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MINIMAL BASICALLY DISCONNECTED COVERS OF SOME EXTENSIONS

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ABSTRACT. Observing that each Tychonoff space X has the minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ and the realcompactification υX, we introduce a concept of stable $\sigma Z(X)^{\#}$-ultrafilters and give internal characterizations of Tychonoff spaces X for which $\Lambda(\upsilon X) = \upsilon(\Lambda X)$.

1. Introduction

It is well-known that each regular space X has the minimal extremally disconnected cover (EX, k_X) and that for a Tychonoff space X, $\beta(EX) = E(\beta X)$ ([5]). Also, internal conditions on a Tychonoff space X that is equivalent to $E(\upsilon X) = \upsilon(EX)$ is known ([5]).

In [6], Vermeer showed that every Tychonoff space X has the minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ and that the minimal basically disconnected cover ΛX of a compact space X is given by the Stone-space $S(\sigma Z(X)^{\#})$ of $\sigma Z(X)^{\#}$. In [3], for any locally weakly Lindelöf space X, ΛX is characterized by the filter space $\{\alpha : \alpha$ is a fixed $\sigma Z(X)^{\#}$-ultrafilter$\}$.

In this paper, we first introduce the notion of stable $\sigma Z(X)^{\#}$-ultrafilters and show that if $\Lambda(\beta X) = \beta(\Lambda X)$, then ΛX is given by the filter space $\{\alpha : \alpha$ is a fixed $\sigma Z(X)^{\#}$-ultrafilter$\}$. Using this, we will show that if $\Lambda(\beta X) = \beta(\Lambda X)$ for a Tychonoff space X, then $\Lambda(\upsilon X) = \upsilon(\Lambda X)$ if and only if every stable $\sigma Z(X)^{\#}$-ultrafilter has the countable intersection property.

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All spaces in this paper are assumed to be Tychonoff spaces and for a space X, β_X : X $\longrightarrow$ βX denotes the Stone-Čech compactification of X. For the terminology, we refer to [1] and [2].

2. The minimal basically disconnected cover

We recall that a continuous map f : Y $\longrightarrow$ X is called a *covering map* if it is an onto, perfect, irreducible map. In case, (Y, f) is called a *cover of X*.

LEMMA 2.1 ([5]). *Consider the following commutative diagram in the category* **Top** *of topological spaces and continuous maps:*

$$\begin{array}{ccc} Z & \xrightarrow{\ f\ } & X \\ {\scriptstyle j_1}\big\downarrow & & \big\downarrow{\scriptstyle j_2} \\ W & \xrightarrow{\ g\ } & Y, \end{array}$$

where j_1, j_2 are dense embeddings. Then we have the following:

(a) *if f is a perfect onto map and g is an onto map, then g(W−Z) = Y−X,*

(b) *if g is a covering map and f is a perfect onto map, then f is a covering map, and*

(c) *if W, Y are compact spaces and f is a covering map, then g is also a covering map.*

Recall that the collection R(X) of all regular closed sets in a space X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows: if A $\in$ R(X) and $\{A_i : i \in I\} \subseteq$ R(X), then

$\bigvee\{A_i : i \in I\} = cl_X(\cup\{A_i : i \in I\})$,

$\bigwedge\{A_i : i \in I\} = cl_X(int_X(\cap\{A_i : i \in I\}))$, and

$A' = cl_X(X-A)$

and a sublattice of R(X) is a subset of R(X) that contains ϕ , X and is closed under finite joins and meets.

LEMMA 2.2 ([3], [5]). (a) *Let f : Y $\longrightarrow$ X be a covering map. Then the map ψ : R(Y) $\longrightarrow$ R(X), defined by ψ(A) = f(A), is a Boolean isomorphism and the inverse map ψ^{-1} of ψ is given by ψ^{-1} (B) = $cl_Y(f^{-1}$ $(int_X$(B))).*

(b) *Let Y be an extension of a space X. Then the map $\phi : R(Y) \longrightarrow R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism and the inverse map ϕ^{-1} is given by $\phi^{-1}(B) = cl_Y(B)$.*

A lattice L is called *σ-complete* if every countable subset of L has join and meet. For a subset M of a complete Boolean algebra L, σM denotes the smallest σ-complete Boolean subalgebra of L containing M. For any space X, Z(X) denotes the set of all zero-sets and let $Z(X)^\# = \{cl_X(int_X(A)) : A \in Z(X)\}$. For a space X and a zero-set Z in X, there is a zero-set A in βX with $A \cap X = Z$. Using this and Lemma 2.2, we have the following:

COROLLARY 2.3. *For any space X, the isomorphism $\phi : R(\beta X) \longrightarrow R(X)$ induces Boolean isomorphisms $\sigma Z(\beta X)^\# \longrightarrow \sigma Z(X)^\#$ and $\sigma Z(\upsilon X)^\# \longrightarrow \sigma Z(X)^\#$.*

A subspace Y of a space X is said to be *C^*-embedded in X* if for any bounded real-valued continuous map $f : Y \longrightarrow R$, there is a bounded real-valued continuous map $g : X \longrightarrow R$ with $g|_Y = f$.

DEFINITION 2.4. A space X is called *basically disconnected* if every cozero-set in X is C^*-embedded in X.

DEFINITION 2.5. Let (Y, f) be a cover of a space X. Then (Y, f) is called

(a) a *basically disconnected cover of X* if Y is a basically disconnected space,

(b) a *minimal basically disconnected cover of X* if it is a basically disconnected cover of X and for any basically disconnected cover (Z, g) of X, there is a covering map $h : Z \longrightarrow Y$ with $f \circ h = g$.

For any lattice (L, $\vee$, $\wedge$) with the bottom element 0 and top element e, a non-empty subset F of L is called an *L-filter* if (i) $0 \notin F$, (ii) $x, y \in F$ implies $x \wedge y \in F$, and (iii) $x \in F$ and $x \leq y \in L$ implies $y \in F$ and a maximal L-filter is called an *L-ultrafilter.*

Let $\mathcal{A}$ be a sublattice of R(X) for a space X and $\sum$ a subset of $\{\mathcal{U} : \mathcal{U}$ is an $\mathcal{A}$-ultrafilter$\}$. For any $A \in \mathcal{A}$, let $A^* = \{\mathcal{U} \in \sum : A \in \mathcal{U}\}$. Note that if $\mathcal{U}$ is an $\mathcal{A}$-ultrafilter and $A \cup B \in \mathcal{U}$, then $A \in \mathcal{U}$ or $B \in \mathcal{U}$. Hence $\{A^* : A \in \mathcal{A}\}$ is a closed base for some topology on $\sum$. In case, the set $\sum$ endowed with the topology generated by $\{\sum - A^* : A \in \mathcal{A}\}$, is called a *filter space*. An $\mathcal{A}$-filter $\mathcal{U}$ is said to be *fixed* if $\cap\mathcal{U} \neq \phi$.

For any space X, $(\Lambda X, \Lambda_X)((\Lambda(\beta X), \Lambda_\beta)$, respectively) denotes the minimal basically disconnected cover of $X(\beta X$, respectively). Vermeer showed that for a compact space X, ΛX is given by the Stone-space $S(\sigma Z(X)^\#)$ of $\sigma Z(X)^\#$ and $\Lambda_X(\alpha) = \cap\alpha$ $(\alpha \in \Lambda X)$([6]).

Recall that a space X is called *weakly Lindelöf* if every open cover $\mathcal{U}$ of X has a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\mathcal{V}$ is dense in X and that a space X is called *locally weakly Lindelöf* if every element of X has a weakly Lindelöf neighborhood. In [3], it is shown that for any locally weakly Lindelöf space X, ΛX is given by the filter space $\{\alpha : \alpha$ is a fixed $\sigma Z(X)^\#$-ultrafilter$\}$ and $\Lambda_X(\alpha) = \cap\alpha$.

REMARK. Let X be a space. Then there is the Stone extension $\Lambda^\beta : \beta(\Lambda X) \longrightarrow \beta X$ such that the following diagram

$$\begin{array}{ccc} \Lambda X & \xrightarrow{\Lambda_X} & X \\ {\scriptstyle \beta_{\Lambda X}}\downarrow & & \downarrow{\scriptstyle \beta_X} \\ \beta(\Lambda X) & \xrightarrow{\Lambda^\beta} & \beta X \end{array}$$

commutes ([5]).

By Lemma 2.1, Λ^β is a covering map. Since $\beta(\Lambda X)$ is basically disconnected([6]), there is a covering map $l_X : \beta(\Lambda X) \longrightarrow \Lambda(\beta X)$ such that $\Lambda^\beta = \Lambda_\beta \circ l_X$. If l_X is a homeomorphism, then we write $\beta(\Lambda X) = \Lambda(\beta X)$ and in case, we will identify $(\beta(\Lambda X), \Lambda^\beta)$ and $(\Lambda(\beta X), \Lambda_\beta)$. In [3], it is shown that if X is a weakly Lindelöf space, then $\beta(\Lambda X) = \Lambda(\beta X)$.

PROPOSITION 2.6. *Let X be a space such that $\beta(\Lambda X) = \Lambda(\beta X)$. Then ΛX is given by the filter space $\{\alpha : \alpha$ is a fixed $\sigma Z(X)^\#$-ultrafilter$\}$ and $\Lambda_X(\alpha) = \cap\alpha$.*

PROOF. Consider the following diagram:

$$\begin{array}{ccccc} \Lambda X & \xrightarrow{l_X^\circ} & \Lambda_\beta^{-1}(X) & \xrightarrow{\Lambda_{\beta X}} & X \\ {\scriptstyle \beta_\Lambda}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \beta_X} \\ \beta(\Lambda X) & \xrightarrow{l_X} & \Lambda(\beta X) & \xrightarrow{\Lambda_\beta} & \beta X, \end{array}$$

where j is the inclusion map and Λ_{β_X} is the restriction and corestriction of Λ_β with respect to $\Lambda_\beta^{-1}(X)$ and X, respectively. Since the right hand rectangle is a pullback in **Top**, there is a continuous map l_X° : $\Lambda X \longrightarrow \Lambda_\beta^{-1}(X)$ such that $\Lambda_{\beta_X} \circ l_X^\circ = \Lambda_X$ and the left hand rectangle is commutative. We will show that l_X° is a homeomorphism. Take any $x \in \Lambda_\beta^{-1}(X)$. Then there is $y \in \beta(\Lambda X)$ with $l_X(y) = x$ and $\Lambda_\beta(x) = \Lambda_{\beta_X}(x) \in X$. Since Λ_X is a covering maps, by Lemma 2.1, $y \in \Lambda X$. Hence l_X° is onto. Since $\Lambda_{\beta_X} \circ l_X^\circ = \Lambda_X$ and Λ_X is perfect, l_X° is a perfect map([5]). Since l_X is 1-1, l_X° is a homeomorphism. Hence $(\Lambda_\beta^{-1}(X), \Lambda_{\beta_X})$ is the minimal basically disconnected cover of X. By Corollary 2.3, $\Lambda_\beta^{-1}(X)$ is the filter space $\{\alpha : \alpha$ is a fixed $\sigma Z(X)^\#$-ultrafilter$\}$ and $\Lambda_X(\alpha) = \cap\alpha$. □

For any space X and $x \in X$, let $\mathcal{F}(x) = \{A : A \in \sigma Z(X)^\#,\ x \in \text{int}_X(A)\}$.

PROPOSITION 2.7. *Let X be a space such that* ΛX *is given by the filter space* $\{\alpha : \alpha$ *is a fixed* $\sigma Z(X)^\#$*-ultrafilter*$\}$ *and* $\Lambda_X(\alpha) = \cap\alpha$, $\mathcal{U}$ *a* $\sigma Z(X)^\#$*-ultrafilter and* $x \in X$. *Then we have the following:*

(a) $\Lambda_X(\mathcal{U}) = x$ *if and only if* $\mathcal{F}(x) \subseteq \mathcal{U}$,

(b) *if* $A \in \sigma Z(X)^\#$, *then* $\Lambda_X(A^*) = A$, *and*

(c) *if* $\{A_n : n \in N\}$ *is a decreasing sequence in* $\sigma Z(X)^\#$, *then* $\Lambda_X(\cap\{A_n^* : n \in N\}) = \cap\{A_n : n \in N\}$.

PROOF. (a) ($\Rightarrow$) Suppose that $\Lambda_X(\mathcal{U}) = \cap\mathcal{U} = x$ and $\mathcal{F}(x) \not\subseteq \mathcal{U}$. Pick $A \in \mathcal{F}(x) - \mathcal{U}$. Since $\mathcal{U}$ is a $\sigma Z(X)^\#$-ultrafilter, $A' \in \mathcal{U}$. Since $\Lambda_X(\mathcal{U}) = x$, $x \in A' = X - \text{int}_X(A)$. This is a contradiction.

($\Leftarrow$) Suppose that $\mathcal{F}(x) \subseteq \mathcal{U}$. Since $\mathcal{F}(x)$ is a local base at x in X, $\Lambda_X(\mathcal{U}) = \cap\mathcal{U} = x$.

(b) Let $A \in \sigma Z(X)^\#$ and $\mathcal{U} \in A^*$. Then $A \in \mathcal{U}$ and so the unique element in $\cap\,\mathcal{U}$ belongs to A. Thus $\Lambda_X(A^*) \subseteq A$. Conversely, suppose that $y \in A$. Let $\mathcal{G} = \mathcal{F}(y) \cup \{A\}$. Then $\mathcal{G}$ has the finite meet property. Hence there is a $\sigma Z(X)^\#$- ultrafilter $\mathcal{V}$ such that $\mathcal{F}(y) \cup \{A\} \subseteq \mathcal{V}$. By (a), $\Lambda(\mathcal{V}) = y$ and $\Lambda_X(A^*) \supseteq A$.

(c) By (b), $\Lambda_X(\cap\{A_n^* : n \in N\}) \subseteq \cap\{A_n : n \in N\}$. Conversely, if $y \in \cap\{A_n : n \in N\}$, then $\mathcal{F}(y) \cup \{A_n : n \in N\}$ has the finite meet property and hence it is contained in a $\sigma Z(X)^\#$-ultrafilter $\mathcal{V}$ and so $\mathcal{V} \in \cap\{A_n^* : n \in N\}$ and $\Lambda_X(\mathcal{V}) = y$. □

3. Minimal basically disconnected covers of υX

Recall that a covering map $f : Y \longrightarrow X$ is called $\sigma Z^\#$*-irreducible* if $\{f(A) : A \in \sigma Z(Y)^\#\} = \sigma Z(X)^\#$ and that a subspace D of a space X is *$\sigma Z^{\#}$-embedded* if for any $B \in \sigma Z(D)^{\#}$, there is $S \in \sigma Z(X)^{\#}$ such that $S \cap D = B$. For any compact space X, Λ_X is $\sigma Z^{\#}$-irreducible([4]) and every dense C^*-embedded subspace of a space is $\sigma Z^{\#}$-embedded.

LEMMA 3.1 ([4]). *For covering maps $g : Y \longrightarrow W$, $h : W \longrightarrow X$, then $h \circ g$ is $\sigma Z^{\#}$-irreducible if and only if h and g are $\sigma Z^{\#}$-irreducible.*

For any space X, $\upsilon_X : X \longrightarrow \upsilon X$ denotes the Hewitt realcompactification of X.

DEFINITION 3.2. Let X be a space. A $\sigma Z(X)^{\#}$- ultrafilter $\mathcal{U}$ is called *stable* if the unique point of $\cap\{cl_{\beta X}(A) : A \in \mathcal{U}\}$ belongs to υX.

In [5], we can find internal characterizations of a space X which $E(\upsilon X) = \upsilon(EX)$. Indeed, for any space X, the following are equivalent:

(a) $E(\upsilon X) = \upsilon(EX)$,

(b) if $\{A_n : n \in N\}$ is a decreasing sequence of members of R(X) and $\cap\{A_n : n \in N\} = \phi$, then $\cap\{cl_{\upsilon X}(A_n) : n \in N\} = \phi$,

(c) if $\{A_n : n \in N\}$ is a decreasing sequence of members of R(X), then $cl_{\upsilon X}(\cap\{A_n : n \in N\}) = \cap\{cl_{\upsilon X}(A_n) : n \in N\}$, and

(d) every stable R(X)-ultrafilter has the countable intersection property.

For any space X, $\upsilon_\Lambda : \Lambda X \longrightarrow \upsilon(\Lambda X)$ denotes the Hewitt realcompactification of ΛX and $(\Lambda(\upsilon X), \Lambda_\upsilon)$ denotes the minimal basically disconnected cover of υX.

REMARK. Let X be a space. Then there is a continuous map $m_X : \upsilon(\Lambda X) \longrightarrow \upsilon X$ such that the following diagram commutes([2]):

$$\begin{array}{ccc} \Lambda X & \xrightarrow{\Lambda_X} & X \\ {\scriptstyle \upsilon_\Lambda}\downarrow & & \downarrow{\scriptstyle \upsilon_X} \\ \upsilon(\Lambda X) & \xrightarrow[m_X]{} & \upsilon X. \end{array}$$

If there is a homeomorphism $h : \upsilon(\Lambda X) \longrightarrow \Lambda(\upsilon X)$ such that $\Lambda_\upsilon \circ h = m_X$, then we write $\Lambda(\upsilon X) = \upsilon(\Lambda X)$ and in case, we will identify $(\upsilon(\Lambda X), m_X)$ and $(\Lambda(\upsilon X), \Lambda_\upsilon)$.

Let X be a space and $p \in \beta X - \upsilon X$. Then there is a zero set Z in βX such that $p \in Z$ and $Z \cap X = \phi$ ([2]). Let $f : \beta X \longrightarrow R$ be a continuous map with $Z = f^{-1}(0)$ and $Z_n = cl_{\beta X}(int_{\beta X}(f^{-1}([-\frac{1}{n}, \frac{1}{n}])))$.

Then $\{Z_n : n \in N\}$ is a decreasing sequence in $\sigma Z(\beta X)^\#$ such that for any $n \in N$, $p \in \text{int}_{\beta X}(Z_n)$ and $(\cap\{Z_n : n \in N\}) \cap X = \phi$. Using this and the fact that a space X is basically disconnected if and only if $Z(X)^\# = \sigma Z(X)^\# = B(X)$([6]), where B(X) is the set of all clopen sets in X, we will show the following lemma:

LEMMA 3.3. *Let X be a space with $\Lambda(\beta X) = \beta(\Lambda X)$. If every stable $\sigma Z(X)^\#$-ultrafilter has the countable intersection property, then m_X is a covering map.*

PROOF. Consider the following diagram:

$$\begin{array}{ccc}
\Lambda X & \xrightarrow{\Lambda_X} & X \\
{\scriptstyle v_\Lambda}\downarrow & & \downarrow{\scriptstyle v_X} \\
v(\Lambda X) & \xrightarrow{m_X} & vX \\
{\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\
\beta(\Lambda X) & \xrightarrow{\Lambda_\beta o l_X} & \beta X,
\end{array}$$

where j_1 and j_2 are inclusion maps. Note that $j_2 \circ m_X \circ v_\Lambda = j_2 \circ v_X \circ \Lambda_X = \Lambda_\beta \circ l_X \circ j_1 \circ v_\Lambda$. Since v_Λ is dense, $j_2 \circ m_X = \Lambda_\beta \circ l_X \circ j_1$. By Lemma 2.1, it is enough to show that m_X is a perfect onto map. Let $p \in vX$. Then $\Lambda_\beta^{-1}(p) \neq \phi$. We will show that $\Lambda_\beta^{-1}(p) \subseteq v(\Lambda X)$. Take any $\alpha \in \Lambda_\beta^{-1}(p)$. Suppose that $\alpha \notin v(\Lambda X)$. Then there is a sequence $\{Z_n : n \in N\}$ in $\sigma Z(\beta(\Lambda X))^\#$ such that for any $n \in N$, $\alpha \in \text{int}_{\beta(\Lambda X)}(Z_n)$ and $(\cap\{Z_n : n \in N\}) \cap \Lambda X = \phi$. Since Λ_β is $\sigma Z^\#$-irreducible, $\Lambda_\beta(Z_n) \in \sigma Z(\beta X)^\#$. Corollary 2.3, $\alpha_X = \{U \cap X : U \in \alpha\}$ is a $\sigma Z(X)^\#$ - ultrafilter. Let $n \in N$. Since $\alpha \in \text{int}_{\beta(\Lambda X)}(Z_n)$ and $\{A^* : A \in \sigma Z(\beta X)^\#\}$ is a base for $\beta(\Lambda X)$, there is $A \in \sigma Z(\beta X)^\#$ with $\alpha \in A^* \subseteq Z_n$ and so $\Lambda_\beta(\alpha) \in \Lambda_\beta(A^*) = A \subseteq \Lambda_\beta(Z_n)$. So $\Lambda_\beta(Z_n) \in \alpha$. Hence, for any $n \in N$, $\Lambda_\beta(Z_n) \cap X \in \alpha_X$. Since $p \in vX$, α_X is stable and so $\cap\{\Lambda_\beta(Z_n) \cap X : n \in N\} \neq \phi$. Pick $x \in \cap\{\Lambda_\beta(Z_n) \cap X : n \in N\}$. Let $n \in N$. Then $\Lambda_\beta^{-1}(x) \cap Z_n \neq \phi$. Since $\Lambda_\beta^{-1}(x) = \Lambda_X^{-1}(x)$, $\Lambda_X^{-1}(x) \cap Z_n \neq \phi$. Since $\Lambda_X^{-1}(x)$ is compact and $\cap\{\Lambda_X^{-1}(x) \cap Z_n : n \in N\}$ is a decreasing family of closed sets in $\Lambda_X^{-1}(x)$ with the finite intersection property, $\cap\{\Lambda_X^{-1}(x) \cap Z_n : n \in N\} \neq \phi$ and hence $(\cap\{Z_n : n \in N\}) \cap \Lambda X \neq \phi$. This is a contradiction. Hence $\alpha \in v(\Lambda X)$. Thus m_X is onto. Since $\Lambda_\beta(v(\Lambda X)) = m_X(v(\Lambda X)) = vX$, $v(\Lambda X) \subseteq \Lambda_\beta^{-1}(vX)$. For any $y \in \Lambda_\beta^{-1}(vX)$, $y \in \Lambda_\beta^{-1}(\Lambda_\beta(y)) \subseteq v(\Lambda X)$

and hence $\Lambda_\beta^{-1}(\upsilon X) \subseteq \upsilon(\Lambda X)$. Hence $\upsilon(\Lambda X) = \Lambda_\beta^{-1}(\upsilon X)$. Using this, we can show that m_X is closed and perfect. Thus m_X is a perfect onto map. □

In the following, we will give internal characterizations of a space X for which $\Lambda(\upsilon X) = \upsilon(\Lambda X)$.

THEOREM 3.4. *Let X be a space with* $\Lambda(\beta X) = \beta(\Lambda X)$. *Then the following are equivalent:*

(a) $\Lambda(\upsilon X) = \upsilon(\Lambda X)$,

(b) *if* $\{A_n : n \in N\}$ *is a decreasing sequence in* $\sigma Z(X)^\#$ *with* $\cap\{A_n : n \in N\} = \phi$, *then* $\cap\{cl_{\upsilon X}(A_n) : n \in N\} = \phi$,

(c) *if* $\{A_n : n \in N\}$ *is a decreasing sequence in* $\sigma Z(X)^\#$, *then* $cl_{\upsilon X}(\cap\{A_n : n \in N\}) = \cap\{cl_{\upsilon X}(A_n) : n \in N\}$, *and*

(d) *every stable* $\sigma Z(X)^\#$*-ultrafilter has the countable intersection property.*

PROOF. (a) ⇒ (b) Let $\{A_n : n \in N\}$ be a decreasing sequence in $\sigma Z(X)^\#$ with $\cap\{A_n : n \in N\} = \phi$. Suppose that $\cap\{cl_{\upsilon X}(A_n) : n \in N\} \neq \phi$. Since $\Lambda(\beta(\upsilon X)) = \Lambda(\beta X) = \beta(\Lambda X) = \beta(\upsilon(\Lambda X)) = \beta(\Lambda(\upsilon X))$ and by Proposition 2.6, $\Lambda(\upsilon X)$ is given by the filter space $\{\alpha : \alpha$ is a fixed $\sigma Z(\upsilon X)^\#$-ultrafilter$\}$. By Corollary 2.3, $cl_{\upsilon X}(A_n) \in \sigma Z(\upsilon X)^\#$ for all $n \in N$. By Proposition 2.7, $\Lambda_\upsilon(\cap\{(cl_{\upsilon X}(A_n))^* : n \in N\}) = \cap\{cl_{\upsilon X}(A_n) : n \in N\}$. Since $\cap\{cl_{\upsilon X}(A_n) : n \in N\} \neq \phi$, $\cap\{(cl_{\upsilon X}(A_n))^* : n \in N\} \neq \phi$. By Lemma 2.2, $(cl_{\upsilon X}(A_n))^* = cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(cl_{\upsilon X}(A_n))))$ for all $n \in N$. Pick $t \in \cap\{cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(cl_{\upsilon X}(A_n)))) : n \in N\}$. Then there is the $Z(\Lambda X)$-ultrafilter $\mathcal{A}$ such that $t \in \cap\{cl_{\Lambda(\upsilon X)}(A) : A \in \mathcal{A}\}$([2]). Since $t \in \upsilon(\Lambda X)$, the $Z(\Lambda X)$-ultrafilter $\mathcal{A}$ has the countable intersection property and for any Z in $Z(\Lambda X)$, $Z \in \mathcal{A}$ if and only if $t \in cl_{\Lambda(\upsilon X)}(Z)$([2]).

Let $n \in N$. By Corollary 2.3, there is $B_n \in \sigma Z(\upsilon X)^\#$ such that $B_n \cap X = A_n$. Note that $\Lambda_X(cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(B_n))) \cap \Lambda X) = \Lambda_X(cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(B_n))) \cap \Lambda_\upsilon^{-1}(X)) = \Lambda_\upsilon(cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(B_n)))) \cap X = B_n \cap X = A_n = \Lambda_\upsilon(cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(cl_{\upsilon X}(A_n))))) \cap X = \Lambda_X(cl_{\Lambda X}(\Lambda_X^{-1}(int_X(A_n))))$. By Lemma 2.2, $cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(B_n))) = cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(cl_{\upsilon X}(A_n))))$ and $cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(B_n))) \cap \Lambda X = cl_{\Lambda X}(\Lambda_X^{-1}(int_X(A_n)))$ and so $cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(B_n))) = cl_{\Lambda(\upsilon X)}(cl_{\Lambda X}(\Lambda_X^{-1}(int_X(A_n))))$. Thus $t \in cl_{\Lambda(\upsilon X)}(cl_{\Lambda X}(\Lambda_X^{-1}(int_X(A_n))))$. Since $cl_{\Lambda(\upsilon X)}(\Lambda_\upsilon^{-1}(int_{\upsilon X}(B_n))) \in \sigma Z(\Lambda(\upsilon X))^\#$ and $\Lambda(\upsilon X)$ is basically disconnected,

$\mathrm{cl}_{\Lambda(\upsilon X)}$ $(\Lambda_\upsilon^{-1}($ $\mathrm{int}_{\upsilon X}(\mathrm{B}_n))) \in \mathrm{B}(\Lambda(\upsilon \mathrm{X}))$. Hence $\mathrm{cl}_{\Lambda X}(\Lambda_X^{-1}(\mathrm{int}_X(\mathrm{A}_n)))$ $\in \mathcal{A}$. Therefore, $\cap\{\mathrm{cl}_{\Lambda X}(\Lambda_X^{-1}(\mathrm{int}_X(\mathrm{A}_n))) : n \in \mathrm{N}\} = \cap\{\mathrm{A}_n : n \in \mathrm{N}\}$ $\neq \phi$. This is a contradiction. Hence $\cap\{\mathrm{cl}_{\upsilon X}(\mathrm{A}_n) : n \in \mathrm{N}\} = \phi$.

(b) $\Rightarrow$ (c) Clearly, $\mathrm{cl}_{\upsilon X}(\cap\{\mathrm{A}_n : n \in \mathrm{N}\}) \subseteq \cap\{\mathrm{cl}_{\upsilon X}(\mathrm{A}_n) : n \in \mathrm{N}\}$. Let $p \notin \mathrm{cl}_{\upsilon X}(\cap\{\mathrm{A}_n : n \in \mathrm{N}\})$. Then there is an $\mathrm{B} \in \sigma\mathrm{Z}(\upsilon\mathrm{X})^\#$ such that $p \in \mathrm{int}_{\upsilon X}(\mathrm{B})$ and $\mathrm{B} \cap (\cap\{\mathrm{A}_n : n \in \mathrm{N}\}) = \phi$. By Corollary 2.3, C $= \mathrm{B} \cap \mathrm{X} \in \sigma\mathrm{Z}(\mathrm{X})^\#$ and $\{\mathrm{C} \wedge \mathrm{A}_n : n \in \mathrm{N}\}$ is a decreasing sequence in $\sigma\mathrm{Z}(\mathrm{X})^\#$ with empty intersection. By the hypothesis, $\cap\{\mathrm{cl}_{\upsilon X}(\mathrm{C} \wedge \mathrm{A}_n)$ $: n \in \mathrm{N}\} = \phi$. Suppose that $p \in \cap\{\mathrm{cl}_{\upsilon X}(\mathrm{A}_n) : n \in \mathrm{N}\}$ and W is a neighborhood of p in υX. Let $n \in \mathrm{N}$. Then $\mathrm{int}_{\upsilon X}(\mathrm{W}) \cap \mathrm{int}_{\upsilon X}(\mathrm{B}) \cap$ $\mathrm{A}_n \neq \phi$. Note that $\mathrm{C} \wedge \mathrm{A}_n = \mathrm{cl}_X(\mathrm{int}_X(\mathrm{C} \cap \mathrm{A}_n)) = \mathrm{cl}_X(\mathrm{int}_X(\mathrm{C}) \cap$ $\mathrm{int}_X(\mathrm{A}_n)) \supseteq \mathrm{int}_X(\mathrm{C}) \cap \mathrm{A}_n = \mathrm{int}_X(\mathrm{B} \cap \mathrm{X}) \cap \mathrm{A}_n \supseteq \mathrm{int}_{\upsilon X}(\mathrm{B}) \cap \mathrm{A}_n$. Thus $(\mathrm{C} \wedge \mathrm{A}_n) \cap \mathrm{W} \supseteq \mathrm{int}_{\upsilon X}(\mathrm{B}) \cap \mathrm{A}_n \cap \mathrm{W} \neq \phi$. Hence $p \in \{\mathrm{cl}_{\upsilon X}(\mathrm{C} \wedge$ $\mathrm{A}_n) : n \in \mathrm{N}\}$. This is a contradiction.

(c) $\Rightarrow$ (d) Let $\mathcal{U}$ be a stable $\sigma\mathrm{Z}(\mathrm{X})^\#$-ultrafilter on X. Let $\{\mathrm{B}_n : n \in$ $\mathrm{N}\} \subseteq \mathcal{U}$. Let $\mathrm{A}_n = \bigwedge\{\mathrm{B}_i : 1 \leq i \leq n\}$. Then $\{\mathrm{A}_n : n \in \mathrm{N}\}$ is a decreasing sequence in $\sigma\mathrm{Z}(\mathrm{X})^\#$. Since $\mathcal{U}$ is stable, there exist a $p \in \upsilon$X such that $p \in \cap\{\mathrm{cl}_{\upsilon X}(\mathrm{A}_n) : n \in \mathrm{N}\}$. By the hypothesis, $p \in \mathrm{cl}_{\upsilon X}(\cap\{\mathrm{A}_n :$ $n \in \mathrm{N}\})$. Hence $\cap\{\mathrm{A}_n : n \in \mathrm{N}\} \neq \phi$ and so $\cap\{\mathrm{B}_i : i \in \mathrm{N}\ \} \neq \phi$. Thus $\mathcal{U}$ has the countable intersection property.

(d) $\Rightarrow$ (a) By Lemma 3.3, $m_X : \upsilon(\Lambda\mathrm{X}) \longrightarrow \upsilon\mathrm{X}$ is a covering map. Since $\upsilon(\Lambda\mathrm{X})$ is basically disconnected, there is a covering map $h : \upsilon(\Lambda\mathrm{X})$ $\longrightarrow \Lambda(\upsilon\mathrm{X})$ with $m_X = \Lambda_\upsilon \circ h$. It remains to show that h is 1-1. Let $x \neq y$ in $\upsilon(\Lambda\mathrm{X})$. Then there are Z_1, Z_2 in $\sigma\mathrm{Z}(\upsilon(\Lambda\mathrm{X}))^\#$ such that $x \in$ Z_1, $y \in \mathrm{Z}_2$ and $\mathrm{Z}_1 \cap \mathrm{Z}_2 = \phi$.

Consider the following commutative diagrams:

$$\begin{array}{ccccc}
\upsilon(\Lambda X) & \xrightarrow{h} & \Lambda(\upsilon X) & \xrightarrow{\Lambda_\upsilon} & \upsilon X \\
{\scriptstyle j_1}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_2} \\
\beta(\Lambda X) & \xrightarrow{l_X} & \Lambda(\beta X) & \xrightarrow{\Lambda_\beta} & \beta X,
\end{array}$$

where j_1, j and j_2 are inclusions maps. Since βX is compact, Λ_β is $\sigma\mathrm{Z}^\#$-irreducible([4]). Since j_1 is C^*-embedded and $\upsilon(\Lambda\mathrm{X})$ is dense in $\beta(\Lambda\mathrm{X})$, j_1 is $\sigma\mathrm{Z}^\#$-embedded. Thus $h \circ \Lambda_\upsilon$ is $\sigma\mathrm{Z}^\#$-irreducible([4]). By Lemma 3.1, h is $\sigma\mathrm{Z}^\#$-irreducible. Since $\mathrm{Z}_1 \wedge \mathrm{Z}_2 = \phi$ and h is a covering map, $h(\mathrm{Z}_1) \wedge h(\mathrm{Z}_2) = \phi$. Since h is $\sigma\mathrm{Z}^\#$-irreducible, $h(\mathrm{Z}_1)$, $h(\mathrm{Z}_2) \in$ $\sigma\mathrm{Z}(\Lambda(\upsilon\mathrm{X}))^\# = \mathrm{B}(\Lambda(\upsilon\mathrm{X}))$ and so $h(\mathrm{Z}_1) \cap h(\mathrm{Z}_2) = \phi$. Hence $h(x) \neq h(y)$ and thus h is 1-1. □

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