In this section we point out a quadrature formula for the Hankel's transform

$$G(g)(\rho) := 2\pi \int_0^1 g(s)\, r J_0(2\pi r\rho)\, dr$$

where g is assumed to be absolutely continuous on $[0,1]$.

Firstly, let us assume that in formula (3.6) we have $0 \le a \le b \le 1$. Then we have the coarser upper bound

$$\left| G(g)(\rho) - \frac{1}{\rho} B_1(2\pi b\rho, 2\pi a\rho) \times \int_a^b g(s)\, ds \right| \le \begin{cases} \frac{2\pi}{3} \|g'\|_\infty (b-a)^2 & \text{if } g' \in L_\infty[a,b]; \\ \frac{2^{1+\frac{1}{q}}\pi \|g'\|_p}{[(q+1)(q+2)]^{\frac{1}{q}}} (b-a)^{\frac{2}{q}} & \text{if } g' \in L_p[a,b],\, p>1,\, \frac{1}{p}+\frac{1}{q}=1; \\ 2\pi \|g'\|_1 (b-a). \end{cases} \tag{4.1}$$

Indeed, from (3.6) we have

$$\left| G(g)(\rho) - \frac{1}{\rho} B_1(2\pi b\rho, 2\pi a\rho) \times \int_a^b g(s)\, ds \right| \le \begin{cases} \frac{\pi}{3} \|g'\|_\infty (a+b)(b-a)^2; \\ \frac{2\pi}{(b-a)(q+1)^{\frac{1}{q}}} \|g'\|_p \left[\int_a^b (r-a)^{q+1} r^q dr + \int_a^b (b-r)^{q+1} r^q dr \right]^{\frac{1}{q}}; \\ \pi \|g'\|_1 (b+a)(b-a). \end{cases}$$

As $0 \le a \le b \le 1$, then $a+b \le 2$ and hence:

$$\frac{\pi}{3} \left\| g' \right\|_\infty (a+b)(b-a)^2 \le \frac{2\pi}{3} \left\| g' \right\|_\infty (b-a)^2$$

and

$$\pi \left\| g' \right\|_1 (b+a)(b-a) \le 2\pi \left\| g' \right\|_1 (b-a).$$

On the other hand,

$$\int_a^b (r-a)^{q+1} r^q dr \le \int_a^b (r-a)^{q+1} dr = \frac{(b-a)^{q+2}}{q+2}$$

and

$$\int_a^b (b-r)^{q+1} r^q dr \le \int_a^b (b-r)^{q+1} dr = \frac{(b-a)^{q+2}}{q+2}.$$

Consequently, we get

$$
\begin{aligned}
& \frac{2\pi}{(b-a)(q+1)^{\frac{1}{q}}}\left\|g'\right\|_p\left[\int_a^b (r-a)^{q+1} r^q dr+\int_a^b (b-r)^{q+1} r^q dr\right]^{\frac{1}{q}} \\
\leq\ & \frac{2\pi}{(b-a)(q+1)^{\frac{1}{q}}}\left\|g'\right\|_p\left[\frac{2(b-a)^{q+2}}{q+2}\right]^{\frac{1}{q}} \\
=\ & \frac{2^{1+\frac{1}{q}}\pi\left\|g'\right\|_p (b-a)^{\frac{2}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}
\end{aligned}
$$

and the second inequality in (4.1) is also proved.

Now, consider $I_n : 0 = x_0 < x_1 < ... < x_{n-1} < x_n = 1$ a division of the interval $[0,1]$. Put $h_i := x_{i+1} - x_i$, $(i = 0, ..., n-1)$ and $\upsilon(h) := \max\{h_i | i = 0, ..., n-1\}$ the norm of the division. Construct the sums

$$
\begin{aligned}
(4.2)\qquad & H(g, I_n, \rho) \\
:\ =\ & \frac{1}{\rho}\sum_{i=0}^{n-1} B_1(2\pi x_{i+1}\rho, 2\pi x_i\rho)\times\int_{x_i}^{x_{i+1}} g(s)\,ds \\
=\ & \frac{1}{\rho}\sum_{i=0}^{n-1}\frac{1}{h_i}\left[x_{i+1}J_1(2\pi x_{i+1}\rho)-x_iJ_1(2\pi x_i\rho)\right]\times\int_{x_i}^{x_{i+1}} g(s)\,ds.
\end{aligned}
$$

We can state the following theorem concerning the approximation of Hankel's transform in terms of the quadrature formula (4.2).

Theorem 7. *Let $g : [0,1] \to \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) be an absolutely continuous mapping on $[0,1]$. Then we have*

$$
(4.3)\qquad G(g)(\rho) = H(g, I_n, \rho) + R(g, I_n, \rho),\ \rho \in (0,1]
$$

where $H(g, I_n, \rho)$ is as given by the formula (4.2) and the remainder $R(g, I_n, \rho)$ satisfies the estimate

$$
(4.4)\qquad |R(g, I_n, \rho)| \leq \begin{cases} \frac{2\pi}{3}\left\|g'\right\|_\infty \sum_{i=0}^{n-1} h_i^2 \ \text{if } g' \in L_\infty[a,b]; \\[2ex] \frac{2^{1+\frac{1}{q}}\pi}{[(q+1)(q+2)]^{\frac{1}{q}}}\left\|g'\right\|_p\left(\sum_{i=0}^{n-1} h_i^2\right)^{\frac{1}{q}} \\ \text{if } g' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] 2\pi\nu(h)\left\|g'\right\|_1 \end{cases}
$$

for all $\rho \in (0,1]$.

Proof. Apply formula (4.1) on the subintervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to get

$$\left|2\pi \int_{x_i}^{x_{i+1}} g(r)\, r J_0(2\pi r\rho)\, dr - \frac{1}{\rho} B_1(2\pi x_{i+1}\rho, 2\pi x_i\rho) \times \int_{x_i}^{x_{i+1}} g(s)\, ds\right|$$
$$\leq \begin{cases} \frac{2\pi}{3} \sup_{r\in[x_i,x_{i+1}]} |g'(r)|\, h_i^2 \\[2ex] \frac{2^{1+\frac{1}{q}}\pi}{[(q+1)(q+2)]^{\frac{1}{q}}} \left(\int_{x_i}^{x_{i+1}} |g'(s)|^p\, ds\right)^{\frac{1}{p}} h_i^{\frac{2}{q}} \\[2ex] 2\pi \left(\int_{x_i}^{x_{i+1}} |g'(s)|\, ds\right) h_i. \end{cases}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$|R(g, I_n, \rho)|$$
$$\leq \sum_{i=0}^{n-1} \left|2\pi \int_{x_i}^{x_{i+1}} g(r)\, r J_0(2\pi r\rho)\, dr - \frac{1}{\rho} B_1(2\pi x_{i+1}\rho, 2\pi x_i\rho) \times \int_{x_i}^{x_{i+1}} g(s)\, ds\right|$$
$$\leq \begin{cases} \frac{2\pi}{3} \sum_{i=0}^{n-1} \sup_{r\in[x_i,x_{i+1}]} |g'(r)|\, h_i^2 \\[2ex] \frac{2^{1+\frac{1}{q}}\pi}{[(q+1)(q+2)]^{\frac{1}{q}}} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} |g'(s)|^p\, ds\right)^{\frac{1}{p}} h_i^{\frac{2}{q}} \\[2ex] 2\pi \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} |g'(s)|\, ds\right) h_i \end{cases}.$$

Now, observe that

$$\sum_{i=0}^{n-1} \sup_{r\in[x_i,x_{i+1}]} \left|g'(r)\right| h_i^2 \leq \left\|g'\right\|_\infty \sum_{i=0}^{n-1} h_i^2$$

and then the first inequality in (4.4) is proved.
Using Hölder's discrete inequality, we deduce

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}}\left|g'(s)\right|^p ds\right)^{\frac{1}{p}} h_i^{\frac{2}{q}} \\
\leq\ &\left(\sum_{i=0}^{n-1}\left[\left(\int_{x_i}^{x_{i+1}}\left|g'(s)\right|^p ds\right)^{\frac{1}{p}}\right]^p\right)^{\frac{1}{p}} \times\left[\sum_{i=0}^{n-1}\left(h_i^{\frac{2}{q}}\right)^q\right]^{\frac{1}{q}} \\
=\ &\left(\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}}\left|g'(s)\right|^p ds\right)\right)^{\frac{1}{p}} \times\left(\sum_{i=0}^{n-1} h_i^2\right)^{\frac{1}{q}} \\
=\ &\left\|g'\right\|_p\left(\sum_{i=0}^{n-1} h_i^2\right)^{\frac{1}{q}}.
\end{aligned}$$

Finally, let us observe that

$$\begin{aligned}
2\pi\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}}\left|g'(s)\right| ds\right) h_i &\leq 2\pi\nu(h)\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}}\left|g'(s)\right| ds\right) \\
&= 2\pi\nu(h)\left\|g'\right\|_1
\end{aligned}$$

and the theorem is completely proved. ∎

Remark 4. *It is obvious that*

$$\sum_{i=0}^{n-1} h_i^2 \leq \nu(h)\sum_{i=0}^{n-1} h_i$$

and then by (4.4) *we deduce the coarser upper bound*

$$|R(g, I_n, \rho)| \leq \begin{cases} \frac{2\pi}{3}\left\|g'\right\|_\infty \nu(h); \\ \\ \frac{2^{1+\frac{1}{q}}\pi}{[(q+1)(q+2)]^{\frac{1}{q}}}\left\|g'\right\|_p[\nu(h)]^{\frac{1}{q}}; \\ \\ 2\pi\nu(h)\left\|g'\right\|_1. \end{cases} \tag{4.5}$$

Now, observe that if $\nu(h) \to 0$, *then every branch on the right hand side of* (4.5) *goes to* 0, *which proves that* $H(g, I_n, \rho)$ *approximates the Hankel's transform with any accuracy.*

In practical problems the interval $[0, 1]$ is devided in equidistant subintervals by the division $I_n : x_i = \frac{i}{n}, i = 0, ..., n$.

If we consider the sum

$$
\begin{aligned}
(4.6) \qquad & H_n(g,\rho) \\
: &= \frac{1}{\rho}\sum_{i=0}^{n-1} B_1\left(\frac{2\pi(i+1)\rho}{n},\frac{2\pi i\rho}{n}\right)\times\int_{\frac{i}{n}}^{\frac{i+1}{n}} g(s)\,ds \\
&= \frac{1}{\rho}\sum_{i=0}^{n-1}\left[(i+1)J_1\left(2\pi\frac{i+1}{n}\rho\right)-iJ_1\left(2\pi\frac{i}{n}\rho\right)\right]\times\int_{\frac{i}{n}}^{\frac{i+1}{n}} g(s)\,ds,
\end{aligned}
$$

then we can state the following corollary.

Corollary 8. *Let g be as in Theorem 7. Then we have*

$$
G(g)(\rho)=H_n(g,\rho)+R_n(g,\rho),\rho\in(0,1] \tag{4.7}
$$

where $H_n(g,\rho)$ is as given in (4.6) and the remainder $R_n(g,\rho)$ satisfies the estimate

$$
|R_n(g,\rho)|\le\begin{cases}\frac{2\pi}{3n}\|g'\|_\infty & \text{if } g'\in L_\infty[0,1]; \\ \\ \frac{2^{1+\frac{1}{q}}\pi}{[(q+1)(q+2)]^{\frac{1}{q}}n^{\frac{1}{q}}}\|g'\|_p & \text{if } g'\in L_p[0,1], p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{2\pi}{n}\|g'\|_1. \end{cases} \tag{4.8}
$$

Remark 5. *Supposing that we know $\|g'\|_\infty$ and would like to approximate $G(g)(\rho)$ with a given error $\varepsilon>0$, then we have to divide the interval $[0,1]$ into at least $n_\varepsilon\in\mathbb{N}^*$ points, where*

$$
n_\varepsilon := \left[\frac{2\pi}{3\varepsilon}\|g'\|_\infty\right]+1
$$

where $[x]$ is the integer part of $x\in\mathbb{R}$.

Using the tools provided in papers [1], [3]-[7] the authors are going to point out other quadrature formulae for Hankel transform for mappings of bounded variation, monotonic, of r-Hölder's type etc.

For a preprint of this paper containing some numerical experiments, see http://rgmia.vu.edu.au/v5(E).html. We omit the details.

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