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TRANSNORMAL SYSTEMS IN SEMI-RIEMANNIAN SPACES

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ABSTRACT. In this paper, we study transnormal systems on the Euclidean and semi-Euclidean spaces. We classified transnormal systems on R_p^{n+1}. We also prove that transnormal systems on R_p^n are algebraic even though there are non-algebraic isoparametric hypersurfaces.

1. Introduction

The study of isoparametric families in space forms was initiated by E. Cartan [2]. Each hypersurface in an isoparametric families has constant principal curvatures and can be expressed as a level set of a real-valued function on its ambient space. Cartan defined an isoparametric function on a Riemannian space form $\overline{M}$ as a smooth function $f : \overline{M} \longrightarrow R$ such that the length of its gradient and its Laplacian are functions of f. If f is isoparametric, then the family of level sets are called an isoparametric family of $\overline{M}$ and each regular level set, called an isoparametric hypersurface, has constant principal curvatures. Conversely, if M is a complete connected isoparametric hypersurface of $\overline{M}$, then there is an isoparametric function with M as a regular level set. This concept is extended to isoparametric submanifolds of arbitrary codimensions and isoparametric functions can be defined on spaces other than space forms such as symmetric spaces (cf. [3], [9], [10]).

A transnormal system on a Riemannian manifold $\overline{M}$ is a partition of $\overline{M}$ into disjoint submanifolds such that any geodesic of $\overline{M}$ cuts these submanifolds orthogonally at none or all of its points. This notion was

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initiated by J. Bolton [1]. Actually transnormality is a weak notion of the isoparametricity. These two notions are equivalent on the Euclidean space R^n and the unit sphere S^n. But it is not true on the hyperbolic space H^n.

A generalization of isoparametric hypersurfaces to the semi-Riemannian spaces has been done by Nomizu [8], Magid [7], Hahn [4]. There are many similarities and many differences between Riemannian spaces and semi-Riemannian spaces. In particular, isoparametric families in R^n is quite simple. But in the semi-Euclidean space R^n_p, there are isoparametric hypersurfaces which is not a level set of an isoparametric function. Also there are isoparametric hypersurfaces in R^n_p with complex principal curvatures.

In this paper, we are going to study transnormal systems in R^n_p. There are interesting differences between isoparametric families and transnormal systems in R^n_p. Note that they are the same objects in the Euclidean space R^n.

2. Preliminaries

Let R^{n+1}_p be the $n+1$ dimensional real vector space with the inner product $<,>$ of index p given by

$$<x, y> = -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_{n+1}y_{n+1}.$$

DEFINITION 2.1. Let $f : R^{n+1}_p \longrightarrow R$ be a smooth function. Let ∇f and Δf denote its gradient and Laplacian.

(1) f is said to be *isoparametric* if $<\nabla f, \nabla f>$ and Δf are functions of f.

(2) f is said to be *transnormal* if $<\nabla f, \nabla f>$ is a function of f.

In R^{n+1}, these two definitions are equivalent. We will show that it is true in R^{n+1}_p. Let M be a hypersurface of R^{n+1}_p with a unit normal vector field η. For a sufficiently small t, define a map

$$\phi_t : M \longrightarrow R^{n+1}_p, \quad x \mapsto exp_x t\eta_x.$$

The image M_t of ϕ_t is called the *parallel surface* of M at distance t. And M is called an *isoparametric* hypersurface if it satisfies one of the following equivalent conditions:

(1) M_t has constant mean curvature for sufficiently small t,

(2) M has constant principal curvatures with constant algebraic multiplicities,

(3) A_η has constant characteristic polynomials on M.

In this case, the family of parallel surfaces is called an *isoparametric family.*

EXAMPLE 2.2. Define $f : R_p^{n+1} \longrightarrow R$ by

$$f(x_1, x_2, \cdots, x_{n+1}) = -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{n+1}^2.$$

Let P be the position vector field on R_p^{n+1}. Then $f = <P, P>$ and hence $\nabla f = 2P$. Thus f is isoparametric and $f^{-1}(\pm r^2) = Q(\pm r)$ are hyperquadrics of R_p^{n+1} for each nonzero r. The hyperquadrics are isoparametric hypersurfaces with one constant principal curvature. The set $\Lambda = f^{-1}(0) - \{0\}$ is the null cone of R_p^{n+1}.

A submanifolds which is the product of a null cone and a k-plane is called a *null cylinder.*

3. Transnormal systems

DEFINITION 3.1. A partition $\Im$ of R_p^{n+1} is called a *transnormal system* if

(1) Each member of $\Im$ is a nondegenerate submanifold or a null cylinder.

(2) Any geodesic cuts these submanifolds orthogonally at none or all of its points.

In the Example 2.2, the regular level sets, the null cone and the origin constitute a transnormal system of R_p^{n+1}. This is a typical transnormal system corresponding to the transnormal system in R^{n+1} given by n-spheres. This means we need the existence of null cylinder in the definition of a transnormal system.

Let $f : R_p^{n+1} \longrightarrow R$ be a transnormal function and $[a, b] \subset f(R^{n+1})$ such that f has no critical values on $[a, b]$. We denote $M_c = f^{-1}(c)$. Since f is transnormal, the causal character of ∇f is constant on each regular level set. Suppose that ∇f is spacelike on M_a. Since there is no critical points on $[a, b]$, ∇f is spacelike on $\cup_{c \in [a,b]} M_c$. For a piecewise smooth spacelike curve $\alpha : [0, 1] \longrightarrow R_p^{n+1}$ such that $\alpha(0) \in M_a$ and $\alpha(1) \in M_b$, we have

$$L(\gamma) \leq L(\alpha),$$

where $L(\gamma)$ denotes the length of the integral curve γ of ∇f. Thus the integral curve of ∇f is the shortest curve between M_a and M_b. That is, the integral curve of ∇f is a geodesic segment.

When α is timelike, we can apply the same argument. Thus we obtain the following result.

PROPOSITION 3.2. *Let $f : R_p^{n+1} \longrightarrow R$ be a transnormal function. Then the family*

$$\{M_c |\ c \text{ is a regular value of } f\}$$

constitutes a transnormal system.

Let $\Im$ be a transnormal system on R_p^{n+1} containing a nondegenerate hypersurface M. Note that geodesics are straight lines. If two normal geodesics on M meet at a point x, consider the plane π determined by these two normal lines. Then any line on π through x is a normal geodesic on M. Let Π_x be the k-plane generated by all normal geodesics through x. If l is a line through x which is orthogonal to Π_x, it can not meet any hypersurface in $\Im$. Thus we obtain an $(n-k+1)$-plane $\Im_x$ as a *focal* submanifold. On Π_x, the partition

$$\Im_\Pi = \{S \cap \Pi_x | S \in \Im\}$$

is also a transnormal system. Note that the members of a transnormal system are equidistant. Thus the transnormal system $\Im_\Pi$ is the canonical system given in the Example 2.2, and hence $\Im_\Pi$ consists of hyperquadrics Q, the only focal submanifold $\{x\}$ and the null cone Λ_Π. Then the system $\Im$ consists of cylinders $Q \times \Im_x$ over hyperquadrics Q, the $(n-k+1)$-plane $\Im_x$ and the null cylinder $\Lambda_\Pi \times \Im F_x$. If every normal geodesics are parallel, then $\Im$ consists of hyperplanes. Thus we obtain the following result.

PROPOSITION 3.3. *Let $\Im$ be a transnormal system on R_p^{n+1} containing a nondegenerate hypersurface M. Then M is a hyperquadric, a cylinder over a hyperquadric or a hyperplane.*

We define three transnormal functions as follows:

(1) $(x_1, x_2, \cdots, x_{n+1}) \mapsto -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots x_{n+1}^2$.

(2) $(x_1, x_2, \cdots, x_{n+1}) \mapsto -x_{i_1}^2 - \cdots - x_{i_s}^2 + x_{j_1}^2 + \cdots + x_{j_t}^2$, where $\{i_1, \cdots, i_s\} \subset \{1, 2, \cdots, p\}$ and $\{j_1, \cdots, j_t\} \subset \{p+1, \cdots, n+1\}$.

(3) $(x_1, x_2, \cdots, x_{n+1}) \mapsto x_1$.

Note that the transnormal systems in the above Proposition are isometric to one of three transnormal systems given by the above transnormal functions. Thus we have the following result.

PROPOSITION 3.4. *Transnormal systems in R_p^{n+1} are algebraic.*

On R^{n+1} the two objects, the transnormal system and the isoparametric family, are equivalent. And isoparametric hypersurfaces are regular level sets of homogeneous functions. But, on R_p^{n+1}, there are isoparametric hypersurfaces which are not algebraic (cf. [4]). Thus an isoparametric hypersurface may not generate a transnormal system in the semi-Euclidean space R_p^{n+1}.

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