

SOME REMARKS ON QUASI AND WEAKLY QUASI CONTINUOUS FUNCTIONS

SEUNGWOOK KIM AND HYUNJUNG SONG

ABSTRACT. Since N. Levin [1] in 1961 defined the weakly continuous function, there is a lot of weakened forms of continuous function. In this paper we consider two forms of the weakened form of continuous function and study the some properties of them.

1. Introduction

N. Levin defined a function $f : X \rightarrow Y$ *weakly continuous at* $x \in X$ if to each open set V in Y containing $f(x)$ there is an open set U in X containing x such that $U \subseteq f^{-1}(\bar{V})$. He called this f *weakly continuous* if it is weakly continuous at every point of X . This new function motivated to research many other weakened forms of continuous functions like almost continuous ([2]), semi continuous functions ([3]). The quasi and weakly quasi continuous functions are one of such functions.

In the present article we will see some properties of the topological spaces which are related with these functions and the relationships of both functions. Throughout this paper X and Y denote the topological spaces unless they are explicitly given.

DEFINITION. $f : X \rightarrow Y$ is said to be *quasi continuous (weakly quasi-continuous)* at $x \in X$ if to each open set V in Y containing $f(x)$ and to each open set U in X containing x there is a non empty open set G contained in U such that $G \subseteq f^{-1}(V)$ ($G \subseteq f^{-1}(\bar{V})$), respectively. $f : X \rightarrow Y$ is said to be *quasi continuous (weakly quasi continuous)* if it is quasi continuous (weakly quasi continuous) at each point $x \in X$, respectively.

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It is well known that the weak continuity does not imply the continuity. It is also well known that the weak continuity and the quasi continuity are independent concepts (5. Examples in [3]) but it is almost trivial to see that both imply the weakly quasi continuity. The Gauss function on \mathbb{R} in the usual sense shows that the weakly quasi continuity does not imply the weak continuity and it is quite simple to construct a counterexample which is weakly quasi continuous but not quasi continuous.

NOTATION. Let $f : X \rightarrow Y$ be a function. Let $x \in X$ and S be a subset of Y containing $f(x)$. Let $G_x^S := \bigcup \{U \mid U \text{ is a non empty open set in } X \text{ such that } U \subseteq f^{-1}(S)\}$.

2. Quasi and weakly quasi continuous functions

THEOREM 1. *The followings are equivalent.*

- (a) $f : X \rightarrow Y$ is quasi continuous (weakly quasi continuous).
- (b) $f|_U : U \rightarrow Y$ is quasi continuous (weakly quasi continuous) for every non empty open set U in X .
- (c) To each $x \in X$ and each open set $V \subseteq Y$ containing $f(x)$, $x \in \overline{G_x^V}$ ($x \in \overline{G_x^V}$), respectively.

PROOF. We only prove the weakly quasi continuity.

“(a) \longrightarrow (b)”: Let U be a non empty open set in X and $x \in U$. Let V be an open set in Y containing $f(x)$ and W be an open set in U containing x . Since W is also open in X and f is weakly quasi continuous at x , there exists a non empty open subset Q of W in X such that $Q \subseteq f^{-1}(\overline{V})$. Since Q is also open in U , we have $Q \subseteq f|_U^{-1}(\overline{V})$.

“(b) \longrightarrow (a)”: Take $U = X$.

“(c) \longleftarrow (a)”: It is trivial. □

From theorem 1 we see immediately the following Lemma we use in the sequel.

LEMMA 2. $f : X \rightarrow Y$ is quasi continuous but not continuous at $x \in X$ (weakly quasi continuous but not weakly continuous at $x \in X$) if and only if there is an open set V in Y containing $f(x)$ such that $x \in \overline{G_x^V} \setminus G_x^V$ ($x \in \overline{G_x^V} \setminus G_x^V$), respectively.

Note that if $f : X \rightarrow Y$ is quasi continuous but not continuous at $x \in X$ (weakly quasi continuous but not weakly continuous at $x \in X$)

then x is a limit point of G_x^V ($G_x^{\bar{V}}$), respectively.

COROLLARY 3. *A weakly quasi continuous function $f : X \rightarrow Y$ is quasi continuous if Y is regular.*

PROOF. The proof is straightforward and is omitted. \square

The next theorem shows the conditions when a quasi continuous function will be continuous. Recall that X is said to be *extremally disconnected* if the closure of every open set is open in X .

THEOREM 4. *A weakly quasi continuous function $f : X \rightarrow Y$ is weakly continuous if X is extremally disconnected and Y is regular.*

PROOF. Suppose there is $x \in X$ at which f is not weakly continuous. Then there is an open set V in Y such that $f(x) \in V$ and $x \in \overline{G_x^{\bar{V}}} \setminus G_x^{\bar{V}}$. Since also $\overline{G_x^{\bar{V}}}$ is open, there is an open set U in X such that $x \in U \subset \overline{G_x^{\bar{V}}}$. But U contains one element y which is not contained in $f^{-1}(\bar{V})$. Otherwise $U \subset f^{-1}(\bar{V})$ and this means $x \in U \subset G_x^{\bar{V}}$. That is contrary to $x \in \overline{G_x^{\bar{V}}} \setminus G_x^{\bar{V}}$. Hence $f(y) \notin \bar{V}$. By the regularity of Y there is an open set W in Y such that $f(y) \in W$ and $\bar{W} \cap \bar{V} = \emptyset$. Thus $G_y^{\bar{W}} \cap G_x^{\bar{V}} = \emptyset$ and also $G_y^{\bar{W}} \cap \overline{G_x^{\bar{V}}} = \emptyset$. But there is a non empty open subset U' of U such that $U' \subset G_y^{\bar{W}}$. It then holds $\emptyset \neq U' \subset G_y^{\bar{W}} \cap \overline{G_x^{\bar{V}}}$ which is contrary. \square

PROPOSITION 5. *Let X be a cofinite topological space. Then $f : X \rightarrow Y$ is quasi continuous (weakly quasi continuous) if and only if f is continuous (weakly continuous), respectively.*

PROOF. We only prove the weakly quasi continuity of f . Necessity. Suppose that X has an element x at which f is not weakly continuous. By Lemma 2 there exists an open set V in Y containing $f(x)$ such that $x \in \overline{G_x^{\bar{V}}} \setminus G_x^{\bar{V}}$. Since X is a cofinite topological space, $X \setminus G_x^{\bar{V}}$ is finite and $x \in X \setminus G_x^{\bar{V}}$. Let U be any open set containing x . Let $U' := \{y \in U \mid f(y) \in \bar{V}\}$. Since $U' \subset X \setminus G_x^{\bar{V}}$, U' is a finite set. Let $U'' := \{y_1, \dots, y_n\}$. Then $U'' = U \setminus U'$ is an open set containing x and $U'' \subset f^{-1}(\bar{V})$. Since $x \in U'' \subset G_x^{\bar{V}}$, we have a contradiction. \square

THEOREM 6. *Let $f : X \rightarrow Y$ be quasi continuous (weakly quasi continuous). Suppose X has an element x at which f is not continuous (not weakly continuous), respectively. If f is injective and Y is a Hausdorff space, then X is infinite.*

PROOF. We only prove the weakly quasi continuity of f at $x \in X$. Suppose that X is finite. By Lemma 2 there exists an open set V in Y containing $f(x)$ such that $x \in \overline{G_x^V} \setminus G_x^V$. Since X is finite, there is $n \in \mathbb{N}$ such that $G_x^V = \{y_1, y_2, \dots, y_n\}$. Thus $x \neq y_i$ so that $f(x) \neq f(y_i)$ for $1 \leq i \leq n$ by injection of f . Since Y is a Hausdorff space, there exists an open set W in Y containing $f(x)$ such that $f(y_i) \notin \overline{W}$, $1 \leq i \leq n$ and $\overline{W} \subset \overline{V}$. Thus $G_x^W = \emptyset$ which contradicts the weakly quasi continuity of f at x . \square

THEOREM 7. *Let $f : X \rightarrow Y$ be quasi continuous (weakly quasi continuous). Let X be a T_1 space and have an element x at which f is not continuous (not weakly continuous), respectively. Then X is infinite.*

PROOF. If X is finite, then X is a cofinite topological space. By Proposition 5, f is weakly continuous. This is a contradiction. \square

P. E. Long and E. E. McGehee, Jr. obtained the following (theorem 2 in [2]). Let $f : X \rightarrow Y$ be almost continuous at $x \in X$ (i.e., $f^{-1}(\overline{V})$ is a neighborhood of x for every open set V in Y containing $f(x)$) where X is T_1 and both X and Y are first countable. If x is a limit point of X , there exists a sequence (x_n) of distinct points on X converging to x such that $(f(x_n))$ converges to $f(x)$. In following we show that the similar statements could be obtained in the case of quasi continuity and weakly quasi continuity.

LEMMA 8. *Let $f : X \rightarrow Y$ be quasi continuous but not continuous at $x_0 \in X$. Then for every pair of open sets $U \subseteq X$ and $V \subseteq Y$ containing x_0 and $f(x_0)$, respectively, there exists an $x \in U \setminus \{x_0\}$ such that $f(x) \in V$ (compare lemma in [2]).*

PROOF. Let $U \subseteq X$ and $V \subseteq Y$ be open sets containing x_0 and $f(x_0)$ respectively. Using Lemma 2, there exists an open set W in Y containing $f(x_0)$ such that $x_0 \in \overline{G_{x_0}^W} \setminus G_{x_0}^W$ so that also $x_0 \in \overline{G_{x_0}^{V \cap W}} \setminus G_{x_0}^{V \cap W}$. Hence there is $x \in U$ different from x_0 such that $f(x) \in V \cap W \subseteq V$. \square

THEOREM 9. *Let $f : X \rightarrow Y$ be quasi continuous but not continuous at $x_0 \in X$ and both X and Y be first countable. There exists a sequence (x_n) different from x_0 converging to x_0 such that $(f(x_n))$ converges to $f(x_0)$.*

PROOF. Let (U_n) and (V_n) be countable descending open bases at the points of x_0 and $f(x_0)$, respectively. By Lemma 8, to each $n \in \mathbb{N}$, there

exists $x_n \in U_n, x_n \neq x_0$ such that $f(x_n) \in V_n$. This (x_n) is obviously the required sequence. □

If moreover X is a T_1 space in the hypothesis of Lemma 8, we will obtain the same result of theorem 2 in [2]. P. E. Long and E. E. McGehee, Jr. defined a function $f : X \rightarrow Y$ to be *finitely closed at* $x_0 \in X$ if for each open set $V \subseteq Y$ containing $f(x_0)$ there exists an open set V_1 in Y such that $f(x_0) \in V_1 \subseteq V$ and $X \setminus \overline{f^{-1}(V_1)}$ consists of finitely many components (definition 3 in [2]). If we write $X \setminus \overline{f^{-1}(V_1)}$ at the place of $X \setminus \overline{f^{-1}(V_1)}$ in the hypothesis of the definition, theorem 5 in [2] can be rewritten as follows.

THEOREM 10. *Let $f : X \rightarrow Y$ be connected. If f is finitely closed at $x_0 \in X$, then f is weakly quasi continuous at x_0 .*

PROOF. Suppose that f is not weakly quasi continuous at x_0 . By Lemma 2 there is an open set V in Y containing $f(x_0)$ such that $x_0 \notin \overline{G_{x_0}^V}$. Hence x_0 is not an interior point of $f^{-1}(\overline{V})$. Thus x_0 is a limit point of $X \setminus f^{-1}(\overline{V})$. Since f is finitely closed at x_0 , there is an open set $V_1 \subset V$ in Y containing $f(x_0)$ such that $X \setminus \overline{f^{-1}(V_1)}$ consists of finitely many components C_1, C_2, \dots, C_n for some $n \in \mathbb{N}$. By $f^{-1}(\overline{V_1}) \subset f^{-1}(\overline{V})$, x_0 is a limit point of $X \setminus \overline{f^{-1}(V_1)}$ and hence a limit point of some $C_k, k \in \{1, 2, \dots, n\}$. Thus $C_k \cup \{x_0\}$ is connected but $f(C_k \cup \{x_0\}) = f(C_k) \cup \{f(x_0)\}$ is not connected in Y because $f(x_0) \in V_1$ while $f(C_k) \subset Y \setminus \overline{V_1}$. This contradicts f being connected. □

With corollary 3 we have

THEOREM 11. *Let $f : X \rightarrow Y$ be connected where Y is regular. If f is finitely closed at $x_0 \in X$, then f is quasi continuous at x_0 .*

A subset S of X is said to be *semi open* if there is an open set U in X such that $U \subseteq S \subseteq \overline{U}$. X is said to be *S-closed* if for every semi open cover of X there exists a finite subfamily such that the union of their closures covers X . X is said to be *H-closed* if X is Hausdorff space and for every open cover of X there exists a finite subfamily whose union covers X .

LEMMA 12. *Let $f : X \rightarrow Y$ be weakly quasi continuous and Y extremally disconnected. Then $f^{-1}(\overline{V})$ is semi-open for all open V in Y .*

PROOF. Let V be an open set in Y . Let $x \in X$ such that $f(x) \in V$. Since f is weakly quasi continuous at x , by theorem 1 there exists an open set $G_x^{\bar{V}}$ in X such that $x \in \overline{G_x^{\bar{V}}}$ and $G_x^{\bar{V}} \subseteq f^{-1}(\bar{V})$. It suffices to show that $f^{-1}(\bar{V}) \subseteq \overline{G_x^{\bar{V}}}$. Let $y \in f^{-1}(\bar{V})$ and $y \notin G_x^{\bar{V}}$. Since \bar{V} is open, there exists open set Q in Y such that $\bar{Q} \subseteq \bar{V}$ and $f(y) \in Q$. Besides, there exists an open set U in X containing y such that $U \cap G_x^{\bar{V}} = \emptyset$. Since f is weakly quasi continuous, there exists a non empty open set $G \subseteq U$ such that $G \subseteq f^{-1}(\bar{Q}) \subseteq f^{-1}(\bar{V})$ so that $G \subseteq G_x^{\bar{V}}$. It is a contradiction. \square

THEOREM 13. *Let $f : X \rightarrow Y$ be weakly quasi continuous. Let X be S -closed and Y be extremally disconnected and Hausdorff space. Then Y is H -closed.*

PROOF. Let $\{Q_\lambda | \lambda \in \Lambda\}$ be an open cover of Y . Then $\{\overline{Q_\lambda} | \lambda \in \Lambda\}$ is also an open cover of Y . By above Lemma 12 $f^{-1}(\overline{Q_\lambda})$ is semi open for all $\lambda \in \Lambda$ and the set $\{f^{-1}(\overline{Q_\lambda}) | \lambda \in \Lambda\}$ is a semi-open cover of X . Since X is S -closed, there exists $n \in \mathbb{N}$ such that $\{f^{-1}(\overline{Q_{\lambda_i}}) | i = 1, \dots, n\}$ is a cover of X . Therefore $Y = \bigcup_{i=1}^n f(f^{-1}(\overline{Q_{\lambda_i}})) \subseteq \bigcup_{i=1}^n \overline{Q_{\lambda_i}}$. \square

References

- [1] N. Levin, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly **68** (1961), 44–46.
- [2] Paul E. Long and Earl E. McGehee, Jr., *Properties of almost continuous functions*, Proc. Amer. Math. Soc. **24** (1970), 175–180.
- [3] T. Noiri, *Properties of some weak forms of continuity*, Intern. J. Math. & Math. Sci. **10** (1987), no. 1, 97–111.

Seungwook Kim
 Department of Mathematics
 Hankuk University of Foreign Studies
 Kyunggido 449-791, Korea
E-mail: swkim@san.hufs.ac.kr

Hyunjung Song
 Graduate School of Science and Technology
 Chiba University
 Chiba-shi 263-8522, Japan
E-mail: shj95@netian.com