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DETERMINANT OF INCIDENCE MATRIX OF NIL-ALGEBRA

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ABSTRACT. The incidence matrices corresponding to a nil-algebra of finite index n can be used to determine the nilpotency. We find the smallest positive integer m such that the sum of the incidence matrices $\sum_P \langle n,\, m\rangle^P$ is invertible. In this paper, we give a different proof of the case that the nil-algebra of index 2 has nilpotency less than or equal to 4.

1. Introduction

Throughout this work, K is a field of characteristic 0. Let A be a K-algebra. If there exists $n \in \mathbb{N}$ such that $a^n = 0$ for all $a \in A$, then A is called a *nil-algebra* and the natural number n is called the *nil-index* of A. A is *nilpotent of index* m or A has *nilpotency* m if $A^m = 0$, but $A^{m-1} \neq 0$. Let $K\langle X\rangle = K\langle x_1,\, x_2,\, x_3,\, \dots\rangle$ be the polynomial ring over K in countably many non-commutative indeterminates x_1, x_2, x_3, ... $\mathcal{I}(f(x_1, x_2, x_3, \dots))$ is the T-ideal of $K\langle x_1,\, x_2,\, x_3,\, \dots\rangle$ generated by $f(x_1, x_2, x_3, \dots)$. The quotient ring $R = K\langle x_1, x_2, x_3, \dots\rangle/\mathcal{I}(x_1^n)$ is a relatively free algebra which is a nil ring of nil-index n. Nagata [4] proved that R is nilpotent, i.e., there exists a positive integer m such that $R^m = 0$. Furthermore, Higman [1] showed that $\frac{n^2}{e^2} \leq m \leq 2^n - 1$. Razmyslov [6] improved Higman's upper bound; he showed $m \leq n^2$. Kuzmin [2] showed that $\frac{n(n+1)}{2} \leq m$ and conjectured that the equality holds. Procesi [5] found that the index of nilpotence m of R is equal to the minimal degree of generating set of the ring of invariants of $n \times n$ generic matrices.

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THEOREM 1.1. [1, 4] (Nagata-Higman Theorem) If we set $\mathcal{I}_n = \mathcal{I}(x^n)$ for all $x \in K\langle X\rangle$, then there exists $m(n)$ such that $a_1a_2\cdots a_m \in \mathcal{I}_n$ for all a_1, a_2, $\ldots$ $a_m \in K\langle X\rangle$.

2. Incidence matrices

If a_1, a_2, $\ldots$, $a_n \in K\langle X\rangle$, we denote by $S_n(a_1, a_2, \ldots, a_n)$ or simply S_n,

$$S_n(a_1, a_2, \ldots, a_n) = \sum_{\sigma\in\mathrm{Sym}(n)} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)},$$

so called the *symmetric polynomial* of $a_1, a_2, \ldots, a_n$, where $\mathrm{Sym}(n)$ is the symmetric group on n letters. For a partition $P = (p_1, \ldots, p_n)$ of m with n parts, the incidence matrix denoted by $\langle n, m\rangle^P$ is constructed as the following. First of all, one labels the columns by the (multilinear) monomials of degree m lexicographically. In other words, the first column is labeled by $x_1\ x_2\ \cdots\ x_{m-2}\ x_{m-1}\ x_m$, the second by $x_1\ x_2\ \cdots\ x_{m-2}\ x_m\ x_{m-1}$ and so on. Thus the last column is indexed by $x_m\ x_{m-1}\ x_{m-2}\ \cdots\ x_2\ x_1$. We use 1, 2, $\cdots$ for x_1, x_2, $\cdots$ if there is no risk of confusion. Suppose that j-th column is indexed by $i_1^j \cdots i_m^j$ or simply $i_1 \cdots i_m$. Fix a partition $P = (p_1, \ldots, p_n)$ of m with n parts where $p_i \geq p_{i+1}$, $1 \leq i \leq n-1$. Then j-th row of the matrix corresponding to the partition $P = (p_1, \ldots, p_n)$ is labeled by

$$S_n(i_1\cdots i_{p_1}, i_{p_1+1}\cdots i_{p_1+p_2}, \ldots, i_{p_1+\cdots+p_{n-1}+1}\cdots i_{p_1+\cdots+p_n}),$$

and the matrix is called $\langle n, m\rangle^P$*-incidence matrix*. In the j-th row one places 1 for the columns labeled by the monomials that appear in that row index, and 0 elsewhere. In other words, if

$$\begin{aligned} &S_n(i_1\cdots i_{p_1}, i_{p_1+1}\cdots i_{p_1+p_2}, \ldots, i_{p_1+\cdots+p_{n-1}+1}\cdots i_{p_1+\cdots+p_n}) \\ &= i_1\cdots i_{p_1} i_{p_1+1}\cdots i_{p_1+p_2}\cdots i_{p_1+\cdots+p_{n-1}+1}\cdots i_{p_1+\cdots+p_n} \\ &\quad + \cdots + i_{p_1+\cdots+p_{n-1}+1}\cdots i_{p_1+\cdots+p_n}\cdots i_{p_1+1}\cdots i_{p_1+p_2} i_1\cdots i_{p_1}, \end{aligned}$$

then put 1 for the columns labeled by

$$i_1\cdots i_{p_1} i_{p_1+1}\cdots i_{p_1+p_2}\cdots i_{p_1+\cdots+p_{n-1}+1}\cdots i_{p_1+\cdots+p_n},$$

$$\ldots,$$

$$i_{p_1+\cdots+p_{n-1}+1}\cdots i_{p_1+\cdots+p_n}\cdots i_{p_1+1}\cdots i_{p_1+p_2} i_1\cdots i_{p_1}.$$

Now we can construct the incidence matrix for any n, m and a partition P of m with n parts.

3. Determinant of incidence matrices

The $\langle 2,3\rangle^{(2,1)}$-incidence matrix is in [3]. The $\langle 2,4\rangle^{(3,1)}$ and $\langle 2,4\rangle^{(2,2)}$-incidence matrices are listed below. The determinants of $\langle 2,4\rangle^{(3,1)}$ and $\langle 2,4\rangle^{(2,2)}$-incidence matrices are 0.

$$\langle 2,4\rangle^{(3,1)} = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}.$$

In other words, the matrices are not invertible. Thus we need both of them to show that Nagata-Higman holds for $n = 2$, $m = 4$.

$$\langle 2,4\rangle^{(2,2)} = \begin{pmatrix}
100000000000000010000000 \\
010000000000000000000010 \\
001000000010000000000000 \\
000100000000000000001000 \\
000010001000000000000000 \\
000001000000001000000000 \\
000000100000000001000000 \\
000000010000000000000001 \\
000010001000000000000000 \\
000000000100000000100000 \\
001000000010000000000000 \\
000000000001100000000000 \\
000000000001100000000000 \\
000000000000010000000100 \\
000001000000001000000000 \\
000000000000000100010000 \\
100000000000000010000000 \\
000000100000000001000000 \\
000000000100000000100000 \\
000000000000000100010000 \\
000100000000000000001000 \\
000000000000010000000100 \\
010000000000000000000010 \\
000000010000000000000001
\end{pmatrix}.$$

The sum of $\langle 2,4\rangle^{(3,1)}$ and $\langle 2,4\rangle^{(2,2)}$ does the work.

THEOREM 3.1. *The determinant of the sum of* $\langle 2,4\rangle^{(3,1)}$ *and* $\langle 2,4\rangle^{(2,2)}$ *is* $16777216 = 2^{24}$, *which means that the matrix is invertible.*

PROOF. The characteristic polynomial of the sum of $\langle 2,4\rangle^{(3,1)}$ and $\langle 2,4\rangle^{(2,2)}$ is

$$\lambda^{24} - 48\lambda^{23} + 1092\lambda^{22} - 15688\lambda^{21} + 159996\lambda^{20} - 1234608\lambda^{19}$$
$$+ 7501584\lambda^{18} - 36874176\lambda^{17} + 149476464\lambda^{16} - 506709888\lambda^{15}$$
$$+ 1451059392\lambda^{14} - 3535648896\lambda^{13} + 7364793664\lambda^{12} - 13147835136\lambda^{11}$$
$$+ 20124694272\lambda^{10} - 26364499968\lambda^{9} + 29437922304\lambda^{8} - 27817291776\lambda^{7}$$
$$+ 22006124544\lambda^{6} - 14343241728\lambda^{5} + 7521632256\lambda^{4} - 3059744768\lambda^{3}$$
$$+ 909115392\lambda^{2} - 176160768\lambda + 16777216.$$

Thus the eigenvalues are $1-i$, $1+i$, 2 and 4, each of multiplicity 6. Hence $x_1x_2x_3x_4 \in \mathcal{I}(x_1{}^2)$. Now we are able to express the monomial $x_1x_2x_3x_4$ explicitly in term of the sum of symmetric polynomials.

$$\begin{aligned}
16x_1x_2x_3x_4 &= 7\{S_2(x_1x_2x_3, x_4) + S_2(x_1x_2, x_3x_4)\} \\
&\quad + 3\{S_2(x_2x_3x_4, x_1) + S_2(x_2x_3, x_4x_1)\} \\
&\quad - \{S_2(x_3x_4x_1, x_2) + S_2(x_3x_4, x_1x_2)\} \\
&\quad - 5\{S_2(x_4x_1x_2, x_3) + S_2(x_4x_1, x_2x_3)\} \\
&= 7\{S_2(x_2x_3x_4, x_1) + S_2(x_1x_2, x_3x_4)\} \\
&\quad - 5\{S_2(x_3x_4x_1, x_2) + S_2(x_2x_3, x_4x_1)\} \\
&\quad - \{S_2(x_4x_1x_2, x_3) + S_2(x_3x_4, x_1x_2)\} \\
&\quad + 3\{S_2(x_1x_2x_3, x_4) + S_2(x_4x_1, x_2x_3)\}.
\end{aligned}$$

Therefore the nil-algebra of index 2 has nilpotency less than or equal to 4. □

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