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LOCAL EXISTENCE AND GLOBAL UNIQUENESS IN ONE DIMENSIONAL NONLINEAR HYPERBOLIC INVERSE PROBLEMS

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ABSTRACT. We prove local existence and global uniqueness in one dimensional nonlinear hyperbolic inverse problems. The basic key for showing the local existence of inverse solution is the principle of contracted mapping. As an application, we consider a hyperbolic inverse problem with damping term.

1. Introduction and main results

We consider in the domain $D = \{(x,t) | -\infty < x < \infty,\ t > 0\}$ the following systems for $u = u(x,t)$;

$$(1.1)\qquad \begin{cases} u_{tt} - u_{xx} = F\left(p(x), u, u_t, u_x\right), & (x,t) \in D, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), & -\infty < x < \infty. \end{cases}$$

Here we set $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $\phi \in C^2(-\infty,\infty)$, $\psi \in C^1(-\infty,\infty)$ and $p \in C(-\infty,\infty)$. We assume that $F = F(\xi_1,\xi_2,\xi_3,\xi_4)$ is of C^2 class in every argument $\xi_i \in \mathbb{R}$, $1 \le i \le 4$ and defined in $\mathbb{R}^4$. Then by successive approximation method we can prove: For any $x_0 \in \mathbb{R}$ we can choose $\eta = \eta(x_0)$ such that $u \in C^2(\Delta(x_0,\eta))$ exists a unique solution to (1.1). Here $\Delta(x_0,\eta) = \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ \mid 0 < t \le \eta,\ x_0 + (t-\eta) \le x \le x_0 - (t-\eta)\}$. Let $u(p) = u(p)(x,t)$ and $u(q) = u(q)(x,t)$ be the solutions to (1.1) with $p = p(x)$ and $q = q(x)$, respectively.

<u>**Nonlinear inverse problem**</u> Let $x_0 \in (-\infty,\infty)$ and $t_0 > 0$ be given.

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Determine $p(x)$ in some interval containing x_0 from $u(p)(x_0,t)$ and $u_x(p)(x_0,t)$, $0 \le t \le t_0$.

The purpose of this paper is to prove local existence and global uniqueness in the inverse problem. Romanov [8] proved local existence, global uniqueness and stability in determining $p = p(x)$ in the case of $F(\xi_1,\xi_2,\xi_3,\xi_4) = \xi_1\xi_2$, that is, $u_{tt}(x,t) = u_{xx}(x,t) + p(x)u(x,t)$ from the same kind of data (Chapter 2 in [8]). In this paper we discuss local existence and global uniqueness in determining $p = p(x)$ for more general $F = F(p(x), u(x,t), u_t(x,t), u_x(x,t))$. In particular F can describe the Sine-Gordon equation and the Klein-Gordon equation. Our methodology is based on Chapter 2 in [8]. In particular, in order to show local existence, we use the principle of contracted mapping. To authors' knowledge, the results for one dimensional inverse problems in nonlinear hyperbolic equations are not many. For one dimensional inverse problems, we further refer for example to Chapter 8 in Isakov [2], Lavrent'ev, Reznitskaya and Yakhno [5] and Romanov [8]. For other results of inverse hyperbolic problems, we refer the Gladwell [1], Khaĭdarov [3], Klibanov [4], Rakesh and Symes [6], Romanov [7] and Yamamoto [10] and references therein. We notice that in multidimensional cases, the existence problem is quite difficult and we can refer for example to Romanov [9].

For the statements of our main results, we consider its linearization as well as (1.1): Set $y(x,t) = u(p)(x,t) - u(q)(x,t)$ and $f(x) = p(x) - q(x)$. $q(x)$ is given. Then in view of Taylor's theorem, we have

$$
\begin{aligned}
y_{tt} - y_{xx} = {} & \frac{\partial F}{\partial \xi_1}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)), \\
& \qquad u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big) \cdot f \\
& + \frac{\partial F}{\partial \xi_2}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)), \\
& \qquad u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big) \cdot y \\
& + \frac{\partial F}{\partial \xi_3}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)), \\
& \qquad u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big) \cdot y_t \\
& + \frac{\partial F}{\partial \xi_4}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)),
\end{aligned}
$$

$$u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big) \cdot y_x,$$

where some $\theta \in (0, 1)$.
We set

$$a_1(x,t) = \frac{\partial F}{\partial \xi_1}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)),$$
$$u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big),$$
$$a_2(x,t) = \frac{\partial F}{\partial \xi_2}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)),$$
$$u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big),$$
$$a_3(x,t) = \frac{\partial F}{\partial \xi_3}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)),$$
$$u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big) \quad \text{and}$$
$$a_4(x,t) = \frac{\partial F}{\partial \xi_4}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)),$$
$$u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big).$$

Thus we obtain a system with respect to y:

$$(1.2) \quad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = a_1(x,t)f(x) + a_2(x,t)y(x,t) \\ \quad + a_3(x,t)y_t(x,t) + a_4(x,t)y_x(x,t), \qquad (x,t) \in D, \\ y(x,0) = y_t(x,0) = 0, \qquad -\infty < x < \infty. \end{cases}$$

Linear inverse problem Let $x_0 \in (-\infty, \infty)$ and $t_0 > 0$ be given and $y(x,t) = y(f)(x,t)$ be a classical solution to (1.2). Determine $f = f(x)$ in some interval containing x_0 from $y(f)(x_0,t)$ and $y_x(f)(x_0,t)$, $0 \le t \le t_0$.

Therefore for local existence and global uniqueness of p it is sufficient to verify:

(Local existence of f) $y(f)(x_0,t)$ and $y_x(f)(x_0,t)$, $0 \le t \le t_0$ determine the existence of $h^* \in [0, t_0]$ such that $f(x)$ exists in $C[x_0 - h^*, x_0 + h^*]$.

(Global uniqueness of f) If $f(x)$ exists in $C[x_0 - t_0, x_0 + t_0]$, then it is unique.

THEOREM 1. (**Local existence of** f) *Let $t_0 > 0$ and $x_0 \in \mathbb{R}$ be given. We assume*

$$a_{it}(x,t) = \frac{\partial a_i}{\partial t}(x,t) \in C(\Delta(x_0,t_0)),\ i = 1,2,3,4 \quad \textit{and}$$

$$|a_1(x,0)| \geq \alpha > 0, \quad x \in [x_0 - t_0, x_0 + t_0]$$

with some constant α. If the solution $y(f)(x,t)$ to (1.2) satisfies

$$y(f)(x_0,t) = f_1(t) \in C^2[0,t_0] \quad \textit{and} \quad y_x(f)(x_0,t) = f_2(t) \in C^1[0,t_0],$$

then there is $h^* \in (0,t_0]$ such that f exists in $C[x_0 - h^*, x_0 + h^*]$.

THEOREM 2. (**Local existence of** p) *Let $t_0 > 0$ and $x_0 \in \mathbb{R}$ and $q(x) \in C[x_0 - t_0, x_0 + t_0]$ be given. We assume*

$$F \in C^2(\mathbb{R}^4),\ \phi(x) \in C^2[x_0 - t_0, x_0 + t_0],\ \psi(x) \in C^1[x_0 - t_0, x_0 + t_0]$$

and

$$\left|\frac{\partial F}{\partial \xi_1}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)),\right.$$
$$\left. u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big)(x,0)\right|$$
$$\geq \alpha > 0, \quad x \in [x_0 - t_0, x_0 + t_0]$$

with some $\theta \in (0,1)$ and some constant α. If the solution $u(p)(x,t)$ to (1.1) satisfies

$$u(p)(x_0,t) \in C^2[0,t_0] \quad \textit{and} \quad u_x(p)(x_0,t) \in C^1[0,t_0],$$

then there is $h^* \in (0,t_0]$ such that $p = p(x)$ exists in $C[x_0 - h^*, x_0 + h^*]$.

THEOREM 3. (**Global uniqueness of** f) *Let $t_0 > 0$ and $x_0 \in \mathbb{R}$ be given. We assume*

$$a_{it}(x,t) = \frac{\partial a_i}{\partial t}(x,t) \in C(\Delta(x_0,t_0)),\ i = 1,2,3,4 \quad \textit{and}$$

$$|a_1(x,0)| \geq \alpha > 0 \quad x \in [x_0 - t_0, x_0 + t_0]$$

with some constant α and the solution $y(f)(x,t)$ to (1.2) satisfies

$$y(f)(x_0,t) = f_1(t) \in C^2[0,t_0] \quad \textit{and} \quad y_x(f)(x_0,t) = f_2(t) \in C^1[0,t_0].$$

If $f \in C[x_0 - t_0, x_0 + t_0]$ exists, then it is unique.

THEOREM 4. (**Global uniqueness of** p) *Let* $t_0 > 0$ *and* $x_0 \in \mathbb{R}$ *and* $q(x) \in C[x_0 - t_0, x_0 + t_0]$ *be given. We assume*

$$F \in C^2(\mathbb{R}^4),\ \ \phi(x) \in C^2[x_0 - t_0, x_0 + t_0],\ \ \psi(x) \in C^1[x_0 - t_0, x_0 + t_0]$$

and

$$\begin{aligned}&\left|\frac{\partial F}{\partial p}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q)),\right.\\ &\qquad \left. u_t(q) + \theta(u_t(p) - u_t(q)), u_x(q) + \theta(u_x(p) - u_x(q))\Big)(x,0)\right|\\ &\geq \alpha > 0, \quad x \in [x_0 - t_0, x_0 + t_0]\end{aligned}$$

with some $\theta \in (0,1)$ *and some constant* α *and the solution* $u(p)(x,t)$ *to* (1.1) *satisfies*

$$u(p)(x_0,t) \in C^2[0,t_0] \quad \text{and} \quad u_x(p)(x_0,t) \in C^1[0,t_0].$$

If $p \in C[x_0 - t_0, x_0 + t_0]$ *exists, then it is unique.*

The remainder of this paper is organized as following:
- Section 2. Proof of main results
- Section 3. Application

2. Proof of main results

PROOF OF THEOREM 1. From the d'Alembert formula in the domain $\Delta(x_0, t_0)$, we obtain the classical solution $y(x,t)$ of (1.2) and its partial derivatives with respect to x, t.

$$\begin{aligned}y(x,t) &= \frac{1}{2}\iint\limits_{\Delta(x,t)} \Big(a_1 f + a_2 y + a_3 y_t + a_4 y_x\Big)\Big|_{\substack{x=\xi\\ t=\tau}} d\tau d\xi\\ &= \frac{1}{2}\int_{x-t}^{x+t}\int_0^{t-|x-\xi|}\Big(a_1 f + a_2 y + a_3 y_t + a_4 y_x\Big)\Big|_{\substack{x=\xi\\ t=\tau}} d\tau d\xi\end{aligned}$$

$$y_t(x,t) = \frac{1}{2}\int_{x-t}^{x+t}\Big(a_1 f + a_2 y + a_3 y_t + a_4 y_x\Big)\Big|_{\substack{x=\xi\\ t=t-|x-\xi|}} d\xi$$

$$y_x(x,t) = \frac{1}{2}\int_{x-t}^{x+t}\Big(a_1 f + a_2 y + a_3 y_t + a_4 y_x\Big)\Big|_{\substack{x=\xi \\ t=t-|x-\xi|}} \cdot \mathrm{sign}(\xi - x)d\xi$$

$$\begin{aligned} y_{tt}(x,t) = &\frac{1}{2}\big[a_1(x+t,0)f(x+t) + a_1(x-t,0)f(x-t)\big] \\ &+ \frac{1}{2}\int_{x-t}^{x+t}\big(a_{1t}f + a_{2t}y + (a_2 + a_{3t})y_t \\ &+ a_{4t}y_x + a_3 y_{tt} + a_4 y_{xt}\big)\Big|_{\substack{x=\xi \\ t=t-|x-\xi|}} d\xi \end{aligned}$$

$$\begin{aligned} y_{xt}(x,t) = &\frac{1}{2}\Big[a_1(x+t,0)f(x+t) - a_1(x-t,0)f(x-t)\Big] \\ &+ \frac{1}{2}\int_{x-t}^{x+t}\big(a_{1t}f + a_{2t}y + (a_2 + a_{3t})y_t \\ &+ a_{4t}y_x + a_3 y_{tt} + a_4 y_{xt}\big)\Big|_{\substack{x=\xi \\ t=t-|x-\xi|}} \cdot \mathrm{sign}(\xi - x)d\xi. \end{aligned}$$

From $y_{tt}(x_0,t) = {f_1}''(t),\ y_{xt}(x_0,t) = {f_2}'(t)$, we can obtain the representation of $f(x)$.
Calculating $y_{tt}(x_0,t)+y_{xt}(x_0,t)$ and setting $x_0+t = x$ $(x_0 \le x \le x_0+t_0)$,

$$\begin{aligned} f(x) = &\frac{1}{a_1(x,0)}\Big[{f_1}''(x-x_0) + {f_2}'(x-x_0)\Big] \\ &- \frac{1}{a_1(x,0)}\int_{x_0}^{x}\big(a_{1t}f + a_{2t}y + (a_2 + a_{3t})y_t \\ &+ a_{4t}y_x + a_3 y_{tt} + a_4 y_{xt}\big)\Big|_{\substack{x=\xi \\ t=x-\xi}} d\xi. \end{aligned}$$

Calculating $y_{tt}(x_0,t) - y_{xt}(x_0,t)$ and setting $x_0 - t = x$ $(x_0 - t_0 \le x \le x_0)$,

$$\begin{aligned} f(x) = &\frac{1}{a_1(x,0)}\Big[{f_1}''(x_0-x) - {f_2}'(x_0-x)\Big] \\ &- \frac{1}{a_1(x,0)}\int_{x}^{x_0}\big(a_{1t}f + a_{2t}y + (a_2 + a_{3t})y_t \\ &+ a_{4t}y_x + a_3 y_{tt} + a_4 y_{xt}\big)\Big|_{\substack{x=\xi \\ t=\xi-x}} d\xi. \end{aligned}$$

Therefore, we get

$$f(x) = f_0(x) + \frac{\operatorname{sign}(x_0 - x)}{a_1(x,0)} \times \int_{x_0}^{x} \Big(a_{1t} f + a_{2t} y + (a_2 + a_{3t}) y_t + a_{4t} y_x + a_3 y_{tt} + a_4 y_{xt}\Big)\Big|_{\substack{x=\xi \\ t=|x-\xi|}} d\xi,$$

where

$$f_0(x) = \frac{1}{a_1(x,0)} \left[f_1{}''(|x - x_0|) + \operatorname{sign}(x - x_0) f_2{}'(|x - x_0|) \right].$$

Despite of the presence of a discontinuous multiplier $\operatorname{sign}(x - x_0)$, the function f_0 is continuous on $[x_0 - t_0, x_0 + t_0]$. Because $f_2{}''(0) = 0$.
In the domain $\Delta(x_0, t_0)$, we define the closed system of integral equations $f,\ y,\ y_t\ y_x,\ y_{tt}$, and y_{xt} by operator equation

$$g = Ag$$

$$g = \Big(g_1, g_2, g_3, g_4, g_5, g_6\Big)(x,t) \in \left(C\left(\Delta(x_0, t_0)\right)\right)^6,$$

where g is the vector function of two variables $x,\ t$ with components in which case

$$g_1 = f,\ g_2 = y,\ g_3 = y_t,\ g_4 = y_x,\ g_5 = y_{tt},\ g_6 = y_{xt}.$$

The operator A is determined by the function g and has the form

$$A = \big(A_1, A_2, A_3, A_4, A_5, A_6\big)$$

$$A_1 g(x,t) = g_{01}(x,t) + \frac{\operatorname{sign}(x_0 - x)}{a_1(x,0)} \int_{x_0}^{x} \Big(G(x,t)\Big)\Big|_{\substack{x=\xi \\ t=|x-\xi|}} d\xi$$

$$A_2 g(x,t) = \frac{1}{2} \iint\limits_{\Delta(x,t)} \Big(a_1 g_1 + a_2 g_2 + a_3 g_3 + a_4 g_4\Big)\Big|_{\substack{x=\xi \\ t=\tau}} d\tau d\xi$$

$$A_3 g(x,t) = \frac{1}{2} \int_{x-t}^{x+t} \Big(a_1 g_1 + a_2 g_2 + a_3 g_3 + a_4 g_4\Big)\Big|_{\substack{x=\xi \\ t=t-|x-\xi|}} d\xi$$

$$A_4g(x,t) = \frac{1}{2}\int_{x-t}^{x+t}\Big(a_1g_1 + a_2g_2 + a_3g_3 + a_4g_4\Big)\Big|_{\substack{x=\xi \\ t=t-|x-\xi|}} \cdot \operatorname{sign}(\xi - x)d\xi$$

$$A_5g(x,t) = g_{05}(x,t) + \frac{\operatorname{sign}(x_0 - x - t)}{2}\int_{x_0}^{x+t}\Big(G(x,t)\Big)\Big|_{\substack{x=\xi \\ t=|x+t-\xi|}} d\xi$$
$$+ \frac{\operatorname{sign}(x_0 - x + t)}{2}\int_{x_0}^{x-t}\Big(G(x,t)\Big)\Big|_{\substack{x=\xi \\ t=|x-t-\xi|}} d\xi$$
$$+ \frac{1}{2}\int_{x-t}^{x+t}\Big(G(x,t)\Big)\Big|_{\substack{x=\xi \\ t=t-|x-\xi|}} d\xi$$

$$A_6g(x,t) = g_{06}(x,t) + \frac{\operatorname{sign}(x_0 - x - t)}{2}\int_{x_0}^{x+t}\Big(G(x,t)\Big)\Big|_{\substack{x=\xi \\ t=|x+t-\xi|}} d\xi$$
$$+ \frac{\operatorname{sign}(x - t - x_0)}{2}\int_{x_0}^{x-t}\Big(G(x,t)\Big)\Big|_{\substack{x=\xi \\ t=|x-t-\xi|}} d\xi$$
$$+ \frac{1}{2}\int_{x-t}^{x+t}\Big(G(x,t)\Big)\Big|_{\substack{x=\xi \\ t=t-|x-\xi|}} \cdot \operatorname{sign}(\xi - x)d\xi,$$

where

$$g_{01} = f_0,\ g_{02} = 0,\ g_{03} = 0,\ g_{04} = 0,$$

$$g_{05} = \frac{1}{2}\begin{bmatrix} f_1''(|x+t-x_0|) + \operatorname{sign}(x+t-x_0)f_2'(|x+t-x_0|) \\ +f_1''(|x-t-x_0|) + \operatorname{sign}(x-t-x_0)f_2'(|x-t-x_0|) \end{bmatrix} \text{ and}$$

$$g_{06} = \frac{1}{2}\begin{bmatrix} f_1''(|x+t-x_0|) + \operatorname{sign}(x+t-x_0)f_2'(|x+t-x_0|) \\ -f_1''(|x-t-x_0)|) - \operatorname{sign}(x-t-x_0)f_2'(|x-t-x_0|) \end{bmatrix}$$

and

$$G(x,t) = \Big(a_{1t}g_1 + a_{2t}g_2 + (a_2 + a_{3t})g_3 + a_{4t}g_4 + a_3g_5 + a_4g_6\Big)(x,t).$$

Denote

$$\|g\|(t_0) = \max_{1\le k\le 6}\ \max_{(x,t)\in\Delta(x_0,t_0)} |g_k(x,t)|$$

and

$$g_0(x,t) = (g_{01}, g_{02}, g_{03}, g_{04}, g_{05}, g_{06})\,(x,t).$$

Consider in the space $C\left(\Delta(x_0,h)\right)^6$, $0 < h \le t_0$, the following set

$$m(h) = \left\{ g(x,t) \mid \|g - g_0\|(h) \le \|g_0\|(t_0) \right\}.$$

Obviously, $m(h)$ is not empty set. Moreover on the set $m(h)$ inequality

$$\|g\|(h) \le 2\|g_0\|(t_0)$$

is valid. To use the principle of contracted mapping on the set $m(h)$ the operator A can be controlled by sufficiently small h. Indeed, for $g \in m(h)$, we obtain

$$\begin{aligned}
|A_1 g - g_{01}| &\le \frac{7C}{\alpha} h\|g\|(h) \le \frac{14C}{\alpha} h\|g_0\|(t_0)\\
|A_2 g - g_{02}| &\le 2Ch^2\|g\|(h) \le 4Ch^2\|g_0\|(t_0)\\
|A_3 g - g_{03}| &\le 4Ch\|g\|(h) \le 8Ch\|g_0\|^2(t_0)\\
|A_4 g - g_{04}| &\le 4Ch\|g\|(h) \le 8Ch\|g_0\|(t_0)\\
|A_5 g - g_{05}| &\le 14Ch\|g\|(h) \le 28Ch\|g_0\|(t_0)\\
|A_6 g - g_{06}| &\le 14Ch\|g\|(h) \le 28Ch\|g_0\|^2(t_0),
\end{aligned}$$

where

$$C = \max_{1\le i\le 4} \|a_i\|_1 \text{ and } \|a_i\|_1 = \sum_{\alpha=0}^{1} \max_{(x,t)\in\Delta(x_0,h)} \left| \frac{\partial^\alpha}{\partial t^\alpha} a_i(x,t) \right|.$$

Hence, we see

$$\|Ag - g\|(h) \le \max\left(\frac{14C}{\alpha} h, 4Ch^2, 28Ch \right) \|g_0\|(t_0) \le \|g_0\|(t_0)$$

for any

$$h \le h^* = \min\left(\frac{\alpha}{14C}, \frac{1}{2\sqrt{C}}, \frac{1}{28C}, t_0 \right).$$

This means the operator A self-maps the set $m(h)$, $h \le h^*$. On the other hands, let $g^{(1)}$, $g^{(2)}$ be any two elements of the set $m(h)$, $h \le h^*$. From the following estimates

$$\left| A_1 g^{(1)} - A_1 g^{(2)} \right| \le \frac{7C}{\alpha} h \left\| g^{(1)} - g^{(2)} \right\| (h)$$

$$\left|A_2 g^{(1)} - A_2 g^{(2)}\right| \le 2Ch^2 \left\|g^{(1)} - g^{(2)}\right\|(h)$$
$$\left|A_3 g^{(1)} - A_3 g^{(2)}\right| \le 4Ch \left\|g^{(1)} - g^{(2)}\right\|(h)$$
$$\left|A_4 g^{(1)} - A_4 g^{(2)}\right| \le 4Ch \left\|g^{(1)} - g^{(2)}\right\|(h)$$
$$\left|A_5 g^{(1)} - A_5 g^{(2)}\right| \le 14Ch \left\|g^{(1)} - g^{(2)}\right\|(h)$$
$$\left|A_6 g^{(1)} - A_6 g^{(2)}\right| \le 14Ch \left\|g^{(1)} - g^{(2)}\right\|(h),$$

we obtain

$$\begin{aligned}\left\|Ag^{(1)} - Ag^{(2)}\right\|(h) &\le \max\left(\frac{7C}{\alpha}h, 2Ch^2, 14Ch\right)\left\|g^{(1)} - g^{(2)}\right\|(h)\\ &\le \frac{h}{2h^*}\left\|g^{(1)} - g^{(2)}\right\|(h),\end{aligned}$$

i.e., the operator A is a contraction mapping for any $h \le h^*$. By the principle of contracted mapping, we uniquely find y, y_t, y_{tt}, y_{xt} and f in the domain $\Delta(x_0, h^*)$. Especially f which is the solution to the linear inverse problem uniquely exists in $C[x_0 - h^*, x_0 + h^*]$. □

Proof of Theorem 2. By setting $y(x,t) = u(p)(x,t) - u(q)(x,t)$, $(x,t) \in \Delta(x_0, t_0)$ and $f(x) = p(x) - q(x)$, $x \in [x_0 - t_0, x_0 + t_0]$, we linearlize (1.1) to (1.2). Here $q(x)$ is given. The regularity of F and the below boundedness of $\frac{\partial F}{\partial \xi_1}$ satisfy the assumption of Theorem 1. Assume that

$$u(p)(x_0,t) = u(q)(x_0,t) \text{ and } u_x(p)(x_0,t) = u_x(q)(x_0,t),\ t \in [0, t_0]$$

that is,

$$y(x_0,t) = y_x(x_0,t) = 0,\ t \in [0,t_0].$$

This means A is a linear operator because $g_0(x,t) = 0$. Therefore the fixed point of A is $g(x,t) = \left(f, y, y_t, y_x, y_{tt}, y_{xt}\right) = 0$. This implies the proof is completed. □

Proof of Theorem 3. Suppose that $f^1(x)$ and $f^2(x)$ are a couple of different solutions to the linear inverse problem in $C[x_0 - t_0, x_0 + t_0]$. Then according to Theorem 1, there is a interval $[x_1, x_2]$ such that

$$\begin{aligned}&x_2 = \sup\{x \mid f^1(\xi) = f^2(\xi),\quad x_0 \le \xi \le x \le x_0 + t_0\} \quad \text{and}\\ &x_1 = \inf\{x \mid f^1(\xi) = f^2(\xi),\quad x_0 - t_0 \le x \le \xi \le x_0\}.\end{aligned}$$

Since $f^1(x) \not\equiv f^2(x)$ on $[x_0 - t_0, x_0 + t_0]$, then at least $x_2 < x_0 + t_0$ or $x_1 > x_0 - t_0$. Without loss of generality, let $x_2 < x_0 + t_0$. Then $f^1(x)$ and $f^2(x)$ coincide on $[x_0, x_2]$. Notice that the function $y(x,t)$ is a solution to (1.2) with $f^1(x)$ or $f^2(x)$ in the domain $\Delta(x_0, t_0)$. Hence we obtain

$$y(x_2, t) = g_1(t) \quad \text{and} \quad y_x(x_2, t) = g_2(t), \quad t \in [0, x_0 + t_0 - x_2].$$

The uniqueness result of local version says that there is a $\tilde{h} \in (0, t_0]$ such that the linear inverse solution uniquely exists on $[x_2, x_2 + \tilde{h}]$, i.e., $f^1(x) = f^2(x)$ on the same interval. This fact contradicts the definition of x_2. The proof is completed. □

PROOF OF THEOREM 4. In the theorem 2, we proved $u(p)(x_0, t) = u(q)(x_0, t)$ and $u(p)_x(x_0, t) = u(q)_x(x_0, t)$ determine $f = 0$ on $[x_0 - h^*, x_0 + h^*]$. Moreover if we apply the argument of proof of theorem 3, then we obtain $f = 0$ on $[x_0 - t_0, x_0 + t_0]$. That is, the global uniqueness of p is proved. □

3. Application

As the first application, we consider the hyperbolic equation with a damping term.

$$(3.1) \qquad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = p(x)u_t(x,t), & (x,t) \in D, \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x), & -\infty < x < \infty. \end{cases}$$

Instantly, we can obtain the local existence of $p(x)$ in (3.1) from inverse data $f_1(t)$ and $f_2(t)$.

THEOREM 3.1. *Let $t_0 > 0$ and $x_0 \in \mathbb{R}$ be given. We assume*

$$\phi(x) \in C^2[x_0 - t_0, x_0 + t_0], \quad \psi(x) \in C^1[x_0 - t_0, x_0 + t_0] \quad \text{and}$$

$$|\psi(x)| \geq \alpha > 0, \quad x \in [x_0 - t_0, x_0 + t_0]$$

with some constant α. If the solution $u(p)(x,t)$ to (3.1) satisfies $u(p)(x_0, t) = f_1(t) \in C^2[0, t_0]$ and $u_x(x_0, t) = f_2(t) \in C^1[0, t_0]$, then there is $h^1 \in (0, t_0]$ such that $p(x)$ exists in $C[x_0 - h^1, x_0 + h^1]$.

PROOF. From the d'Alembert formula in the domain $\Delta(x_0, t_0)$ and assumptions, we obtain

$$u(x,t) = u_0(x,t) + \frac{1}{2} \iint\limits_{\Delta(x,t)} p(\xi) u_t(\xi,\tau) d\xi d\tau \qquad (x,t) \in \Delta(x_0, t_0),$$

where

$$u_0(x,t) = \frac{1}{2}\Big[\phi(x+t) - \phi(x-t)\Big] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi) d\xi.$$

Next we write only a closed integral system,

$$\begin{aligned}
p(x) &= \frac{\partial}{\partial t} u_0(x,t) + \frac{\operatorname{sign}(x_0 - x)}{\psi(x)} \int_{x_0}^{x} p(\xi) u_{tt}(\xi, |\xi - x|) d\xi, \\
u_t(x,t) &= \frac{\partial}{\partial} u_0(x,t) + \frac{1}{2}\int_{x-t}^{x+t} p(\xi) u_t(\xi, t - |\xi - x|) d\xi \quad \text{and} \\
u_{tt}(x,t) &= \frac{\partial^2}{\partial t^2} u_0(x,t) + \frac{1}{2}\Big\{p(x+t)u_t(x+t,0) + p(x-t)u_t(x-t,0)\Big\} \\
&\quad + \frac{1}{2}\int_{x-t}^{x+t} p(\xi) u_{tt}(\xi, t - |\xi - x|) d\xi,
\end{aligned}$$

where

$$p_0(x) = \frac{1}{\psi(x)} \left\{ \begin{array}{l} f_1{}''(|x - x_0|) + \operatorname{sign}(x - x_0) f_2{}'(|x - x_0|) \\ - \left(\dfrac{\partial^2}{\partial t^2} + \operatorname{sign}(x - x_0) \dfrac{\partial^2}{\partial x \partial t} \right) u_0(\xi, \tau) \Big|_{\substack{\xi = x_0 \\ \tau = |x - t_0|}} \end{array} \right\}.$$

The closed integral system above $p(x)$, $u_t(x,t)$ and $u_{tt}(x,t)$ determine an operator equation $g = Ag$ like as the proof of Theorem 1. To apply the principle of contracted mapping to the operator A on a suitable set, we can find $h^1 \in (0, t_0]$. □

REMARK 3.2. The proof of global uniqueness depends on the result of local existence. It is the reason that we state only Theorem 3.1.

The second application is the following

$$\text{(3.2)} \qquad \begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = F(p(x), u(x,t)), & (x,t) \in D, \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x), & -\infty < x < \infty. \end{cases}$$

If $F = F(\xi_1, \xi_2)$ is C^2 class, then we obtain (3.3) as the modification of (3.2).

$$(3.3)\quad \begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = r(x,t)y(x,t) + R(x,t)f(x), & (x,t) \in D, \\ y(x,0) = y_t(x,0) = 0, & -\infty < x < \infty. \end{cases}$$

In this case,

$$r(x,t) = \frac{\partial F}{\partial \xi_2}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q))\Big) \quad \text{and}$$
$$R(x,t) = \frac{\partial F}{\partial \xi_1}\Big(q + \theta(p-q), u(q) + \theta(u(p) - u(q))\Big),$$

where some $\theta \in (0,1)$. We notice that a relation between the nonlinear inverse problem (3.2) and the linear inverse problem (3.3) is the same as it between (1.1) and (1.2).

THEOREM 3.3. *Let $t_0 > 0$ and $x_0 \in \mathbb{R}$ be given. We assume*

$$\frac{\partial}{\partial t} r(x,t), \quad \frac{\partial}{\partial t} R(x,t) \in C(\Delta(x_0, t_0)) \quad \textit{and}$$
$$|R(x,0)| \geq \alpha > 0, \quad x \in C[x_0 - t_0, x_0 + t_0]$$

with some constant α. If the solution $y = y(f)(x,t)$ to (3.3) satisfies

$$y(f)(x_0,t) = f_1(t) \in C^2[0,t_0] \quad \textit{and} \quad y_x(f)(x_0,t) = f_2(t) \in C^1[0,t_0],$$

then there is $\bar{h} \in (0, t_0]$ such that $f(x)$ exists in $C[x_0 - \bar{h}, x_0 + \bar{h}]$.

REMARK 3.4. Theorem 3.3 can be proved along the same process in the proof of Theorem 1.

REMARK 3.5. As more concrete example of the nonlinear term $F(\xi_1, \xi_2, \xi_3, \xi_4)$, there are the following

$$p(x)\sin u(x,t)(\text{Sine-Gordon eq.}), p(x)e^{u(x,t)}, p(x)\left|u(x,t)^{\alpha-1}\right|u(x,t) \text{ etc.}$$

in many applied fields. Inverse problems to one dimensional hyperbolic equations containing the nonlinear terms above can be considered. From our results, we can establish local existence and global uniqueness of inverse solution $p(x)$ for the inverse problem from the same kind of datum.

References

[1] G. M. L. Gladwell, *Inverse problems in vibration*, Martin Nijhoff Publishers, Dordrecht, 1986.
[2] V. Isakov, *Inverse Problems for Partial Differential Equations*, Applied Mathematical Sciences Vol. 127, Springer, 1998.
[3] A. Khaĭdarov, *On stability estimates in multidimensional inverse problems for differential equations*, (English translation), Soviet Math. Dokl. **38** (1989), 614–617.
[4] M. V. Klibanov, *Inverse problems and Carleman estimates*, Inverse Problems **8** (1992), 575–596.
[5] M. M. Lavrent'ev, K. G. Reznitskaya and V. G. Yakhno, *One-dimensional inverse problems of mathematical physics*, Translations Vol. 130, AMS, Probidence, R.I., 1986.
[6] Rakesh and Symes, *Uniqueness for an inverse problem for the wave equation*, Commun. in Partial Differential Equations **13** (1988), 87–96.
[7] V. G. Romanov, *Integral Geometry and Inverse Problems for Hyperbolic Equations*, Springer-Verlag, Berlin, 1974.
[8] ———, *Inverse Problems of Mathematical Physics*, (English translation), VNU Science Press, Utrecht, The Netherlands, 1987.
[9] ———, *Local solvability of some multidimensional inverse problems for hyperbolic equations*, translation in Differential Equations **25** (1989), no. 2, 203–209.
[10] M. Yamamoto, *Uniqueness and stability in multidimensional hyperbolic inverse problems*, J. Math. Pures. Appl. **78** (1999), 65–98.

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