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CATEGORICAL PROPERTIES OF SEMI-CONTINUOUS QUASI-ORDERED SPACES

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Abstract. We study categorical properties of the category **SWQOS** ($\mathbf{S^UWQOS}$, $\mathbf{S^LWQOS}$, resp.) of (upper, lower, resp.) semi-continuous quasi-ordered spaces and the subcategory **SWPOS** ($\mathbf{S^UWPOS}$, $\mathbf{S^LWPOS}$, resp.) of (upper, lower, resp.) semi-continuous partially ordered spaces. We show that the categories $\mathbf{S^UWPOS}$, $\mathbf{S^LWPOS}$ and **SWPOS** are closed under the formation of initial mono-sources in the category **TQOS** of topological quasi-ordered spaces, and they are mono-topological, complete and cocomplete epireflective subcategories of the category **TQOS**. We obtain their MacNeille and universal initial completions, as well as those of subcategories $\mathbf{S^UWQOS}$($\mathbf{S^LWQOS}$) and **SWQOS**, in which the topologies are T_0 and T_1, respectively.

0. Introduction

A topological quasi-ordered space $(X, \tau, \leq)$ is a set X endowed with both a topology τ and a quasi-order $\leq$. The class of topological quasi-ordered spaces and continuous isotones forms a topological category which we denote by **TQOS**.

The purpose of this paper is to study categorical properties of well behaved subcategories of the category **TQOS**, i.e., categories $\mathbf{S^UWQOS}$ ($\mathbf{S^LWQOS}$, resp.) of upper (lower, resp.) semi-continuous quasi-ordered spaces and **SWQOS** of semi-continuous quasi-ordered spaces. In fact, an upper (lower, resp.) semi-continuous partially ordered space is a T_0-space, but an upper (lower, resp.) semi-continuous quasi-ordered space need not be so. Using this fact, we obtain various epireflective subcategories of the categories $\mathbf{S^UWQOS}$, $\mathbf{S^LWQOS}$ and **SWQOS**.

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Moreover, extending Schauerte's idea ([9]), we have their MacNeille and universal initial completions.

For the terminology not introduced in this paper, we refer to [1], [6] and [7] for the category theory, [3] for topology and [5] for the order theory. Also we assume throughout this paper that a subcategory of a category is full and isomorphism closed.

1. Categories of semi-continuous partially ordered spaces

The following definition is due to Ward, Jr ([11]).

DEFINITION 1.1. Let $(X, \tau, \leq)$ be a topological quasi-ordered space. Then

(1) $\leq$ is said to be *upper* (*lower*, resp.) *semi-continuous* if for any $a \not\leq b$ in X, there is an open neighborhood U of b (a, resp.) such that $a \not\leq x$ ($x \not\leq b$, resp.) for all $x \in U$ (we write $a \not\leq U$ ($U \not\leq b$, resp.)).

(2) $\leq$ is said to be *semi-continuous* if it is both upper and lower semi-continuous.

In each case, $(X, \tau, \leq)$ is called an *upper* (*lower*, resp.) *semi-continuous quasi-ordered space* and *semi-continuous quasi-ordered space*, respectively.

We note that the categories **TQOS**, $\mathbf{S^UWQOS}$ ($\mathbf{S^LWQOS}$, resp.) and **SWQOS** of topological quasi-ordered spaces, upper (lower, resp.) semi-continuous quasi-ordered spaces and semi-continuous quasi-ordered spaces, respectively, and continuous isotones are topological ([10]); hence complete and cocomplete.

Let **TPOS**, $\mathbf{S^UWPOS}$ ($\mathbf{S^LWPOS}$, resp.) and **SWPOS** be their full subcategories determined by partial order relation instead of quasi-order relation, respectively.

PROPOSITION 1.2. (1) *If* $(X, \tau, \leq) \in \mathbf{S^UWPOS}(\mathbf{S^LWPOS})$, *then* $(X, \tau) \in \mathbf{Top_0}$.

(2) *If* $(X, \tau, \leq) \in \mathbf{SWPOS}$, *then* $(X, \tau) \in \mathbf{Top_1}$, *where* $\mathbf{Top_0}(\mathbf{Top_1}$, *resp.) is the category of* T_0*-spaces* (T_1*-spaces, resp.) and continuous maps.*

PROOF. (1) For any distinct x and y in X, $x \not\leq y$ or $y \not\leq x$, because $\leq$ is a partial order. Suppose that $x \not\leq y$. Then there is an open neighborhood of y such that $x \not\leq U$; hence $x \notin U$. Thus $(X, \tau) \in \mathbf{Top_0}$.

(2) For any distinct x and y in X, $x \nleq y$ or $y \nleq x$. Suppose that $x \nleq y$. Then there are open neighborhoods U and V of x and y, respectively, such that $U \nleq y$ and $x \nleq V$. So $y \notin U$ and $x \notin V$. Hence $(X, \tau) \in \mathbf{Top_1}$ (see also [4]). □

The converse of the above proposition is not true. For example, (1) consider the two-point chain $X = \{0, 1\}$ with $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$. Then $(X, \tau) \in \mathbf{Top_0}$ but $(X, \tau, \leq) \notin \mathbf{S^UWPOS}$. (2) For the real line $\mathbb{R}$ with the usual order $\leq$ and the cofinite topology τ on $\mathbb{R}$, $(\mathbb{R}, \tau) \in \mathbf{Top_1}$. But $(\mathbb{R}, \tau, \leq) \notin \mathbf{SWPOS}$, because for all $x \in \mathbb{R}$, $\uparrow x = \{a \in \mathbb{R} | x \leq a\}$ is not closed.

For any $(X, \tau) \in \mathbf{Top_0}$, $(X, \tau, =) \in \mathbf{S^UWPOS}$ ($\mathbf{S^LWPOS}$, resp.) and hence we have an embedding functor $E_0 : \mathbf{Top_0} \hookrightarrow \mathbf{S^UWPOS}$ ($\mathbf{S^LWPOS}$, resp.). Similarly, for any $(X, \tau) \in \mathbf{Top_1}$, we obtain an embedding functor $E_1 : \mathbf{Top_1} \hookrightarrow \mathbf{SWPOS}$.

Since the discrete order is a partial order, we have the following:

PROPOSITION 1.3. *The categories* $\mathbf{Top_0}$ *and* $\mathbf{Top_1}$ *are coreflective subcategories of* $\mathbf{S^UWPOS}$($\mathbf{S^LWPOS}$) *and* $\mathbf{SWPOS}$, *respectively.*

PROOF. For any $(X, \tau, \leq) \in \mathbf{S^UWPOS}$ ($\mathbf{S^LWPOS}$, $\mathbf{SWPOS}$), $((X, \tau), 1_X)$ is a coreflection of $(X, \tau, \leq)$. □

LEMMA 1.4. *The category* $\mathbf{TPOS}$ *is closed under the formation of initial mono-sources in* $\mathbf{TQOS}$.

PROOF. Let $(f_\alpha : (X, \tau, \leq) \longrightarrow (X_\alpha, \tau_\alpha, \leq_\alpha))_{\alpha \in J}$ be an initial mono-source in $\mathbf{TQOS}$ such that for all $\alpha \in J$, $(X_\alpha, \tau_\alpha, \leq_\alpha) \in \mathbf{TPOS}$. Suppose that $x \leq y$ and $y \leq x$ in $(X, \tau, \leq)$, then $f_\alpha(x) \leq_\alpha f_\alpha(y)$ and $f_\alpha(y) \leq_\alpha f_\alpha(x)$ $(\alpha \in J)$; therefore $f_\alpha(x) = f_\alpha(y)$ for all $\alpha \in J$, because $\leq_\alpha$ is a partial order. Hence $x = y$, for $(f_\alpha)_{\alpha \in J}$ is a mono-source. Thus $(X, \tau, \leq) \in \mathbf{TPOS}$. □

By the results in [10], $\mathbf{S^UWQOS}$, $\mathbf{S^LWQOS}$ and $\mathbf{SWQOS}$ are all closed under the formation of initial sources in $\mathbf{TQOS}$. Furthermore, $\mathbf{S^UWPOS} = \mathbf{S^UWQOS} \bigcap \mathbf{TPOS}$, $\mathbf{S^LWPOS} = \mathbf{S^LWQOS} \bigcap \mathbf{TPOS}$ and $\mathbf{SWPOS} = \mathbf{SWQOS} \bigcap \mathbf{TPOS}$ and therefore we have the following by the above lemma.

THEOREM 1.5. *The categories* $\mathbf{S^UWPOS}$, $\mathbf{S^LWPOS}$ *and* $\mathbf{SWPOS}$ *are closed under the formation of initial mono-sources in* $\mathbf{TQOS}$.

COROLLARY 1.6. (1) *The categories* $\mathbf{S^UWPOS}$, $\mathbf{S^LWPOS}$ *and* $\mathbf{SWPOS}$ *are mono-topological, complete and cocomplete.*

(2) *The above categories are epireflective subcategories of* $\mathbf{TQOS}$.

(3) *The above categories are hereditary and productive subcategories of* $\mathbf{TPOS}$.

(4) $\mathbf{S^UWPOS}(\mathbf{S^LWPOS})$ *and* $\mathbf{SWPOS}$ *are reflective subcategories of* $\mathbf{TPOS}$ *and* $\mathbf{S^UWPOS}(\mathbf{S^LWPOS})$, *respectively.*

2. Subcategories between $\mathbf{S^UWQOS}$ (SWQOS) and $\mathbf{S^UWPOS}$ (SWPOS, respectively)

An upper semi-continuous partially ordered space is a T_0-space, but an upper semi-continuous quasi-ordered space need not be so. Noting this fact, we define the following full subcategories of $\mathbf{S^UWQOS}$:

DEFINITION 2.1. (1) $\mathbf{T_0S^UWQOS}$ consists of those objects $(X,\tau,\leq)$ such that (X,τ) is a T_0-space.

(2) $\mathbf{T_0^{-1}S^UWQOS}$ consists of those objects $(X,\tau,\leq)$ which satisfy the condition: for $x,y\in X$, $x\leq y$ and $y\leq x$ if and only if $\mathcal{N}(x)=\mathcal{N}(y)$, where $\mathcal{N}(x)$ is the neighborhood filter of x.

REMARK 2.2. (1) $\mathbf{S^UWPOS}\subsetneq\mathbf{T_0^{-1}S^UWQOS}\subsetneq\mathbf{S^UWQOS}$.
(2) $\mathbf{S^UWPOS}\subsetneq\mathbf{T_0S^UWQOS}\subsetneq\mathbf{S^UWQOS}$.

THEOREM 2.3. *The category* $\mathbf{T_0S^UWQOS}$ *is an epireflective subcategory of* $\mathbf{S^UWQOS}$.
Moreover, $\mathbf{T_0S^UWQOS}$ *is initially dense in* $\mathbf{S^UWQOS}$.

PROOF. The $\mathbf{T_0S^UWQOS}$-reflection of $(X,\tau,\leq)\in\mathbf{S^UWQOS}$ can be constructed as follows: let $E:\mathbf{T_0S^UWQOS}\hookrightarrow\mathbf{S^UWQOS}$ be the embedding functor. For an $(X,\tau,\leq)\in\mathbf{S^UWQOS}$, we define a relation $\mathcal{R}=\{(x,y)\in X\times X|\mathcal{N}(x)=\mathcal{N}(y)\}$. On the quotient set $X/\mathcal{R}$, let $\tau_{\mathcal{R}}$ be the final topology with respect to the quotient map $q:X\longrightarrow X/\mathcal{R}$ defined by $q(x)=[x]$, and $\leq_{\mathcal{R}}$ be the order with the graph $G_{\leq_{\mathcal{R}}}=\{([x],[y])\in X/\mathcal{R}\times X/\mathcal{R}|$ there are $a,b\in X$ such that $a\leq b$, $q(a)=[x]$ and $q(b)=[y]\}$. Clearly $\leq_{\mathcal{R}}$ is a quasi-order and $(X/\mathcal{R},\tau_{\mathcal{R}})\in\mathbf{Top_0}$. We note that $[a]=[b]$ implies $a\leq b$ and $b\leq a$, since $\leq$ is upper semi-continuous. We claim that $q^{-1}(\uparrow[x])=\uparrow x$ for all $x\in X$. In fact, if $y\in\uparrow x$, then $x\leq y$; hence $[x]\leq_{\mathcal{R}}[y]$. So $[y]\in\uparrow[x]$. Hence $y\in q^{-1}(\uparrow[x])$. Conversely, if $y\in q^{-1}(\uparrow[x])$, then $q(y)\in\uparrow[x]$; hence $[x]\leq_{\mathcal{R}}[y]$. By the definition of $\leq_{\mathcal{R}}$, there are a,b in X such that $a\leq b$, $[a]=[x]$

and $[b] = [y]$. So $x \leq a \leq b \leq y$; hence $x \leq y$. Hence $y \in \uparrow x$. Since $(X, \tau, \leq) \in \mathbf{S^UWQOS}$, for all $x \in X$, $\uparrow x$ is closed in X. So for all $[x] \in X/\mathcal{R}$, $\uparrow [x]$ is closed in $X/\mathcal{R}$. In all, $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}) \in \mathbf{T_0S^UWQOS}$. For any $(Y, \tau', \leq') \in \mathbf{T_0S^UWQOS}$ and $f : (X, \tau, \leq) \longrightarrow E((Y, \tau', \leq'))$ in $\mathbf{S^UWQOS}$, $\mathcal{R} \subseteq ker(f)$. By a generalization of the Fundamental Theorem of Factorization for **Top** ([10]), there is a unique continuous isotone $\overline{f} : (X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}) \longrightarrow (Y, \tau', \leq')$ defined by $\overline{f}([x]) = f(x)$ with $E(\overline{f}) \circ q = f$, i.e., $\overline{f} \in \mathbf{T_0S^UWQOS}$. Thus $(q, (X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}))$ is an E-universal map for $(X, \tau, \leq)$. In all, $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is the $\mathbf{T_0S^UWQOS}$-reflection of $(X, \tau, \leq)$.

For the second statement, we note that, in the above proof, $G_{\leq} = (q \times q)^{-1}(G_{\leq_{\mathcal{R}}})$, i.e., $\leq$ is the initial order for q. Also the topology on X is initial for q, because for any open U in X, $U = q^{-1}(q(U))$; hence the reflection of $\mathbf{S^UWQOS}$ to $\mathbf{T_0S^UWQOS}$ is initial. This completes the proof. □

PROPOSITION 2.4. $\mathbf{T_0^{-1}S^UWQOS} = \{(X, \tau, \leq) \in \mathbf{S^UWQOS} |$ $\mathbf{T_0S^U}$ **WQOS**-*reflection* $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ *of* $(X, \tau, \leq)$ *is in* $\mathbf{S^UWPOS}\}$.

PROOF. For any $(X, \tau, \leq)$ in $\mathbf{S^UWQOS}$ such that the $\mathbf{T_0S^UWQOS}$ -reflection $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ is in $\mathbf{S^UWPOS}$, suppose $x \leq y$ and $y \leq x$ in $(X, \tau, \leq)$, then $[x] \leq_{\mathcal{R}} [y]$ and $[y] \leq_{\mathcal{R}} [y]$, and thus $[x] = [y]$, since $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}}) \in \mathbf{S^UWPOS}$; hence $\mathcal{N}(x) = \mathcal{N}(y)$. Conversely, if $\mathcal{N}(x) = \mathcal{N}(y)$, then $[x] = [y]$; hence $x \leq y$ and $y \leq x$. Thus $(X, \tau, \leq) \in \mathbf{T_0^{-1}S^UWQOS}$.

On the other hand, take any $(X, \tau, \leq) \in \mathbf{T_0^{-1}S^UWQOS} \subsetneq \mathbf{S^UWQOS}$, let $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$ be the $\mathbf{T_0S^UWQOS}$-reflection of $(X, \tau, \leq)$. It is enough to show that $\leq_{\mathcal{R}}$ satisfies antisymmetry. Suppose $[x] \leq_{\mathcal{R}} [y]$ and $[y] \leq_{\mathcal{R}} [x]$ in $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$. Then there are $a, b \in X$ with $a \leq b$ such that $[a] = [x]$ and $[b] = [y]$, and $c, d \in X$ with $c \leq d$ such that $[c] = [y]$ and $[d] = [x]$; hence $[a] = [d]$ and $[b] = [c]$. So $a \leq d$ and $d \leq a$, $b \leq c$ and $c \leq b$; hence $a \leq c$ and $c \leq a$. Since $(X, \tau, \leq) \in \mathbf{T_0^{-1}S^UWQOS}$, $\mathcal{N}(a) = \mathcal{N}(c)$. Hence $[a] = [c]$, i.e., $[x] = [y]$ in $(X/\mathcal{R}, \tau_{\mathcal{R}}, \leq_{\mathcal{R}})$. □

COROLLARY 2.5. *The category* $\mathbf{S^UWPOS}$ *is an epireflective subcategory of* $\mathbf{T_0^{-1}S^UWQOS}$.

Moreover, $\mathbf{S^UWPOS}$ *is initially dense in* $\mathbf{T_0^{-1}S^UWQOS}$.

It is known that the category of partially ordered sets is initially and finally dense in the category of quasi-ordered sets ([2]).

REMARK 2.6. Let $((X_\alpha, \tau_\alpha, \leq_\alpha))_{\alpha \in J}$ be a class in **TQOS** and $(f_\alpha : X_\alpha \longrightarrow X)_{\alpha \in J}$ a sink in **Set**. Let τ be the final topology on X with respect to $(f_\alpha : (X_\alpha, \tau_\alpha) \longrightarrow X)_{\alpha \in J}$ and $\leq$ the transitive hull of $\Delta_X \bigcup \bigcup_{\alpha \in J} (f_\alpha \times f_\alpha)(G_{\leq_\alpha}) = \Delta_X \bigcup \bigcup_{\alpha \in J} \{(f_\alpha(a), f_\alpha(b)) | \ a \leq_\alpha b$ in $(X_\alpha, \leq_\alpha)\}$. Then $(X, \tau, \leq) \in$ **TQOS**, and clearly each $f_\alpha : (X_\alpha, \tau_\alpha, \leq_\alpha) \longrightarrow (X, \tau, \leq)$ is continuous isotone. Take any map $g : (X, \tau, \leq) \longrightarrow (Y, \tau^{'}, \leq^{'})$ such that for all $\alpha \in J$, $g \circ f_\alpha$ is continuous isotone, then clearly g is continuous, and for all $\alpha \in J$, $(g \times g) \circ (f_\alpha \times f_\alpha)(G_{\leq_\alpha}) \subseteq G_{\leq'}$ which imply $(f_\alpha \times f_\alpha)(G_{\leq_\alpha}) \subseteq (g \times g)^{-1}(G_{\leq'})$. Since the latter is reflexive and transitive, $G_\leq \subseteq (g \times g)^{-1}(G_{\leq'})$, i.e., $(g \times g)(G_\leq) \subseteq G_{\leq'}$. Thus g is isotone. In all, $(f_\alpha : (X_\alpha, \tau_\alpha, \leq_\alpha) \longrightarrow (X, \tau, \leq))_{\alpha \in J}$ is the final sink in **TQOS**.

REMARK 2.7. We note that the category $\mathbf{S^U WQOS}$ is topological ([10]) and hence cotopological ([1]). Thus for any family $((X_\alpha, \tau_\alpha, \leq_\alpha))_{\alpha \in J}$ in $\mathbf{S^U WQOS}$, a sink $(f_\alpha : X_\alpha \longrightarrow X)$ has the final lift. In fact, let $(f_\alpha : (X_\alpha, \tau_\alpha, \leq_\alpha) \longrightarrow (X, \tau, \leq))_{\alpha \in J}$ be the final lift in **TQOS** of $(f_\alpha)_{\alpha \in J}$ and $1_X : (X, \tau, \leq) \longrightarrow (X, \tau^{'}, \leq^{'})$ the $\mathbf{S^U WQOS}$-reflection of $(X, \tau, \leq)$, then $(f_\alpha : (X_\alpha, \tau_\alpha, \leq_\alpha) \longrightarrow (X, \tau^{'}, \leq^{'}))_{\alpha \in J}$ is the final sink in $\mathbf{S^U WQOS}$.

PROPOSITION 2.8. *The category* $\mathrm{S^U WPOS}$ *is finally dense in* $\mathrm{S^U WQOS}$.

PROOF. Let $(X, \tau, \leq) \in \mathbf{S^U WQOS}$ and $\mathcal{S} = \{f \in \mathbf{S^U WQOS} | f : (\tilde{X}, \tilde{\tau}, \tilde{\leq}) \longrightarrow (X, \tau, \leq), (\tilde{X}, \tilde{\tau}, \tilde{\leq}) \in \mathbf{S^U WPOS}\}$. We note that every topological space is a quotient of a Hausdorff space ([3]). So there is a Hausdorff space $(X^{'}, \tau^{'})$ and a final $q : (X^{'}, \tau^{'}) \longrightarrow (X, \tau)$. Then $(X^{'}, \tau^{'}, =) \in \mathbf{S^U WPOS}$ and $q : (X^{'}, \tau^{'}, =) \longrightarrow (X, \tau, \leq)$ is continuous isotone. Hence $q \in \mathcal{S}$, and so $\mathcal{S} \neq \emptyset$. We claim that $\mathcal{S}$ is a final sink. Let $G_0 = \bigcup \{(f \times f)(G_{\tilde{\leq}}) | f : (\tilde{X}, \tilde{\tau}, \tilde{\leq}) \longrightarrow (X, \tau, \leq)$ in $\mathcal{S}\}$. Then $G_0 = G_\leq$. In fact, if $x \leq y$ in X, and let $Y = \{a, b\}$ and $G_{\leq'} = \Delta_Y \bigcup \{(a, b)\}$, then $(Y, \mathcal{D}, \leq^{'}) \in \mathbf{S^U WPOS}$, where $\mathcal{D}$ is the discrete topology on Y, and the map $f : (Y, \mathcal{D}, \leq^{'}) \longrightarrow (X, \tau, \leq)$ defined by $f(a) = x$ and $f(b) = y$ is continuous isotone. Hence $f \in \mathcal{S}$, i.e., $(x, y) \in G_0$. Conversely, since each $f \in \mathcal{S}$ is isotone, $G_0 \subseteq G_\leq$. Since q is final in **Top** and $q \in \mathcal{S}$, the sink $(f : (\tilde{X}, \tilde{\tau}) \longrightarrow (X, \tau))_{f \in \mathcal{S}}$ is final in **Top**. Thus by the above two remarks, $\mathcal{S}$ is a final sink in **TQOS** and hence in $\mathbf{S^U WQOS}$. This completes the proof. □

COROLLARY 2.9. *For any full subcategories* **A** *and* **B** *of* $\mathbf{S^U WQOS}$ *satisfying* $\mathbf{S^U WPOS} \subseteq \mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{S^U WQOS}$, **A** *is finally dense in* **B**.

PROOF. For any $(X, \tau, \leq) \in \mathbf{B} \subseteq \mathbf{S^UWQOS}$, there is a final sink $\mathcal{S}= \{f \in \mathbf{S^UWQOS} | f : (\tilde{X}, \tilde{\tau}, \tilde{\leq}) \longrightarrow (X, \tau, \leq), (\tilde{X}, \tilde{\tau}, \tilde{\leq}) \in \mathbf{S^UWPOS}\}$. Since $\mathbf{S^UWPOS} \subseteq \mathbf{A}$, $(\tilde{X}, \tilde{\tau}, \tilde{\leq}) \in \mathbf{A}$. Hence $\mathbf{A}$ is finally dense in $\mathbf{B}$. □

COROLLARY 2.10. *The categories* $\mathbf{T_0S^UWQOS}$ *and* $\mathbf{S^UWPOS}$ *are finally dense in* $\mathbf{S^UWQOS}$ *and* $\mathbf{T_0^{-1}S^UWQOS}$, *respectively.*

Collecting the above facts, one can get the following diagram of reflective subcategories in $\mathbf{S^UWQOS}$:

$$\begin{array}{ccccc} & & \mathbf{S^UWQOS} & & \\ & r\nearrow & & \nwarrow d & \\ \mathbf{T_0^{-1}S^UWQOS} & & & & \mathbf{T_0S^UWQOS}, \\ & \nwarrow d & & \nearrow r & \\ & & \mathbf{S^UWPOS} & & \end{array}$$

where d is the inclusion functor with initial and final dense reflection and r is the inclusion functor with epireflection.

By replacing the category $\mathbf{S^UWQOS}$ with the category $\mathbf{S^LWQOS}$ in this section, dually we get an epireflective subcategories $\mathbf{T_0S^LWQOS}$, $\mathbf{S^LWPOS}$ which is initially and finally dense in $\mathbf{S^LWQOS}$, $\mathbf{T_0^{-1}S^LWQOS}$, respectively.

Since the topology in a semi-continuous partially ordered space is T_1, one can define the following full subcategories of $\mathbf{SWQOS}$:

DEFINITION 2.11. (1) $\mathbf{T_0SWQOS}$($\mathbf{T_1SWQOS}$, resp.) consists of those objects $(X, \tau, \leq)$ such that (X, τ) is a T_0-space (T_1-space, resp.).

(2) $\mathbf{R_0SWQOS}$ consists of those objects which satisfy the condition : for any $x, y \in X$, if there is an open set U with $x \in U$ and $y \notin U$, then there exists an open set V with $y \in V$ and $x \notin V$, or equivalently $y \in \overline{\{x\}}$ implies $x \in \overline{\{y\}}$.

(3) $\mathbf{T_0^{-1}SWQOS}$ consists of those objects which satisfy the condition : for any $x, y \in X, x \leq y$ and $y \leq x$ if and only if $\mathcal{N}(x) = \mathcal{N}(y)$.

REMARK 2.12. (1) $\mathbf{SWPOS} \subsetneq \mathbf{T_1SWQOS} \subsetneq \mathbf{T_0SWQOS} \subsetneq \mathbf{SWQOS}$.

(2) $\mathbf{SWPOS} \subsetneq \mathbf{T_0^{-1}SWQOS} \subsetneq \mathbf{R_0SWQOS} \subsetneq \mathbf{SWQOS}$.

THEOREM 2.13. *The category* $\mathbf{T_0SWQOS}$ *is epireflective and initially dense in* $\mathbf{SWQOS}$.

PROOF. For a $(X,\tau,\leq)$ in **SWQOS**, let $q:(X,\tau,\leq)\longrightarrow(X/\mathcal{R},\tau_{\mathcal{R}},\leq_{\mathcal{R}})$ be the $\mathbf{T_0S^UWQOS}$-reflection of $(X,\tau,\leq)$. For any $x\in X$, $q^{-1}(\downarrow[x])=\bigcup\{[y]|[y]\leq_{\mathcal{R}}[x]\}=\downarrow x$. Indeed, $\downarrow x\subseteq q^{-1}(\downarrow[x])$ and for any $z\in[y]$ with $[y]\leq_{\mathcal{R}}[x]$, there are $a,b\in X$ such that $a\leq b$, $[a]=[y]=[z]$ and $[b]=[x]$. By the construction of $(X/\mathcal{R},\tau_{\mathcal{R}},\leq_{\mathcal{R}})$, $z\leq a$ and $b\leq x$; hence $z\leq x$. Since $(X,\tau,\leq)\in\mathbf{SWQOS}$, $\downarrow x$ is closed in X, so that $\downarrow[x]$ is closed in $X/\mathcal{R}$. Hence $\leq_{\mathcal{R}}$ is a also lower semi-continuous quasi-order. Thus $(X/\mathcal{R},\tau_{\mathcal{R}},\leq_{\mathcal{R}})\in\mathbf{T_0SWQOS}$, so that q is the $\mathbf{T_0SWQOS}$-reflection of $(X,\tau,\leq)$, which is also initial as shown in Theorem 2.3. This completes the proof. □

PROPOSITION 2.14. (1) $\mathbf{R_0SWQOS}=\{(X,\tau,\leq)\in\mathbf{SWQOS}|$ $\mathbf{T_0S\ WQOS}$*-reflection* $(X/\mathcal{R},\tau_{\mathcal{R}},\leq_{\mathcal{R}})$ *of* $(X,\tau,\leq)$ *is in* $\mathbf{T_1SWQOS}\}$.

(2) $\mathbf{T_0^{-1}SWQOS}=\{(X,\tau,\leq)\in\mathbf{SWQOS}|\mathbf{T_0SWQOS}$*-reflection* $(X/\mathcal{R},\tau_{\mathcal{R}},\leq_{\mathcal{R}})$ *of* $(X,\tau,\leq)$ *is in* $\mathbf{SWPOS}\}$.

PROOF. (1) For $(X,\tau,\leq)\in\mathbf{R_0SWQOS}$, let $q:(X,\tau,\leq)\longrightarrow(X/\mathcal{R},\tau_{\mathcal{R}},\leq_{\mathcal{R}})$ be the $\mathbf{T_0SWQOS}$-reflection of $(X,\tau,\leq)$. For $[x]\neq[y]$ in $X/\mathcal{R}$, we may assume that there is an open neighborhood U of $[x]$ in $X/\mathcal{R}$ with $[y]\notin U$. Then $q^{-1}(U)$ is an open neighborhood of x in X with $y\notin q^{-1}(U)$. By the assumption, there is an open neighborhood V of y in X with $x\notin V$. Then $q(V)$ is an open neighborhood of $[y]$ in $X/\mathcal{R}$ with $[x]\notin q(V)$; hence $(X/\mathcal{R},\tau_{\mathcal{R}})\in\mathbf{Top_1}$. Thus $(X/\mathcal{R},\tau_{\mathcal{R}},\leq_{\mathcal{R}})\in\mathbf{T_1SWQOS}$. Conversely, take any $(X,\tau,\leq)$ in the right handside. For any x, $y\in X$, suppose that there is an open neighborhood U of x with $y\notin U$. Then $\mathcal{N}(x)\neq\mathcal{N}(y)$, i.e., $[x]\neq[y]$ in $(X/\mathcal{R},\tau_{\mathcal{R}},\leq_{\mathcal{R}})$. Since $(X/\mathcal{R},\tau_{\mathcal{R}},\leq_{\mathcal{R}})\in\mathbf{T_1SWQOS}$, there are open neighborhoods V and W of $[x]$ and $[y]$, respectively, in $X/\mathcal{R}$ such that $[x]\notin W$ and $[y]\notin V$. Then $q^{-1}(W)$ is an open neighborhood of y in X such that $x\notin q^{-1}(W)$. Thus $(X,\tau,\leq)\in\mathbf{R_0SWQOS}$.

(2) It is obtained by the similar way as the proof of Proposition 2.4. □

COROLLARY 2.15. *The categories* $\mathbf{T_1SWQOS}$ *and* $\mathbf{SWPOS}$ *are epireflective subcategories of* $\mathbf{R_0SWQOS}$ *and* $\mathbf{T_0^{-1}SWQOS}$, *respectively.*

Moreover, $\mathbf{T_1SWQOS}$ *and* $\mathbf{SWPOS}$ *are initially dense in* $\mathbf{R_0SWQOS}$ *and* $\mathbf{T_0^{-1}SWQOS}$, *respectively.*

Using the fact that semi-continuous quasi-ordered space is both upper and lower semi-continuous quasi-ordered space, we apply the same method in the previous and then have the following proposition easily.

Proposition 2.16. *The categories* $\mathbf{T_0SWQOS}, \mathbf{T_1SWQOS}$ *and* $\mathbf{SWPOS}$ *are finally dense in* $\mathbf{SWQOS}$, $\mathbf{R_0SWQOS}$ *and* $\mathbf{T_0^{-1}SWQOS}$, *respectively.*

Collecting the above facts, one has the following diagram of reflective subcategories in $\mathbf{SWQOS}$:

$$
\begin{array}{ccccccccc}
 & & & & & & \mathbf{SWQOS} & & \\
 & & & & & r\nearrow & & \nwarrow d & \\
 & & & \mathbf{R_0SWQOS} & & & & & \mathbf{T_0SWQOS}, \\
 & r\nearrow & & & \nwarrow d & & & \nearrow r & \\
\mathbf{T_0^{-1}SWQOS} & & & & & & \mathbf{T_1SWQOS} & & \\
 & \nwarrow d & & & \nearrow r & & & & \\
 & & & \mathbf{SWPOS} & & & & &
\end{array}
$$

where d is the inclusion functor with initial and final dense reflection and r is the inclusion functor with epireflection.

3. MacNeille and universal initial completions

A completion of a category $\mathbf{C}$ that is simultaneously an initial completion and a final completion is called a *MacNeille completion* of $\mathbf{C}$ ([1], [9]). MacNeille completions are characterized as the smallest initial completions of $\mathbf{C}$. It is known that the category of quasi-ordered sets is the MacNeille completion of the category of partially ordered sets ([2]).

Proposition 3.1. $\mathbf{T_0^{-1}S^UWQOS}$ *and* $\mathbf{T_0^{-1}SWQOS}$ *are topological categories.*

Proof. We note that $\mathbf{TQOS}$ is a topological category ([10]), so it is enough to show that $\mathbf{T_0^{-1}S^UWQOS}$ is closed under initial sources in $\mathbf{TQOS}$. Let $(f_\alpha : (X,\tau,\leq) \longrightarrow (X_\alpha,\tau_\alpha,\leq_\alpha))_{\alpha\in J}$ be an initial source in $\mathbf{TQOS}$ such that for all $\alpha \in J$ $(X_\alpha,\tau_\alpha,\leq_\alpha) \in \mathbf{T_0^{-1}S^UWQOS}$. Then $(X,\tau,\leq) \in \mathbf{S^UWQOS}$ by Proposition 3.7 in [11]. Thus it remains to show that for $x,y \in X$, $x \leq y$ and $y \leq x$ if and only if $\mathcal{N}(x) = \mathcal{N}(y)$. But $x \leq y$ and $y \leq x$ if and only if $f_\alpha(x) \leq_\alpha f_\alpha(y)$ and $f_\alpha(y) \leq_\alpha f_\alpha(x)$ $(\alpha \in J)$ if and only if $\mathcal{N}(f_\alpha(x)) = \mathcal{N}(f_\alpha(y))$ $(\alpha \in J)$ if and only if $\mathcal{N}(x) = \mathcal{N}(y)$, since $(f_\alpha)_{\alpha\in J}$ is initial in $\mathbf{Top}$ and $(X_\alpha,\tau_\alpha,\leq_\alpha) \in \mathbf{T_0^{-1}S^UWQOS}$ $(\alpha \in J)$.

Similarly, we obtain the result for $\mathbf{T_0^{-1}SWQOS}$. □

PROPOSITION 3.2. $\mathbf{R_0SWQOS}$ *is a topological category.*

PROOF. We show that $\mathbf{R_0SWQOS}$ is closed under initial sources in $\mathbf{TQOS}$. Let $(f_\alpha : (X, \tau, \leq) \longrightarrow (X_\alpha, \tau_\alpha, \leq_\alpha))_{\alpha \in J}$ be an initial source in $\mathbf{TQOS}$ such that each $(X_\alpha, \tau_\alpha, \leq_\alpha) \in \mathbf{R_0SWQOS}$ $(\alpha \in J)$. For any $x, y \in X$, suppose U is an open set with $x \in U$ and $y \notin U$. Then $x \in \bigcap_{\alpha \in K} f_\alpha^{-1}(U_\alpha) \subseteq U$, where U_α is open in X_α and K is a finite subset of J, since τ is initial. So $x \in f_\alpha^{-1}(U_\alpha)$ for all $\alpha \in K$ and $y \notin f_{\alpha_o}^{-1}(U_{\alpha_o})$ for some $\alpha_o \in K$; hence $f_{\alpha_o}(x) \in U_{\alpha_o}$ and $f_{\alpha_o}(y) \notin U_{\alpha_o}$. Since $X_{\alpha_o} \in \mathbf{R_0SWQOS}$, there is an open set V_{α_o} with $f_{\alpha_o}(y) \in V_{\alpha_o}$ and $f_{\alpha_o}(x) \notin V_{\alpha_o}$; hence $y \in f_{\alpha_o}^{-1}(V_{\alpha_o})$ and $x \notin f_{\alpha_o}^{-1}(V_{\alpha_o})$. Thus $(X, \tau, \leq) \in \mathbf{R_0SWQOS}$. □

By the above facts and the results in the previous section, the following proposition is immediate.

PROPOSITION 3.3. (1) *The categories* $\mathbf{S^UWQOS}$ *and* $\mathbf{T_0^{-1}S^UWQOS}$ *are MacNeille completions of* $\mathbf{T_0S^UWQOS}$ *and* $\mathbf{S^UWPOS}$, *respectively.*

(2) *The categories* $\mathbf{SWQOS}, \mathbf{R_0SWQOS}$ *and* $\mathbf{T_0^{-1}SWQOS}$ *are Mac-Neille completions of* $\mathbf{T_0SWQOS}$, $\mathbf{T_1SWQOS}$ *and* $\mathbf{SWPOS}$, *respectively.*

PROPOSITION 3.4. $\mathbf{T_0S^UWQOS}$, $\mathbf{T_0SWQOS}$ *and* $\mathbf{T_1SWQOS}$ *are mono-topological categories.*

PROOF. It follows from the fact that $\mathbf{S^UWQOS}$ and $\mathbf{SWQOS}$ are topological ([10]), and $\mathbf{Top_0}$ and $\mathbf{Top_1}$ are closed under mono-sources in $\mathbf{Top}$. □

The universal initial completion of $\mathbf{C}$ is the initially complete reflection of $\mathbf{C}$ in the category of small categories and initially preserving functors; equivalently, it is the largest initiallity preserving, initially dense, full extension of $\mathbf{C}$ ([1], [9]).

It is known that the MacNeille completion and the universal initial completion of a mono-topological category always coincide ([2]).

Using Corollary 1.6(1) and the above results, we obtain the following immediately.

PROPOSITION 3.5. (1) *The categories* $\mathbf{S^UWQOS}$ *and* $\mathbf{T_0^{-1}S^UWQOS}$ *are universal initial completions of* $\mathbf{T_0S^UWQOS}$ *and* $\mathbf{S^UWPOS}$, *respectively.*

(2) *The categories* **SWQOS**, $\mathbf{R_0}$**SWQOS** *and* $\mathbf{T_0^{-1}}$**SWQOS** *are universal initial completions of* $\mathrm{T_0}$**SWQOS**, $\mathbf{T_1}$**SWQOS** *and* **SWPOS**, *respectively.*

By replacing the category $\mathbf{S^U}$**WQOS** with the category $\mathbf{S^L}$**WQOS** in this section, dually we get $\mathbf{S^L}$**WQOS** and $\mathbf{T_0^{-1}S^L}$**WQOS** are MacNeille and universal initial completions of $\mathbf{T_0S^L}$**WQOS** and $\mathbf{S^L}$**WPOS**, respectively.

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