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SMOOTHING ANALYSIS IN MULTIGRID METHOD FOR THE LINEAR ELASTICITY FOR MIXED FORMULATION

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Abstract. We introduce an assumption about smoothing operator for mixed formulations and show that convergence of Multigrid method for the mixed finite element formulation for the Linear Elasticity. And we show that Richardson and Kaczmarz smoothing satisfy this assumption.

1. Introduction and Preliminary

We consider Multigrid method for the pure displacement and pure traction problems in planar linear elasticity by using mixed formulation. The resulting algebraic linear operators by discretization of mixed formulation for the linear elasticity are not positive definite but nonsingular. So, we cannot use the Jacobi and Gauss-Seidel smoother but can use Richardson type smoother and Kaczmarz smoother for solving algebraic linear system.

Richardson type smoother is a very simple and convergence of Multigrid method with this was easily shown([1],[2],[10],[11], [12], [13],[14], [18]), but Multigrid method with this has a slow convergence. For the positive definite problem, authors show that the convergence of Multigrid method with various smoothing by using some assumptions concerning smoothing and show that Jacobi smoothing and Gauss-Seidel smoothing satisfy these assumptions([3], [4], [5], [6], [7], [8], [9], [19]). In [16], authors introduce weaker assumptions and show that convergence of Multigrid method. In this paper, we introduce an assumption concerning smoother and show that Multigrid algorithm converge under this assumption and Richardson and Kaczmarz smoother satisfy this assumption.

From here and after, a boldfaces is used to denote vector-valued functions, operators, and their associated spaces. Upper characters and Greece characters are used for matrix-valued functions and operators. We define

$$\mathbf{grad}p = \begin{pmatrix} \partial p/\partial x_1 \\ \partial p/\partial x_2 \end{pmatrix}, \quad \mathbf{curl}p = \begin{pmatrix} \partial p/\partial x_2 \\ -\partial p/\partial x_1 \end{pmatrix},$$

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$$
\begin{aligned}
&\mathbf{div}\tau = \begin{pmatrix} \partial\tau_{11}/\partial x_1 + \partial\tau_{12}/\partial x_2 \\ \partial\tau_{21}/\partial x_1 + \partial\tau_{22}/\partial x_2 \end{pmatrix}, \\
&\mathrm{div}\mathbf{v} = \partial v_1/\partial x_1 + \partial v_2/\partial x_2, \quad \mathrm{rot}\mathbf{v} = -\partial v_1/\partial x_2 + \partial v_2/\partial x_1, \\
&\mathbf{Gradv} = \begin{pmatrix} \partial v_1/\partial x_1 & \partial v_1/\partial x_2 \\ \partial v_2/\partial x_1 & \partial v_2/\partial x_2 \end{pmatrix}, \quad \mathbf{Curlv} = \begin{pmatrix} \partial v_1/\partial x_2 & -\partial v_1/\partial x_1 \\ \partial v_2/\partial x_2 & -\partial v_2/\partial x_1 \end{pmatrix}.
\end{aligned}
$$

We also define

$$
\delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \chi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{tr}(\tau) = \tau : \delta,
$$

where

$$
\tau : \delta = \sum_{i=1}^{2}\sum_{j=1}^{2} \delta_{ij}\tau_{ij}.
$$

Finally, let

$$
\varepsilon(\mathbf{v}) = \frac{1}{2}[\mathbf{Gradv} + (\mathbf{Gradv})^t] = \mathbf{Gradv} - \frac{1}{2}(\mathrm{rot}\mathbf{v})\chi.
$$

Let Ω be a bounded convex polygonal domain in $\mathbf{R}^2$. $\mathbf{u}$ denotes the displacement, $\mathbf{f}$ the body force, $\mathbf{g}$ the boundary traction, $\mu > 0$, $\lambda > 0$ the Lamé constants, and $\mathbf{n}$ is the outer normal. We assume that the Lamé constants (μ, λ) belong to the range $[\mu_1, \mu_2] \times [\lambda_0, \infty)$, where μ_1, μ_2, λ_0 are fixed positive constants.

Here we define two elasticity problems and give the well known properties concerning solution of the problems. The **pure displacement** boundary value problem for planar linear elasticity is given by

$$
\begin{aligned}
-\mathrm{div}\{2\mu\varepsilon(\mathbf{u}) + \lambda\mathrm{tr}(\varepsilon(\mathbf{u}))\delta\} &= \mathbf{f}, \quad \text{in } \Omega, \\
\mathbf{u} &= 0, \quad \text{on } \partial\Omega.
\end{aligned} \tag{1.1}
$$

It is well known ([13]) that, for $\mathbf{f} \in \mathbf{L}^2(\Omega)$, equation (1.1) has a unique solution $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)$. Moreover, there exists a positive constant C independent of μ and λ such that

$$
\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \lambda\|\mathrm{div}\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \le C\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}.
$$

The **pure traction** boundary value problem for planar linear elasticity is given by

$$
\begin{aligned}
-\mathbf{div}\left\{2\mu\varepsilon(\mathbf{u}) + \lambda\mathrm{tr}\left(\varepsilon(\mathbf{u})\right)\delta\right\} &= \mathbf{f}, \qquad \text{in } \Omega, \\
\left(2\mu\varepsilon(\mathbf{u}) + \lambda\mathrm{tr}\left(\varepsilon(\mathbf{u})\right)\delta\right)\mathbf{n} &= \mathbf{g}, \qquad \text{on } \partial\Omega.
\end{aligned} \tag{1.2}
$$

Since the domain Ω is a polygon which has corners, the boundary conditions (1.2) must be carefully handled. We shall denote by S_i, $1 \le i \le n$, the vertices of $\partial\Omega$; by $\Gamma_i, 1 \le i \le n$, be open line segment joining S_i to S_{i+1}; by $\mathbf{t}_i$ the positively oriented unit tangent along Γ_i; and by $\mathbf{n}_i$ the unit outer normal along Γ_i. Let $p \in H^{1/2}(\Gamma_i)$ and $q \in H^{1/2}(\Gamma_{i+1})$. We say that $p \equiv q$ at S_{i+1} if

$$
\int_0^t |q(s) - p(-s)|^2 \frac{ds}{s} < \infty,
$$

where s is the oriented arc length measured from S_{i+1} and t is a positive number less than $\min\{|\Gamma_i| : 1 \leq i \leq n\}$.

We are able to write equation (1.2) more precisely as

$$\begin{aligned} -\mathbf{div}\left\{2\mu\varepsilon(\mathbf{u}) + \lambda \mathrm{tr}\left(\varepsilon(\mathbf{u})\right)\delta\right\} &= \mathbf{f}, \qquad \text{in } \Omega, \\ \left(2\mu\varepsilon(\mathbf{u}) + \lambda \mathrm{tr}\left(\varepsilon(\mathbf{u})\right)\delta\right)\mathbf{n}|_{\Gamma_i} &= \mathbf{g}, \qquad 1 \leq i \leq n, \end{aligned} \tag{1.3}$$

where $\mathbf{f} \in \mathbf{L}^2(\Omega)$, and $\mathbf{g}_i \in \mathbf{H}^{1/2}(\Gamma_i)$ satisfy

$$\mathbf{g}_i \cdot \mathbf{n}_{i+1} \equiv \mathbf{g}_{i+1} \cdot \mathbf{n}_i \qquad \text{at } S_{i+1} \text{ for } 1 \leq i \leq n.$$

In order to exist a solution of (1.3), $\mathbf{f}$ and $\mathbf{g}_i$ must satisfy the compatibility condition

$$\int_\Omega \mathbf{f} \cdot \mathbf{v} dxdy + \sum_{i=1}^n \int_{\Gamma_i} \mathbf{g}_i \cdot \mathbf{n}_i ds = 0, \qquad \forall \mathbf{v} \in \mathbf{RM},$$

where $\mathbf{RM}$, the space of rigid motions, is defined by

$$\mathbf{RM} := \{\mathbf{v} : \mathbf{v} = (a + bx, c - by), \quad a, b, c, \in \mathbf{R}\}.$$

When this compatibility condition holds, the pure traction boundary value problem (1.3) has a unique solution ([14],[15]) $\mathbf{u} \in \mathbf{H}^2_\perp(\Omega)$ where

$$\mathbf{H}^k_\perp(\Omega) := \left\{\mathbf{u} \in \mathbf{H}^k(\Omega) : \int_\Omega \mathbf{u} \cdot \mathbf{v} dxdy = 0, \quad \forall \mathbf{v} \in \mathbf{RM}\right\}.$$

Moreover, there exists a positive constant C independent of μ and λ such that

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \lambda \|\mathrm{div}\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq C\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}.$$

Here, $\mathbf{H}^k(\Omega), k \geq 0$, denotes the usual L^2-based Sobolev spaces of vector-valued functions. The space $\mathbf{L}^2_\perp(\Omega)$ is interpreted as $\mathbf{H}^0_\perp(\Omega)$. Note that $|\mathbf{u}|_{\mathbf{H}^k(\Omega)}$ becomes a norm on $\mathbf{H}^k_\perp(\Omega)$.

In Section 2, we consider the mixed formulations of (1.1) and (1.3) and its finite discretizations. In Section 3, we consider Multigrid methods and its convergence analysis with smoothing assumption. In Section 4, we show that simple Richardson type smoother and Kaczmarz smoother satisfy the above assumption concerning smoother. We give numerical experiment of (1.1) in Section 5.

2. Mixed Formulations and its Finite Discretizations

First, we consider the pure displacement boundary value problem. The boundary value problem (1.1) can be written as

$$\begin{aligned} -\mu\Delta\mathbf{u} - (\mu + \lambda)\mathbf{grad}(\mathrm{div}\mathbf{u}) &= \mathbf{f}, \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0}, \quad \text{on } \partial\Omega. \end{aligned}$$

Hence, we have the following weak formulation:

Find $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$ such that

$$\mu \int_\Omega \mathbf{Grad}\mathbf{u} : \mathbf{Grad}\mathbf{v} dx + (\mu + \lambda) \int_\Omega (\mathrm{div}\mathbf{u})(\mathrm{div}\mathbf{v}) dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx, \tag{2.1}$$

for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$. Let $\gamma = \frac{\mu+\lambda}{\mu}$ and $p = \gamma \mathrm{div}\mathbf{u}$. It is clear that (2.1) is equivalent to the following mixed formulation:

Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \mathbf{Grad u} : \mathbf{Grad v} dx + \int_\Omega p \mathrm{div}\mathbf{v} dx &= \frac{1}{\mu}\int_\Omega \mathbf{f}\cdot\mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ \int_\Omega (\mathrm{div}\mathbf{u}) q - \frac{1}{\gamma}\int_\Omega pq dx &= 0, \qquad q \in L^2(\Omega). \end{aligned} \tag{2.2}$$

Equation (2.2) can be written concisely as

$$\mathcal{B}((\mathbf{u}, p), (\mathbf{v}, q)) = \frac{1}{\mu}\int_\Omega \mathbf{f}\cdot\mathbf{v} dx, \quad \forall (\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega), \tag{2.3}$$

where the symmetric bilinear form $\mathcal{B}(\cdot,\cdot) : \mathbf{H}_0^1(\Omega) \times L^2(\Omega) \to \mathbf{R}$ is defined by

$$\begin{aligned} &\mathcal{B}((\mathbf{v}_1, q_1), (\mathbf{v}_2, q_2)) \\ &\quad := \int_\Omega \left\{ \mathbf{Grad v}_1 : \mathbf{Grad v}_2 + q_1(\mathrm{div}\mathbf{v}_2) + (\mathrm{div}\mathbf{v}_1) q_2 - \frac{1}{\gamma} q_1 q_2 \right\} dx. \end{aligned}$$

It is clear from the definition of $\mathcal{B}$ that

$$\begin{aligned} &|\mathcal{B}((\mathbf{v}_1, q_1),(\mathbf{v}_2, q_2))| \\ &\qquad \le \sqrt{2}(|\mathbf{v}_1|_{\mathbf{H}^1(\Omega)} + \|q_1\|_{L^2(\Omega)})(|\mathbf{v}_2|_{\mathbf{H}^1(\Omega)} + \|q_2\|_{L^2(\Omega)}). \end{aligned}$$

Let $\mathcal{T}_k$ be a sequence of triangulations of Ω, where $\mathcal{T}_{k+1}$ is obtained by connecting the midpoints of the triangles in $\mathcal{T}_k$. We will denote $\max\{\mathrm{diam}T : T \in \mathcal{T}_k\}$ by h_k. Let

$$Q_k = \{q : q \in L^2(\Omega) \text{ and } q|T \text{ is a constant for all } T \in \mathcal{T}_k\}.$$

The nonconforming finite element spaces $\mathbf{V}_k$ are defined as follows.

$$\begin{aligned} \mathbf{V}_k = \{\mathbf{v} :& \mathbf{v} \in \mathbf{L}^2(\Omega), \mathbf{v}|_T \text{ is linear for all } T \in \mathcal{T}_k, \\ & \mathbf{v} \text{ is continuous at the midpoints} \\ & \text{ of interelement boundaries} \\ & \text{and } \mathbf{v} = \mathbf{0} \text{ at the midpoints of edges along } \partial\Omega.\}. \end{aligned}$$

The discretized problem for (2.3) is : Find $(\mathbf{u}_k, p_k) \in \mathbf{V}_k \times Q_k$ such that

$$\mathcal{B}_k((\mathbf{u}_k, p_k), (\mathbf{v}, q)) = \frac{1}{\mu}\int_\Omega \mathbf{f}\cdot\mathbf{v} dx, \quad \forall (\mathbf{v}, q) \in \mathbf{V}_k \times Q_k. \tag{2.4}$$

Here the symmetric bilinear form $\mathcal{B}_k(\cdot,\cdot) : (\mathbf{H}_0^1(\Omega) + \mathbf{V}_k) \times Q_k \to \mathbf{R}$ is defined by

$$\begin{aligned} &\mathcal{B}_k((\mathbf{v}_1, q_1), (\mathbf{v}_2, q_2)) \\ &\quad := \int_\Omega \left\{ \mathbf{Grad}_k \mathbf{v}_1 : \mathbf{Grad}_k \mathbf{v}_2 + q_1(\mathrm{div}_k\mathbf{v}_2) + (\mathrm{div}_k\mathbf{v}_1) q_2 - \frac{1}{\gamma} q_1 q_2 \right\} dx, \end{aligned}$$

where

$$(\mathbf{Grad}_k \mathbf{v})|_T = \mathbf{Grad}(\mathbf{v}|_T), \qquad (\mathrm{div}_k\mathbf{v})|_T = \mathrm{div}(\mathbf{v}|_T), \quad \forall T \in \mathcal{T}_k.$$

In [13], auther show that (2.4) is uniquely solvable and derive the following discretization error estimate:

$$\|\mathbf{u}-\mathbf{u}_k\|_{\mathbf{L}^2(\Omega)} + h_k\left(\|\mathbf{u}-\mathbf{u}_k\|_k + \|p-p_k\|_{L^2(\Omega)}\right) \leq Ch_k^2\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)},$$

where the nonconforming energy norm $\|\cdot\|_k$ on $\mathbf{H}_0^1(\Omega)+\mathbf{V}_k$ is defined by

$$\|\mathbf{v}\|_k := \|\mathbf{Grad}_k\mathbf{v}\|_{\mathbf{L}^2(\Omega)}.$$

Second, we consider the pure traction problem. Let $\gamma = \frac{\lambda}{2\mu}$ and $p = \gamma \text{div}\mathbf{u}$, we consider the mixed weak formulation for (1.3) as follows:

Find $(\mathbf{u}, p) \in \mathbf{H}_\perp^1(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} &\int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})dx + \int_\Omega p(\text{div}\mathbf{v})dx = \frac{1}{2\mu}\left[\int_\Omega \mathbf{f}\cdot\mathbf{v}dx + \sum_{i=1}^n \int_{\Gamma_i} \mathbf{g}_i\cdot\mathbf{v}|_{\Gamma_i}ds\right], \\ &\int_\Omega (\text{div}\mathbf{u})qdx - \frac{1}{\gamma}\int_\Omega pqdx = 0, \end{aligned} \tag{2.5}$$

for all $(\mathbf{v}, q) \in \mathbf{H}_\perp^1(\Omega) \times L^2(\Omega)$.

Replacing p and q by $\sqrt{\omega}p$ and $\sqrt{\omega}q(\omega \geq 1)$, respectively, we obtain the following formulation which is equivalent to (2.5):

Find $(\mathbf{u}, p) \in \mathbf{H}_\perp^1(\Omega) \times L^2(\Omega)$ such that

$$\mathcal{B}_\omega\left((\mathbf{u},p),(\mathbf{v},q)\right) = \frac{1}{2\mu}\left[\int_\Omega \mathbf{f}\cdot\mathbf{v}dx + \sum_{i=1}^n \int_{\Gamma_i} \mathbf{g}_i\cdot\mathbf{v}|_{\Gamma_i}ds\right] \tag{2.6}$$

for all $(\mathbf{v}, q) \in \mathbf{H}_\perp^1(\Omega) \times L^2(\Omega)$, where

$$\mathcal{B}_\omega\left((\mathbf{u},p),(\mathbf{v},q)\right) := \int_\Omega \left\{\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) + \sqrt{\omega}p(\text{div}\mathbf{v}) + \sqrt{\omega}(\text{div}\mathbf{u})q - \frac{\omega}{\gamma}pq\right\}dx.$$

The quantity ω is called the weighting factor. Equation (2.6) has a unique solution on $\mathbf{H}_\perp^1(\Omega) \times L^2(\Omega)$.

Let $\{\mathcal{T}^k\}$ be a family of triangulations of Ω, where $\mathcal{T}^{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}^k$. Let $h_k := \max_{T\in\mathcal{T}^k} \text{diam}T$, then $h_k = 2h_{k+1}$. Now let us define the conforming finite element spaces.

$$\begin{aligned} \mathbf{W}_k &:= \{\mathbf{u} : \mathbf{u}|_T \text{ is linear for all } T \in \mathcal{T}^k, \mathbf{u} \text{ is continuous on } \Omega\}, \\ \mathbf{W}_k^\perp &:= \left\{\mathbf{u} \in \mathbf{W}_k : \int_\Omega \mathbf{u}\cdot\mathbf{v}dxdy = 0, \quad \forall \mathbf{v} \in \mathbf{RM}\right\}. \end{aligned}$$

To describe the mixed finite element method, we define

$$Q_k := \{q : q \in L^2(\Omega) \text{ and } q|_T \text{ is a constant for all } T \in \mathcal{T}^k\}.$$

For each k, define the bilinear form $\mathcal{B}_{\omega,k}$ on $\mathbf{H}^1(\Omega) \times L^2(\Omega)$ by

$$\begin{aligned} &\mathcal{B}_{\omega,k}\left((\mathbf{u},p),(\mathbf{v},q)\right) \\ &:= \int_\Omega \left\{\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) + \sqrt{\omega}p(P_{k-1}\text{div}\mathbf{v}) + \sqrt{\omega}(P_{k-1}\text{div}\mathbf{u})q - \frac{\omega}{\gamma}pq\right\}dx, \end{aligned}$$

where P_{k-1} is the L^2-orthogonal projection onto Q_{k-1}. Note that the bilinear forms $\mathcal{B}_{\omega,k}$ are symmetric but indefinite.

The following discretization of (2.6) is a modification of one introduced by Falk in [15].

Find $(\mathbf{u}_k, p_k) \in \mathbf{W}_k^{\perp} \times Q_{k-1}$ such that

$$\mathcal{B}_{\omega,k}\left((\mathbf{u}_k, p_k), (\mathbf{v}, q)\right) = \frac{1}{2\mu}\left[\int_\Omega \mathbf{f}\cdot\mathbf{v}dx + \sum_{i=1}^n \int_{\Gamma_i} \mathbf{g}_i \cdot \mathbf{v}|_{\Gamma_i} ds\right] \tag{2.7}$$

for all $(\mathbf{v}, q) \in \mathbf{W}_k^{\perp} \times Q_{k-1}$.

In [17], author showed the uniqueness of the solution of the discretization (2.7) with $\omega = 1$ and derived the following discretization error estimate:

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_k\|_{L^2(\Omega)} &+ h_k\left(|\mathbf{u}-\mathbf{u}_k|_{H^1(\Omega)} + \|p-p_k\|_{L^2(\Omega)}\right)\\ &\le Ch_k^2\left\{\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^n \|\mathbf{g}_i\|_{\mathbf{H}^{1/2}(\Gamma_i)}\right\}.\end{aligned}$$

3. Multigrid algorithm and convergence analysis

We define the intergrid transfer operators and the mesh-dependent norms for each problems. Next, we present Multigrid algorithm MG with Kaczmarz or Richardson type smoother and prove the convergence of the algorithm at the same time. Some of them are rephrases of lemmas in [13] and [18] and we give these lemmas without proof.

For the pure displace problem, since the $\mathbf{V}_k$'s are nonconforming, the intergrid transfer operators are defined by averaging.

Let $J_{k-1}^k : \mathbf{V}_{k-1} \to \mathbf{V}_k$ be defined by

$$J_{k-1}^k(\mathbf{v})(m_e) = \begin{cases} \mathbf{v}(m_e), & \text{if } m_e \in \text{int}T \\ & \quad\text{for some } T \in \mathcal{T}_{k-1}, \\ \frac{1}{2}[\mathbf{v}|_{T_1}(m_e) + \mathbf{v}|_{T_2}(m_e)], & \text{if } e = T_1 \cap T_2 \\ & \quad\text{for some } T_1, T_2 \in \mathcal{T}_{k-1}, \end{cases}$$

at the midpoints m_e of internal edges e in $\mathcal{T}_k$.

The coarse-to-fine operator $I_{k-1}^k : \mathbf{V}_{k-1} \times Q_{k-1} \to \mathbf{V}_k \times Q_k$ is defined by

$$I_{k-1}^k(\mathbf{v}, q) = (J_{k-1}^k \mathbf{v}, q).$$

Define the mesh dependent inner product by

$$((\mathbf{u}, p), (\mathbf{v}, q))_k := (\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega)} + h_k^2 (p, q)_{L^2(\Omega)}.$$

The intergrid transfer operators $I_k^{k-1} : \mathbf{V}_k \times Q_k \to V_{k-1} \times Q_{k-1}$ is defined by

$$\left(I_k^{k-1}(\mathbf{u}, p), (\mathbf{v}, q)\right)_{k-1} = \left((\mathbf{u}, p), I_{k-1}^k(\mathbf{v}, q)\right)_k,$$

for all $(\mathbf{u}, p) \in \mathbf{V}_k \times Q_k$, and $(\mathbf{v}, q) \in \mathbf{V}_{k-1} \times Q_{k-1}$.

Let $\hat{Q}_k = \{q \in Q_k : \int_\Omega q dx = 0\}$.

The following three lemmas which concerned the pure displacement problems came from [13].

Lemma 3.1. *The following properties of* I_{k-1}^{k} *and* I_{k}^{k-1} *hold.*

(i) Given any $(\mathbf{v},q)\in\mathbf{V}_k\times Q_k$, $(\mathbf{v},q)\in\mathbf{V}_k\times\hat{Q}_k$ *only if* $((\mathbf{v},q),(\mathbf{0},1))_k=0$.

(ii) $\left(I_{k-1}^{k}(\mathbf{v},q),(\mathbf{0},1)\right)_k=\frac{1}{4}\left((\mathbf{v},q),(\mathbf{0},1)\right)_{k-1}$, *for all* $(\mathbf{v},q)\in\mathbf{V}_{k-1}\times Q_{k-1}$.

(iii) $I_{k-1}^{k}:\mathbf{V}_{k-1}\times\hat{Q}_{k-1}\to\mathbf{V}_k\times\hat{Q}_k$.

(iv) $I_{k}^{k-1}:\mathbf{V}_k\times\hat{Q}_k\to\mathbf{V}_{k-1}\times\hat{Q}_{k-1}$.

Let $B_k:\mathbf{V}_k\times Q_k\to\mathbf{V}_k\times Q_k$ be defined by

$$(B_k(\mathbf{u},p),(\mathbf{v},q))_k=\mathcal{B}_k\left((\mathbf{u},p),(\mathbf{v},q)\right),\quad\forall(\mathbf{u},p),(\mathbf{v},q)\in\mathbf{V}_k\times Q_k.$$

Lemma 3.2. $B_k:\mathbf{V}_k\times\hat{Q}_k\to\mathbf{V}_k\times\hat{Q}_k$.

Let $\hat{B}_k=B_k|_{\mathbf{V}_k\times\hat{Q}_k}$.

Lemma 3.3. *The spectral radius of* $\hat{B}_k\le Ch_k^{-2}$ *for* $k=1,2,\ldots$.

For the pure traction problem, because the $\mathbf{W}_k$ś are conforming, the intergrid transfer operators I_{k-1}^{k} are defined by the natural way.

Define the mesh dependent inner product by

$$((\mathbf{u},p),(\mathbf{v},q))_k:=(\mathbf{u},\mathbf{v})_{\mathbf{L}^2(\Omega)}+h_k^2(p,q)_{L^2(\Omega)}.$$

The intergrid transfer operators $I_k^{k-1}:\mathbf{W}_k\times Q_{k-1}\to\mathbf{W}_{k-1}\times Q_{k-2}$ is defined by

$$\left(I_k^{k-1}(\mathbf{u},p),(\mathbf{v},q)\right)_{k-1}=((\mathbf{u},p),(\mathbf{v},q))_k\,,$$

for all $(\mathbf{u},p)\in\mathbf{W}_k\times Q_{k-1}$, and $(\mathbf{v},q)\in\mathbf{W}_{k-1}\times Q_{k-2}$.

The following three lemmas which concerns the pure traction problems came from [18].

Lemma 3.4. *(i)* $\mathbf{RM}\subset\mathbf{W}_k,\qquad\forall k=1,2,\ldots$.

(ii) Given $(\mathbf{u},p)\in\mathbf{W}_k\times Q_{k-1}$,

$$(\mathbf{u},p)\in\mathbf{W}_k^{\perp}\times Q_{k-1}\Leftrightarrow((\mathbf{u},p),(\mathbf{v},0))_k\,,\quad\forall\mathbf{v}\in\mathbf{RM}.$$

(iii) $I_k^{k-1}:\mathbf{W}_k^{\perp}\times Q_{k-1}\to\mathbf{W}_{k-1}^{\perp}\times Q_{k-2}$.

Define $B_{k,\omega}:\mathbf{W}_k\times Q_{k-1}\to\mathbf{W}_k\times Q_{k-1}$ by

$$(B_{k,\omega}(\mathbf{u},p),(\mathbf{v},q))_k=\mathcal{B}_{k,\omega}\left((\mathbf{u},p),(\mathbf{v},q)\right),\quad\forall(\mathbf{u},p),(\mathbf{v},q)\in\mathbf{V}_k\times Q_{k-1}.$$

Lemma 3.5. $B_k:\mathbf{W}_k^{\perp}\times Q_{k-1}\to\mathbf{W}_k^{\perp}\times Q_{k-1}$.

Let $B_{k,\omega}^{\perp}=B_{k,\omega}|_{\mathbf{W}_k^{\perp}\times Q_{k-1}}$.

Lemma 3.6. *The spectral radius of* $B_{k,\omega}^{\perp}\le Ch_k^{-2}$ *for* $k=1,2,\ldots$.

Here and after in this chapter, we only consider the case of the pure displacement problems. So we only use $\mathbf{V}_k$, $\hat{Q}_k$ and $\hat{B}_k$, but the following lemmas and theorem are satisfied the case of pure traction problems by replacing $\mathbf{V}_k$ as $\mathbf{W}_k^{\perp}$, $\hat{Q}_k$ as Q_{k-1} and $\hat{B}_k$ as $B_k^{\perp}$.

The mesh-dependent norms on $\mathbf{V}_k \times \hat{Q}_k$ are defined as follows:

$$|||(\mathbf{u},p)|||_{s,k} := \sqrt{\left((\hat{B_k}^2)^{s/2}(\mathbf{u},p),(\mathbf{u},p)\right)_k}, \qquad \forall(\mathbf{u},p) \in \mathbf{V}_k \times \hat{Q}_k.$$

This norm is well-defined. Moreover, for all $(\mathbf{u},p),(\mathbf{v},q) \in \mathbf{V}_k \times \hat{Q}_k$,

$$|||(\mathbf{u},p)|||_{0,k} = \sqrt{\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + h_k^2\|p\|^2_{L^2(\Omega)}},$$
$$|\mathcal{B}_k\left((\mathbf{u},p),(\mathbf{v},q)\right)| \le |||(\mathbf{u},p)|||_{2,k}|||(\mathbf{v},q)|||_{0,k},$$

and

$$|||(\mathbf{u},p)|||_{2,k} = \sup_{(\mathbf{v},q)\in\mathbf{V}_k\times\hat{Q}_k-(\mathbf{0},0)} \frac{|\mathcal{B}_k\left((\mathbf{u},p),(\mathbf{v},q)\right)|}{|||(\mathbf{v},q)|||_{0,k}}.$$

Define $P_k^{k-1} : \mathbf{V}_k \times \hat{Q}_k \to \mathbf{V}_{k-1} \times \hat{Q}_{k-1}$ by

$$\mathcal{B}_{k-1}\left(P_k^{k-1}(\mathbf{u},p),(\mathbf{v},q)\right) = \mathcal{B}_k\left((\mathbf{u},p), I_{k-1}^k(\mathbf{v},q)\right),$$

for all $(\mathbf{u},p) \in \mathbf{V}_k \times \hat{Q}_k$ and $(\mathbf{v},q) \in \mathbf{V}_{k-1} \times \hat{Q}_{k-1}$.

We are now ready to state the basic Lemmas which are essential in the approximation property of the multigrid algorithm.

Lemma 3.7. *Given $\mathbf{w} \in \mathbf{L}^2(\Omega)$, let $(\mathbf{u}_k,p_k) \in \mathbf{V}_k \times \hat{Q}_k$ be the solution of*

$$\mathcal{B}_k\left((\mathbf{u}_k,p_k),(\mathbf{v},q)\right) = \int_\Omega \mathbf{w}\cdot\mathbf{v}dx, \quad \forall(\mathbf{v},q) \in \mathbf{V}_k \times \hat{Q}_k$$

and $(\mathbf{u}_{k-1},p_{k-1}) \in \mathbf{V}_{k-1} \times \hat{Q}_{k-1}$ be the solution of

$$\mathcal{B}_{k-1}\left((\mathbf{u}_{k-1},p_{k-1}),(\mathbf{v},q)\right) = \int_\Omega \mathbf{w}\cdot\mathbf{v}dx, \quad \forall(\mathbf{v},q) \in \mathbf{V}_{k-1} \times \hat{Q}_{k-1}.$$

Then

$$|||P_k^{k-1}(\mathbf{u}_k,p_k) - (\mathbf{u}_{k-1},p_{k-1})|||_{0,k-1} \le Ch_k^2\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}.$$

Lemma 3.8. *Given $w \in L^2(\Omega)$, let $(\mathbf{u}_k,p_k) \in \mathbf{V}_k \times \hat{Q}_k$ be the solution of*

$$\mathcal{B}_k\left((\mathbf{u}_k,p_k),(\mathbf{v},q)\right) = \int_\Omega wqdx, \quad \forall(\mathbf{v},q) \in \mathbf{V}_k \times \hat{Q}_k$$

and $(\mathbf{u}_{k-1},p_{k-1}) \in \mathbf{V}_{k-1} \times \hat{Q}_{k-1}$ be the solution of

$$\mathcal{B}_{k-1}\left((\mathbf{u}_{k-1},p_{k-1}),(\mathbf{v},q)\right) = \int_\Omega wqdx, \quad \forall(\mathbf{v},q) \in \mathbf{V}_{k-1} \times \hat{Q}_{k-1}.$$

Then

$$|||(\mathbf{u}_k,p_k) - I_{k-1}^k(\mathbf{u}_{k-1},p_{k-1})|||_{0,k} \le Ch_k\|w\|_{L^2(\Omega)}.$$

Finally, to define the k-th level Multigrid algorithm MG_k, we need linear smoothing operators $R_k : \mathbf{V}_k \times \hat{Q}_k \to \mathbf{V}_k \times \hat{Q}_k$ for all k. For the analysis of Multigrid algorithms, we assume the following condition concerning the smoothing operators. To describe this, we first define $K_k = I - R_k\hat{B}_k$.

Smoothing assumption(SM). There exists a constant C, independent of h_k and m, such that

$$|||K_k^m(\mathbf{u},p)|||_{2,k} \le Ch_k^{-2}\frac{1}{\sqrt{m}}|||(\mathbf{u},p)|||_{0,k}, \quad \forall(\mathbf{u},p)\in \mathbf{V}_k\times\hat{Q}_k. \tag{3.1}$$

Here, we only consider the W-cycle nonsymmetric multigrid algorithm.

The k-th level itration scheme of Multigrid algorithm $\mathbf{MG}_k$: For $k=1$, $\mathrm{MG}_k((\mathbf{y}_0,s_0),(\mathbf{w},r))$ is the solution obtained from a direct method, i.e.,

$$\mathrm{MG}_1((\mathbf{y}_0,s_0),(\mathbf{w},r)) = \left(\hat{B}_1\right)^{-1}(\mathbf{w},r).$$

We assume that MG_{k-1} is defined. The k-th level iteration with initial guess $(\mathbf{y}_0,s_0)\in \mathbf{V}_k\times\hat{Q}_k$ yields
$\mathrm{MG}_k((\mathbf{y}_0,s_0),(\mathbf{w},r))$ as a approximate solution to the following problem.

Find $(\mathbf{y},s)\in\mathbf{V}_k\times\hat{Q}_k$ such that

$$\hat{B}_k(\mathbf{y},s) = (\mathbf{w},r), \text{ where } (\mathbf{w},r)\in\mathbf{V}_k\times\hat{Q}_k.$$

Smoothing Step : The approximation $(\mathbf{y}_m,s_m)\in\mathbf{V}_k\times\hat{Q}_k$ is constructed recursively from the initial guess $(\mathbf{y}_0,s_0)$ and the equations

$$(\mathbf{y}_l,s_l) = (\mathbf{y}_{l-1},s_{l-1}) + R_kB_k\left((\mathbf{w},r) - B_k(\mathbf{y}_{l-1},s_{l-1})\right), \quad 1\le l\le m.$$

Correction Step : The coarse-grid correction in $\mathbf{V}_{k-1}\times\hat{Q}_{k-1}$ is obtained by applying the $(k-1)$-th level iteration twice. More precisely,

$$\begin{aligned}(\mathbf{v}_0,q_0) &= (\mathbf{0},0) \quad \text{and}\\ (\mathbf{v}_i,q_i) &= \mathrm{MG}_{k-1}((\mathbf{v}_{i-1},q_{i-1},(\bar{\mathbf{w}},\bar{r})), \quad i=1,2\end{aligned}$$

where $(\bar{\mathbf{w}},\bar{r})\in\mathbf{V}_{k-1}\times\hat{Q}_{k-1}$ is defined by $(\bar{\mathbf{w}},\bar{r}) := I_k^{k-1}\left((\mathbf{w},r) - B_k(\mathbf{y}_m,s_m)\right)$.

Then

$$\mathrm{MG}_k((\mathbf{y}_0,s_0),(\mathbf{w},r)) = (\mathbf{y}_m,s_m) + (\mathbf{v}_2,q_2).$$

Let the final output of the two-grid algorithm be

$$(\mathbf{y}^\sharp,s^\sharp) := (\mathbf{y}_m,s_m) + (\mathbf{v}^\sharp,q^\sharp)$$

where

$$\begin{aligned}(\mathbf{v}^\sharp,q^\sharp) &= \left(\widehat{B_{k-1}}\right)^{-1}(\bar{\mathbf{w}},\bar{r})\\ &= \left(\hat{B}_{k-1}\right)^{-1} I_k^{k-1}\left((\mathbf{w},r) - B_k(\mathbf{y}_m,s_m)\right)\\ &= \left(\hat{B}_{k-1}\right)^{-1} I_k^{k-1}B_k(\mathbf{y}-\mathbf{y}_m,s-s_m).\end{aligned}$$

The following two lemmas are found in [13].

Lemma 3.9.

$$(\mathbf{v}^\sharp,q^\sharp) = P_k^{k-1}(\mathbf{y}-\mathbf{y}_m,s-s_m).$$

From the definitions, we have

$$(\mathbf{y}-\mathbf{y}_m, s-s_m) = K_k^m(\mathbf{y}-\mathbf{y}_0, s-s_0),$$
$$(\mathbf{y}-\mathbf{y}^\sharp, s-s^\sharp) = (I-P_k^{k-1})K_k^m(\mathbf{y}-\mathbf{y}_0, s-s_0).$$

Lemma 3.10. *There exists a constant C, independent of h_k and m, such that*

$$|||(I-P_k^{k-1})(\mathbf{u},p)|||_{0,k} \le Ch_k^2|||(\mathbf{u},p)|||_{2,k}, \quad \forall(\mathbf{u},p)\in \mathbf{V}_k\times\hat{Q}_k. \tag{3.2}$$

Theorem 3.11. *[Convergence of the Two-Grid Algorithm] There exists a constant C, independent of k and m, such that*

$$|||(\mathbf{y}-\mathbf{y}^\sharp, s-s^\sharp)|||_{0,k} \le \frac{C}{\sqrt{m}}|||(\mathbf{y}-\mathbf{y}_0, s-s_0)|||_{0,k}.$$

Proof. From the definition, (3.2), and (3.1), we get

$$\begin{aligned}|||(\mathbf{y}-\mathbf{y}^\sharp, s-s^\sharp)|||_{0,k} &= |||(I-P_k^{k-1})K_k^m(\mathbf{y}-\mathbf{y}_0, s-s_0)|||_{0,k}\\ &\le Ch_k^2|||K_k^m(\mathbf{y}-\mathbf{y}_0, s-s_0)|||_{2,k}\\ &\le \frac{C}{\sqrt{m}}|||(\mathbf{y}-\mathbf{y}_0, s-s_0)|||_{0,k}.\end{aligned}$$

Theorem 3.12. *[Convergence of the k-th level Iteration] There exists a constant C, independent of k and m, such that*

$$|||(\mathbf{y},s)-MG_k((\mathbf{y}_0,s_0),(\mathbf{w},r))|||_{0,k} \le \frac{C}{\sqrt{m}}|||(\mathbf{y}-\mathbf{y}_0, s-s_0)|||_{0,k}.$$

4. Verification of the Smothing assuption

In this section, we consider two smoothing, one is a Richardson type smoothing for nonsymmetric or indefinite operator and other is a Kaczmarz smoothing.

The Richardson type smoothing is defined by

$$R_k := \frac{1}{\Lambda_k^2}\hat{B}_k^{\,2},$$

where Λ_k^2 be the largest eigenvalue of $\hat{B}_k^2$. This smoothing are considered in [13] and [18].

Lemma 4.1. *The above R_k satisfy Smoothing assumption (SM).*

Now, we consider the Kaczmarz smoothing. Let $\hat{B}_k = (b_{ij})_{i,j=1}^l$. Then Kaczmarz smoother is defined by the following algorithm.

Kaczmarz Algorithm. Let $(\mathbf{w},r)\in\mathbf{V}_k\times\hat{Q}_k$. We define $R_k(\mathbf{w},r)\in\mathbf{V}_k\times\hat{Q}_k$ as follows:

(i) Set $(\phi_0,\zeta_0)=(\mathbf{0},0)$.

(ii) Define (ϕ_i,ζ_i) for $i=1,\ldots,l$ by

$$(\phi_i,\zeta_i) = (\phi_{i-1},\zeta_{i-1}) - \frac{b_i}{b_i^t b_i}(b_i^t(\phi_{i-1},\zeta_{i-1}) - (\mathbf{w},r))$$

where $b_i^t = i^{\text{th}}$ row of $B_k^\perp$, i.e., $b_i^t = (b_{i1}, b_{i2}, \ldots, b_{il})$.

(iii) Set $R_k(\mathbf{w}, r) = (\phi_l, \zeta_l)$.

From the above algorithm, we obtain $K_k = I - \hat{B}_k(D+L)^{-1}\hat{B}_k$ where $B_k B_k^T = D + L + L^T$, L is a strictly lower triangular matrix, and D is a diagonal matrix.

The following theorem are in [6].

Theorem 4.2. *Let A_k be a sparse symmetric positive definite operator from $\mathcal{M}_k$ to $\mathcal{M}_k$ and let $A_k = l + d + l^t$ where l is a lower triangular part and d is a diagonal part of A. Let $R_k = (d+l)^{-1}$. Then R_k satisfy the following property: There is a constant C_R which does not depend on k such that*

$$\frac{\|u\|_k^2}{\lambda_k} \leq C_R(\bar{R}_k u, u)_k, \quad \text{for all } u \in \mathcal{M}_k.$$

Here, $\bar{R}_k$ is either $(I - K_k^* K_k)A_k^{-1}$ or $(I - K_k K_k^*)A_k^*$ and $K_k = I - R_k A_k$. λ_k is the largest eigenvalue of A_k.

Lemma 4.3. *The Kaczmarz smoother is satisfied the smoothing assumption (SM).*

Proof. Let $A_k = (\hat{B}_k)^2$ and $\mathcal{M}_k = \mathbf{V}_k \times \hat{Q}_k$ in Theorem 4.2, then we have

$$\begin{aligned}
&\frac{\|(\mathbf{u}, p)\|_k^2}{\Lambda_k^2} \leq C_R \Big[((B_k^{\perp})^2(\mathbf{u}, p), (\mathbf{u}, p)) \\
&-((\hat{B}_k)^2(I - (D+L)^{-2}(\hat{B}_k)^2)(\mathbf{u}, p), (I - (D+L)^{-2}(\hat{B}_k)^2)(\mathbf{u}, p))\Big],
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{|||\hat{B}_k(\mathbf{u}, p)|||_{2,k}^2}{\Lambda_k^2} \leq \Big[(\hat{B}_k(\mathbf{u}, p), \hat{B}_k(\mathbf{u}, p)) \\
&-((I - \hat{B}_k(D+L)^{-2}\hat{B}_k)\hat{B}_k(\mathbf{u}, p), (I - \hat{B}_k(D+L)^{-2}\hat{B}_k)\hat{B}_k(\mathbf{u}, p)\Big] \\
&= \Big((\hat{B}_k(\mathbf{u}, p), \hat{B}_k(\mathbf{u}, p)) - (K_k\hat{B}_k(\mathbf{u}, p), K_k\hat{B}_k(\mathbf{u}, p))\Big).
\end{aligned}$$

In above, let $\hat{B}_k(\mathbf{u}, p) = (\mathbf{v}, q)$, then we have

$$\begin{aligned}
\frac{|||(\mathbf{v}, q)|||_{2,k}^2}{\Lambda_k^2} &\leq C_R\left(((\mathbf{v}, q), (\mathbf{v}, q)) - (K_k(\mathbf{v}, q), K_k\mathbf{v}, q))\right) \\
&= C_R((I - K_k^t K_k)(\mathbf{v}, q), (\mathbf{v}, q)).
\end{aligned} \tag{4.1}$$

In (4.1), we let $(\mathbf{v}, q) = K_k^m(\mathbf{w}, s)$, then we have

$$\begin{aligned}
|||K_k^m(\mathbf{w}, s)|||_{2,k}^2 &\le C_R \Lambda_k^2((I - K_k^t K_k) K_k^m(\mathbf{w}, s), K_k^m(\mathbf{w}, s)) \\
&= C_R \Lambda_k^2((I - K_k^t K_k)(K_k^t K_k)^m(\mathbf{w}, s), (\mathbf{w}, s)) \\
&\le C_R \Lambda_k^2 \frac{1}{m} \sum_{i=0}^{m-1}((I - K_k^t K_k)(K_k^t K_k)^i(\mathbf{w}, s), (\mathbf{w}, s)) \\
&= C_R \Lambda_k^2 \frac{1}{m}((I - (K_k^t K_k)^m)((\mathbf{w}, s), (\mathbf{w}, s)) \\
&\le C_R \Lambda_k^2 \frac{1}{m} |||(\mathbf{w}, s)|||_{0,k}^2
\end{aligned}$$

because spectral radius of K_k is less than 1 and $\Lambda_k = Ch_k^{-2}$.

5. Numerical experiments

Multigrid algorithm described in Section 4 was applied to the pure displacement boundary value problem (1.1) with $\mu = 1$. The domain Ω is the unit square.

In Table I and II, $\nu = \lambda/(2(1+\lambda))$ is the Poisson ratio, h represents the lengths of the horizontal and vertical sides of the triangles in the triangulation, the numbers n represent Multigrid iterations required to achieved an L^2 relative error of less than 1% in the displacements. In the first row, smoothing number represent the number of smoothing steps in Multigrid algorithm. Table I represent the number of Multigrid iterations with Kaczmarz smoother and Table II represent the number Multigrid iterations with Richardson smoother.

The results clearly illustrate that the number of Multigrid iterations is independent of the Poisson ratio and Multigrid algorithm with Kaczmarz smoother is slightly better than Multigrid algorithm with Richardson smoother.

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Table I. Multigrid iterations with Kaczmarz smoother.

smoothing number		1	2	3	4
ν	h	n	n	n	n
0	$(0.5)^4$	38	34	31	30
	$(0.5)^5$	74	67	62	60
0.45	$(0.5)^4$	25	23	22	21
	$(0.5)^5$	49	46	43	41
0.495	$(0.5)^4$	22	21	21	20
	$(0.5)^5$	43	43	42	40
0.4995	$(0.5)^4$	18	18	18	18
	$(0.5)^5$	40	41	40	38

Table II. Multigrid iterations with Richardson smoother.

smoothing number		1	2	3	4
ν	h	n	n	n	n
0	$(0.5)^4$	63	49	44	42
	$(0.5)^5$	125	98	89	85
0.45	$(0.5)^4$	71	51	42	36
	$(0.5)^5$	136	99	84	75
0.495	$(0.5)^4$	67	44	35	38
	$(0.5)^5$	125	81	65	58
0.4995	$(0.5)^4$	95	66	54	48
	$(0.5)^5$	122	78	63	57

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