

## BOUNDARY ELEMENT TECHNIQUE IN TORSION PROBLEMS OF BEAMS WITH MULTIPLY CONNECTED CROSS-SECTIONS

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ABSTRACT. This paper shows how boundary element method can be used to calculate torsion geometrical stiffness of cross-sections of various beams and airfoil profiles. Using the BEM direct formulation, the technique for determining bending and torsional geometrical characteristics of arbitrary multiply connected cross-sections is presented. The application limits of several well-known formulae on some test problems have been demonstrated and discussed.

Beam theory is widely applied to dynamics and strength problems of various structures. Being developed for calculation of naturally pre-twisted beams with non-symmetric alternating cross-section [1, 2], it has taken certain place in analysis of turbine and compressor blades of gas turbine engines. Furthermore, an important phase in preparing the initial data for calculation is to determine geometrical characteristics of blade cross-sections, since the obtained results validity depends on accuracy of these characteristics calculation. Separate problem is to determine geometrical torsion stiffness. At present, there are several methods of determining this geometrical characteristic of the elastic homogeneous beams with arbitrary cross-section.

First of all, a whole series of analogies between the elastic beam torsion problems and hydrodynamics, electrostatics and some other mechanics problems given by Greenhill, Heaveside, Prandtl et al [3] should be named. Having opened new applications of direct methods, particularly variational methods, in numerous problems of prismatic beam torsion, the membrane analogy proposed by Prandtl has played a significant role.

Of all experimental papers, the researches conducted by Weber, Schmieden and Föppl [3, 4, 5] are worth of special attention. In these investigations the sum of rigidities of all rectangles composing the profile under consideration is taken as an initial expression for the determination of the profile torsion stiffness. Then, using experiments, either a correction factor or an additional summand dependant on the profile dimensions are determined and added to the initial value of stiffness.

A significant role in solving problems of prismatic beam torsion is played by the variational methods of the elasticity theory. Among them, the methods of Kantorovich, Trefftz, Ritz and Galerkin are applied most widely. Ref [3] deserves special attention of all papers on this subject. In this paper, to determine the stress function in the torsion problem for beams with polygonal cross-section, in particular for rolled sections (channel, I-beam, etc.), the authors have used the method, which essence is in reducing the

problem of stress function definition to solving the completely regular infinite systems of linear equations by means of ancillary functions expanding in series, with structure determined by the homogeneous boundary conditions. One can also note Ref [6] where the authors have applied the variational method with  $R$ -functions as approximation functions to solve the torsion problems.

Some substantial results for beams with cross-sections bounded by functions with piecewise continuous derivatives to  $n$ -th order inclusively have been obtained by conformal map method. The theory of complex variable functions has found application for solving the problems of elastic beam torsion in papers of Muskhelishvili [7]. He has shown that the problem of elastic beam torsion can be solved if the function mapping the cross-section domain into the circle in case of continuous beam, and in case of hollow beam – into the circular ring, is known. Development of this method with illustrations of many particular problems is given by Weber and Günter [8].

To estimate the torsion stiffness the approximate relationships are usually applied, among them the Vlasov's formula [1] and the formulae of Bredt and Griffiths-Prescott for thin-walled section [3] are worth of most attention. The finite difference method, the relaxation method, which is a certain derivation of the Zeidel's iterative scheme, and the finite element method (FEM) are the most widely applied of all numerical methods whose results are sufficiently accurate for the practical use. For instance, Ref [9] contains the FEM solution of the torsion problem for the beams with hollow square cross-section. One should also note Ref [10] where errors of geometrical torsion stiffness calculation by various approximate formulae are estimated versus specific blade geometrical parameters, and Ref [11] where the extent of influence of the internal canal form in cooled blades on their geometrical characteristics is estimated. For more detailed review of literature dedicated to elastic beam torsion problems see Ref [3].

Present paper has an aim to demonstrate the capacities of the boundary element method (BEM) as an alternative to the mentioned above techniques of determining the geometrical torsion stiffness. This approach is implemented in the designed by authors automatic system of geometrical characteristics determining for the arbitrary one- and multiply connected cross-sections [12].

As it is known, the prismatic beam torsion problem is reduced to the problem of the equation integration [13]

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0, \quad \frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} = -2G\Theta, \quad (1)$$

where  $\tau_{xz}$  and  $\tau_{yz}$  are the shear stresses in the beam section (Fig. 1),  $\Theta$  is the angle of twist per length, it is constant for all fibers parallel to the axis  $Z$  of the twisted beam,  $G$  is the shear modulus of the beam material.

Assume the solution of (1) in the form

$$\tau_{xz} = G\Theta \frac{\partial \Phi}{\partial y}, \quad \tau_{yz} = -G\Theta \frac{\partial \Phi}{\partial x}, \quad (2)$$

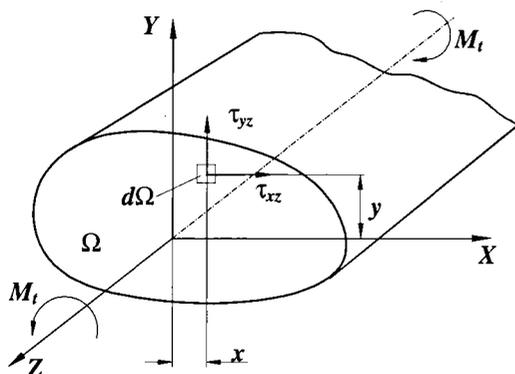


FIGURE 1. Boundary conditions on the beam edge surface.

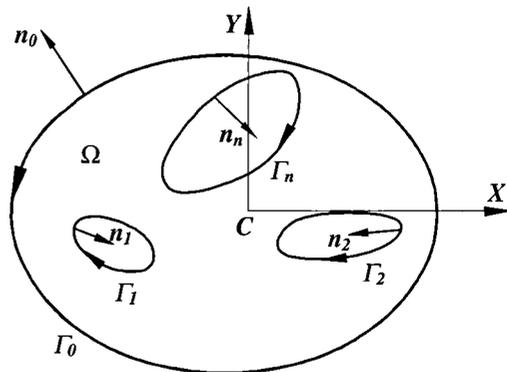


FIGURE 2. The go round rule of the multiply connected cross-section.

where  $\Phi(x, y)$  is to be defined function of coordinates  $x$  and  $y$ , it is called the stress torsion function or the Prandtl's function. In the cross-section domain of the twisted beam, this function must satisfy the Poisson's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \nabla^2 \Phi(x, y) = -2. \quad (3)$$

For the one connected domain the stress function must satisfy the following boundary condition

$$\Phi(x, y) = 0 \quad \forall x, y \in \Gamma_0. \quad (4)$$

The relationship for the torsion moment in the arbitrary multiply connected cross-section may be presented in the following way

$$M_t = G\Theta J_t, \quad (5)$$

where

$$J_t = 2 \int_{\Omega} \Phi(x, y) d\Omega. \quad (6)$$

Value  $J_t$  is called the geometrical torsion stiffness of the homogeneous beam, and as well as the stress function  $\Phi(x, y)$  it is the function of the beam cross-section coordinates.

For the multiply connected cross-section (Fig. 2) the stress function  $\Phi(x, y)$  keeps values  $C_0, C_1, C_2, C_3, \dots, C_n$  constant at each of the contours  $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \dots, \Gamma_n$ . Meanwhile, only one of those constants can be chosen arbitrary ( $C_0 = 0$ ) and then

$$\Phi = 0 \quad \forall x, y \in \Gamma_0, \quad \Phi = C_k \quad \forall x, y \in \Gamma_k, \quad \text{where } k = \overline{1, n}. \quad (7)$$

Let  $\Phi_0(x, y)$  denote the solution of the differential equation (3) in the domain  $\Omega$  with the boundary conditions  $\Phi_0 = 0 \quad \forall x, y \in \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \dots, \Gamma_n$ . Let also  $\Phi_k(x, y)$

denote functions that are harmonic in the domain  $\Omega$ , i.e.

$$\nabla^2 \Phi_k = 0, \quad (8)$$

and satisfy the boundary conditions  $\Phi_k = 0 \quad \forall x, y \in \Gamma_0, \dots, \Gamma_{k-1}, \Gamma_{k+1}, \dots, \Gamma_n$ ;  
 $\Phi_k = 1 \quad \forall x, y \in \Gamma_k$ .

Then, the stress function may be presented in the form [14]

$$\Phi(x, y) = \Phi_0(x, y) + \sum_{k=1}^n C_k \Phi_k(x, y). \quad (9)$$

The unknown constants  $C_k$  are determined by the Bredt's theorem about shear stress circulation [3]

$$\oint_{\Gamma_i} \frac{\partial \Phi}{\partial n} ds = 2F_i, \quad (10)$$

here  $F_i$  is the area of the  $i$ -th contour.

Using condition (10) for every internal contour, we obtain the system of linear equations that in the matrix form can be represented in the following way

$$[A]\{C\} = \{Y\}, \quad (11)$$

where  $A_{ij} = \oint_{\Gamma_i} \frac{\partial \Phi_j}{\partial n} ds$ ,  $Y_i = 2F_i - \oint_{\Gamma_i} \frac{\partial \Phi_0}{\partial n} ds$ .

For the multiply connected cross-section, formula for  $J_t$  has a little more complicated form

$$J_t = 2 \int_{\Omega} \Phi(x, y) d\Omega + 2 \sum_{i=1}^n C_i F_i. \quad (12)$$

Let us find the solution of (3) and (8) using the boundary element method (BEM). The essence of this method is in reducing the partial differential equation that describes the unknown function behavior within the domain and on its boundary to the integral equation that defines only the boundary values, the next step being to take the numerical solution of this equation. Therefore, it is possible to use less unknown values and the problem order decreases by one, because all approximations due to the numerical calculation are related only to the boundary, and there are no unknowns within the domain.

Let use the method of weighted residuals to establish the integral relationships of the BEM. For the domain under consideration and its boundary the residual functions in (8) may be defined as

$$\begin{aligned} R_0 &= \nabla^2 \Phi_k \neq 0 \quad \forall x, y \in \Omega, \\ R_i &= \Phi_k - \overline{\Phi}_i \neq 0 \quad \forall x, y \in \Gamma_i, \end{aligned} \quad (13)$$

where  $\overline{\Phi}_j = 0$  if  $i \neq k$  and  $\overline{\Phi}_i = 1$  if  $i = k$ .

Our aim is to minimize the residual functions on the boundary  $\Gamma$  as well as in the domain  $\Omega$ . It can be proposed to minimize these functions by making their average values go to zero [15]. Assume now that the unknown function  $\Phi_k$  satisfies all boundary

conditions  $R_i = 0$ , except one ( $R_0 \neq 0$ ). Furthermore, assume that the residual function is distributed within the domain  $\Omega$  according to the form of the weighting function  $w$  [16]. Thus, using function  $w$ , one can consider the domain-averaged value of the residual function  $R$

$$\int_{\Omega} R w d\Omega = \int_{\Omega} (\nabla^2 \Phi_k) w d\Omega = 0. \quad (14)$$

For further transformations we need to use the second Green's formula

$$\int_{\Omega} (w \nabla^2 \Phi_k - \Phi_k \nabla^2 w) d\Omega = \oint_{\Gamma} \left( w \frac{\partial \Phi_k}{\partial n} - \Phi_k \frac{\partial w}{\partial n} \right) d\Gamma, \quad (15)$$

where  $n$  is the exterior normal to the contour  $\Gamma$  which bounds the domain  $\Omega$  and which is gone round anti-clockwise.

Then, taking into account (14) expression (15) can be rewritten in the form

$$\int_{\Omega} (\nabla^2 w) \Phi_k d\Omega = \oint_{\Gamma} \Phi_k \frac{\partial w}{\partial n} d\Gamma - \oint_{\Gamma} w \frac{\partial \Phi_k}{\partial n} d\Gamma. \quad (16)$$

If the system of basis functions, which makes the domain integrals go to zero, is chosen as the weighting function, there are only the contour integrals in (16).

According to BEM, the fundamental solution corresponding to the concentrated potential given at a point is usually taken as the weighting function  $w$ . For instance, for the Laplace's equation the fundamental solution has the form [15]

$$\Phi^* = \frac{1}{2\pi} \ln \left( \frac{1}{r} \right), \quad (17)$$

where  $r = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ , and it is the solution of the equation

$$\nabla^2 \Phi^* = -\delta_i, \quad (18)$$

here  $\delta_i$  is the delta function.

It should be noted that  $\Phi^*$  is the function of two points: one point is  $(x_i, y_i)$ , where the considered delta function has a singularity, the other point  $(x, y)$  is the point of observation, it is an independent variable in equation under investigation.

Therefore, using (18), left side of (16) may be transformed as

$$\int_{\Omega} (\nabla^2 \Phi^* \Phi_k) d\Omega = - \int_{\Omega} \delta_i \Phi_k d\Omega = -\Phi_k(x_i, y_i). \quad (19)$$

Expression (16) may be now rewritten in the following form

$$\Phi_k(x_i, y_i) + \oint_{\Gamma} \Phi_k Q^* d\Gamma = \oint_{\Gamma} \Phi^* Q_k d\Gamma, \quad (20)$$

where  $Q^* = \partial \Phi^* / \partial n$  and  $Q_k = \partial \Phi_k / \partial n$ .

Expression (20) is correct if point  $(x_i, y_i)$  is in the domain  $\Omega$ , but to formulate the problem using the contour integrals point  $(x_i, y_i)$  must lie on the contour  $\Gamma$ . The merit of this formulation is in moderation of smoothness limitations (in the sense of Lyapounov) of the domain boundary, therefore, this formulation is applicable for more general form

of regular boundaries including boundaries with corners [15]. Thus, considering point  $(x_i, y_i)$  on the boundary and taking into account the integral sudden in the left side of (20), we have the following boundary integral equation

$$\zeta_i \Phi_k(x_i, y_i) + \oint_{\Gamma} \Phi_k Q^* d\Gamma = \oint_{\Gamma} \Phi^* Q_k d\Gamma. \quad (21)$$

The value  $\zeta_i$  is equal to the interior boundary angle at the point  $(x_i, y_i)$  (in radians) per value of  $2\pi$ . Eq. (21) is also known as the basic equation of the BEM direct formulation for the problem (8). It provides functional constraint between functions  $\Phi_k$  and  $Q_k$  on the boundary  $\Gamma$ , this fact itself proves their values match on the boundary. Since for our case a Dirichlet's problem is to be solved and the function values are given on the boundary, we obtain Fredholm's equation of the first kind for determining unknown values of function normal derivatives  $Q_k$  on the boundary.

Consider now the Poisson's equation (3) from which the function  $\Phi_0$  should be defined. The initial relationship of the weighted residual method for this equation has the form

$$\int_{\Omega} (\nabla^2 \Phi_0 + 2) w d\Omega = 0. \quad (22)$$

Performing transformations analogous to those conducted earlier for the Laplace's equation and choosing the fundamental solution in the form of (17) we get the equation of the BEM direct formulation for Poisson's problem (3)

$$\zeta_i \Phi_0(x_i, y_i) + \oint_{\Gamma} \Phi_0 Q^* d\Gamma - 2 \int_{\Omega} \Phi^* d\Omega = \oint_{\Gamma} \Phi^* Q_0 d\Gamma. \quad (23)$$

where  $Q_0 = \partial\Phi_0/\partial n$ .

The left side of (23) contains the domain integral of the function  $\Phi^*$ . In the general case, it can be evaluated by dividing the domain into finite triangular or quadrilateral elements and applying to them numerical integration formulae. However, alternative and more effective method to transform the domain integral into the equivalent contour integral using second Green's formula (15) can be suggested. Indeed, if there is a function  $V^*$  such that

$$\nabla^2 V^* = \Phi^*, \quad (24)$$

then, using expressions (15) and (24), we have from (23)

$$\int_{\Omega} \Phi^* d\Omega = \int_{\Omega} (\nabla^2 V^*) d\Omega = \oint_{\Gamma} \frac{\partial V^*}{\partial n} d\Gamma. \quad (25)$$

One of these functions  $V^*$  is presented in Ref [15] in the form

$$V^* = \frac{r^2}{8\pi} \left( \ln \frac{1}{r} + 1 \right). \quad (26)$$

Hence, expression (23) may be presented in the following form, which contains only the contour integrals

$$\zeta_i \Phi_0(x_i, y_i) + \oint_{\Gamma} \Phi_0 Q^* d\Gamma - 2 \oint_{\Gamma} \frac{\partial V^*}{\partial n} d\Gamma = \oint_{\Gamma} \Phi^* Q_0 d\Gamma. \quad (27)$$

Taking into account the boundary conditions for function  $\Phi_0$ , which goes to zero on all contours, we have definitive form of (27)

$$-2 \oint_{\Gamma} \frac{\partial V^*}{\partial n} d\Gamma = \oint_{\Gamma} \Phi^* Q_0 d\Gamma. \quad (28)$$

For the particular case of the beam cross-section geometry, it is necessary to reduce the initial integral equation (21) or (28) to the algebraic system of equations in order to use further the numerical approach. To do that the following stages should be performed.

- (1) The boundary  $\Gamma$  is divided into the elements, within these elements the function  $\Phi_0$  or  $\Phi_k$  and its normal derivative vary according to the chosen interpolation functions. Line segments with the linear distribution law for named functions within each of them are considered below as these elements.
- (2) According to the collocation method, the discrete form of the equation connecting the function and its normal derivative values in each node is written down for isolated nodal point distributed within the element.
- (3) The integrals around each element are calculated using the standard formulae of the non-dimensional Gauss quadratures.
- (4) By imposing the boundary conditions, one gets the system of linear algebraic equations, which can be solved using the direct or iteration method and then the unknown values of the function normal derivative in the nodal points can be determined.
- (5) At last, the values of the function  $\Phi_0$  or  $\Phi_k$  in the arbitrary interior point may be determined by the known values of the function and its normal derivative on the boundary using numerical integration of (20) or (27).

As it was shown, the geometrical torsion stiffness  $J_t$  of the beam multiply connected section can be calculated by the well-known formula (12). The integral around the domain  $\Omega$  in (12) may be reduced to the integral around the boundary  $\Gamma$  by the second Green's formula using function  $U$ , such that

$$\nabla^2 U = 1. \quad (29)$$

For instance,

$$U = \frac{1}{4} (x^2 + y^2). \quad (30)$$

If the origin of the coordinate system  $XY$  is placed in the mass center of the beam section, then, taking into account (29) and (30), formula (12) is easily reduced to the

following form

$$J_t = -2 \oint_{\Gamma} U \frac{\partial \Phi(x, y)}{\partial n} d\Gamma - J_p = -2 \oint_{\Gamma} U \left( Q_0 + \sum_{k=1}^n C_k Q_k \right) d\Gamma - J_p, \quad (31)$$

here  $n$  is the number of the interior contours,  $J_p$  is the section polar moment of inertia, its value is equal to the sum of the axial moments of inertia

$$J_p = J_x + J_y, \quad J_x = \int_{\Omega} y^2 d\Omega, \quad J_y = \int_{\Omega} x^2 d\Omega. \quad (32)$$

To transform the integrals around domain in expression (32) to the contour integrals we may use the Green's formula for the plane domain

$$\int_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d\Omega = \oint_{\Gamma} P dx + Q dy. \quad (33)$$

Thus, geometrical bending characteristics of the section could be defined, for example, as

$$J_x = -\frac{1}{3} \oint_{\Gamma} y^3 dx, \quad J_y = -\oint_{\Gamma} x^2 y dx. \quad (34)$$

#### NUMERICAL EXAMPLES

*Circle.* Geometrical torsion stiffness is defined as  $J_t = J_p = \pi d^4/32 = 0.09817d^4$  [13]. Table 1 contains the results obtained by BEM versus the number of boundary elements.

Table 1

Number of elements	4	10	20	30	50	70	100	200	300
$J_t^{(p)} \times 10^{-2}/d^4$	4.528	8.745	9.543	9.695	9.774	9.795	9.807	9.815	9.816

*Equal-side angle.* The geometrical torsion stiffness formula proposed by Arutunyan and Abramyan [3] has the form

$$J_t^{(1)} = \frac{2}{3} b d^3 \left( 1 - 0.942 \frac{d}{b} \right) \quad \text{at } \frac{b}{d} \geq 2. \quad (35)$$

Formulae offered by Weber and Schmieden for the equal-side angle have the following forms respectively

$$J_t^{(2)} = \frac{2}{3} b d^3 \left( 1 - 0.8 \frac{d}{b} \right) \quad (36)$$

$$J_t^{(3)} = \frac{2}{3} b d^3 \left( 1 - 0.2166 \frac{d}{b} \right) \quad (37)$$

Table 2 presents the dependency of non-dimensional value  $J_t^{(p)}/d^4$  on the boundary element number used for BEM calculation and the condensation of elements to the base line ends for the case  $b/d = 4$ .

Table 2

Number of elements	15	20	30	50	70	100	200	300
Without condensation	1.960	2.018	2.103	2.158	2.176	2.185	2.193	2.194
With condensation	1.942	2.033	2.151	2.168	2.186	2.185	2.193	2.195

The results obtained by approximate formulae (35)-(37) for different values of dimension ratio  $b/d$  and the BEM results are presented in Fig. 3.

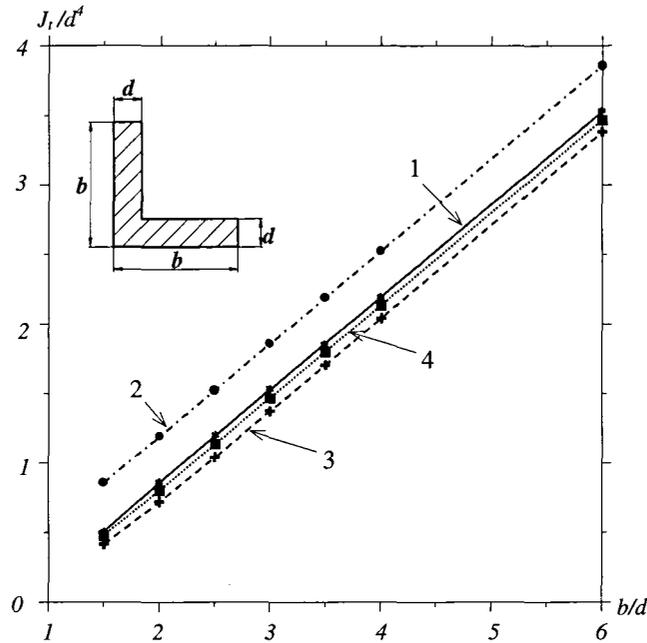


FIGURE 3. The relationship between the section geometrical torsion stiffness and the dimension ratio  $b/d$  for the equal-side angle: 1 — by formula (35), 2 — by formula (36), 3 — by formula (37), 4 — calculation by BEM.

*Hollow square.* Table 3 contains the calculation results for  $J_t/d^4$  by the known FEM program ANSYS [9] using linear quadrilateral isoparametric finite element. Because of the symmetry only a quarter of the profile was considered. The finite element number decreased with increase of ratio  $b/d$  (where  $b$  is a half of the side length and  $d$  is the wall thickness) from 128 elements for  $b/d = 1.5$  to 32 elements for  $b/d = 12$ . The geometrical torsion stiffness was calculated using numerical integration around the finite element mesh. Table 3 presents also the BEM results versus the node number and condensation of elements to the base line ends. It should be noted that the results obtained by FEM and BEM show excellent agreement. It is also important that BEM application gives sufficiently accurate result even with a very small element number (20 nodes) over all

range of values  $b/d$ . This problem evidently shows the advantages of BEM over other numerical methods and in this case over FEM.

Table 3

$b/d$		12	6	4	3	2.4	2	1.7	1.5	1
Calculation by FEM [9]		11968	1408	362.2	140.8	64.02	32.95	18.47	11.15	2.243
Calculation by BEM										
Without condensation	20 nodes	11630	1440	372.4	144.4	65.41	33.38	18.73	11.31	2.248
	50 nodes	11740	1383	365.5	140.4	63.94	32.95	18.49	11.17	2.248
	100 nodes	11940	1401	367.9	141.0	64.12	33.01	18.52	11.18	2.25
	200 nodes	12060	1406	368.8	141.2	64.20	33.04	18.53	11.19	2.25
With condensation	20 nodes	11670	1444	381.9	145.2	65.69	33.73	18.89	11.40	2.262
	50 nodes	11940	1398	367.7	141.2	64.32	33.13	18.58	11.22	2.251
	100 nodes	12050	1406	368.8	141.3	64.23	33.06	18.55	11.20	2.25
	200 nodes	12090	1408	369.2	141.3	64.24	33.06	18.54	11.19	2.25

*Compressor blade.* The compressor blade cross-section (Fig. 4) has the chord  $b = 76.417 \text{ mm}$  and the maximal profile thickness  $c_{max} = 4.586 \text{ mm}$ .

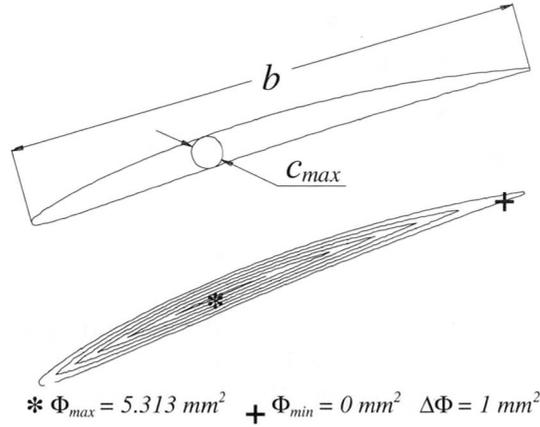


FIGURE 4. Isolines of the stress function  $\Phi(x, y)$  for the compressor blade.

To define the approximate upper bound for the geometrical torsion stiffness the Vlasov's formula [1] is used

$$J_t^{(1)} = \frac{4J_1^{(pr)} J_2^{(pr)}}{J_1^{(pr)} + J_2^{(pr)}}, \quad (38)$$

where  $J_1^{(pr)}$  and  $J_2^{(pr)}$  are the principal axial moments of inertia, for the given profile  $J_1^{(pr)} = 415.9 \text{ mm}^4$  and  $J_2^{(pr)} = 8.759 \times 10^4 \text{ mm}^4$ .

The particular case of the Griffiths-Prescott's formula obtained by Labensone [2] is also often used for the compressor blades

$$J_t^{(2)} = 0.162bc_{max}^3. \quad (39)$$

The results obtained by formulae (38) and (39) and using the BEM technique (100 elements) are presented in Table 4. It should be noted that the Vlasov's formula (38) sets off the excess of the considered value of 21.8%, while the formula (39) sets off the result less of 8.3%.

Table 4

Profile	Calculation by BEM	Formula (38)	Formula (39)
Fig. 4	1294	1656	1194
Fig. 5a	1623	5814	3865
Fig. 5b	2498	6060	3865

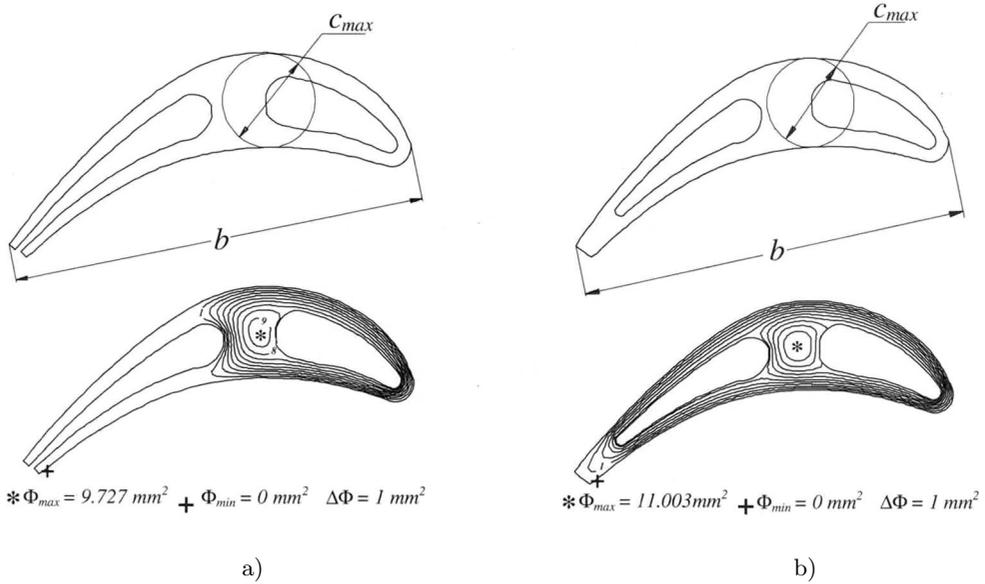


FIGURE 5. Isolines of the stress function  $\Phi(x, y)$  for the a) two-connected and b) three-connected profiles of the turbine blade.

#### Turbine blades.

Fig. 5a presents the turbine blade cross-section with chord  $b = 37.9 \text{ mm}$  and the maximal profile thickness  $c_{max} = 8.57 \text{ mm}$ . The profile principal axial moments of inertia are  $J_1^{(pr)} = 1635 \text{ mm}^4$  and  $J_2^{(pr)} = 1.314 \times 10^4 \text{ mm}^4$ .

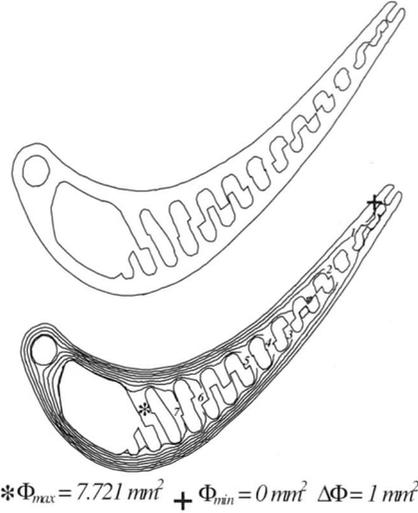


FIGURE 6. Isolines of the stress function  $\Phi(x, y)$  for the turbine blade cross-section of complex form.

Comparison between the results obtained by the numerical formulae and the BEM technique shows that the formulae (38) and (39) are not applicable to the definition of geometrical torsion stiffness in the case of the multiply connected blade profile.

Fig. 5b presents another cross-section of the same turbine blade with the same principal geometrical dimensions. For the BEM calculation 250 elements were used.

It should be noted that the blade cross-section geometry in Fig. 5b has only negligible variations in comparison with the profile shown in Fig. 5a. The bending stiffness of the profile didn't almost change ( $J_1^{(pr)} = 1693 \text{ mm}^4$  and  $J_2^{(pr)} = 1.44 \times 10^4 \text{ mm}^4$ ). At the same time, the geometrical torsion stiffness increased of more than half (53.3%) because of the transformation of the cross-section from the two connected domain into the three connected domain. Thus, in the calculation of the natural frequencies and mode shapes of such blade, the effect of section connectivity changing owing to the presence of the varying along the blade length spaces doesn't affect much the bending frequencies and mode shapes, but influences greatly on the torsion frequencies and mode shapes. Therefore, the geometrical torsion stiffness should be calculated with high accuracy by the methods taking into account the blade cross-section details of construction.

The turbine blade cross-section shown in Fig. 6 was chosen as an another example to determine geometrical characteristics of complex form cross-sections [17]. The BEM technique gives the following values of the bending and torsion stiffnesses:  $J_1^{(pr)} = 1712 \text{ mm}^4$ ,  $J_2^{(pr)} = 1.487 \times 10^4 \text{ mm}^4$  and  $J_t^{(p)} = 1954 \text{ mm}^4$ .

Fig. 4-6 also present the stress function isolines for all profiles under consideration.

As it is seen from the results presented above, for airfoil profiles ranging from compressor blades to cooled turbine blades, BEM has been alternative and efficient technique for calculating geometrical torsion stiffness.

#### ACKNOWLEDGEMENTS

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#### REFERENCES

- [1] Yu.S.Vorobiev, B.F.Shorr, *Theory of Twisted Beams*, Naukova Dumka, Kiev, 1983, [in Russian].
- [2] *Strength Analysis of Aviation Gas-Turbine Engines*, I.A.Birger and N.I.Koterov (ed.), Mashinostroenie, Moscow, 1984, [in Russian].
- [3] N.Kh.Arutunyan, B.L.Abramyan, *Torsion of Elastic Bodies*, Fizmatgiz, Moscow, 1963 [in Russian].
- [4] D.V.Bychkov, A.K.Mroshinsky, *Torsion of Steel Bars*, Stroyizdat, Moscow, 1944 [in Russian].
- [5] C.Weber, *Die Lehre von der Drehungstestigkeit*, Forschungsarbeiten Jng.-Wes H., 249(1921), Berlin.
- [6] V.L.Rvachev, I.V.Goncharuk, *Torsion of Beams with Intricate Profile*, KhSU, Kharkov, 1973, [in Russian].
- [7] N.N.Muskhelishvili, *Several Basic Problems of Mathematical Theory of Elasticity*, Nauka, Moscow, 1966, [in Russian].
- [8] C.Weber, W.Günther, *Torsionstheorie*, Fredr.Vieweg & Sohn. Braunschweig, 1958.
- [9] J.S.Lamancusa, D.A.Saravanos, *The torsional analysis of bars with hollow square cross-sections*, in Finite elements in Analysis and Design, vol.6, 1989, p.71-79.
- [10] Yu.S.Vorobiev, Z.V.Sapelkina, A.I.Shepel', *Fields of application for different methods of determining torsion stiffness of turbomachine blades*, in Dynamics and Strength of Machines, Visha shkola, Kharkov, vol.41, 1985, p.38-43, [in Russian].
- [11] Yu.S.Vorobiev, M.V.Bekh, A.I.Shepel', Z.V.Sapelkina, *Natural frequencies of the bladed disks with cooled blades*, in Dynamics and Strength of Machines, Visha shkola, Kharkov, vol.47, 1988, p.62-66, [in Russian].
- [12] Yu.M.Temis, V.V.Karaban, *Determining geometrical torsion stiffness of engineering and blade one- and multiply connected profiles with specialities using BEM*, in Applied Problems of Strength and Plasticity. Numerical simulation of physic-mechanical processes, Moscow, vol.55, 1997, p.137-149, [in Russian].
- [13] I.A.Birger, R.R.Mavlutov, *Strength of Materials*, Nauka, Moscow, 1986 [in Russian].
- [14] A.I.Lurie, *Elasticity Theory*, Nauka, Moscow, 1970, [in Russian].
- [15] C.A.Brebbia, J.C.F.Telles and L.C.Wroubel, *Boundary Element Techniques*, Verlag Berlin, Heidelberg, 1984.
- [16] C.A.Brebbia and S.Walker, *Boundary Element Techniques in Engeneering*, Butterworths, London, 1980.
- [17] *Turbines*, in Scientific Contribution in Creation of Aviation Motors, vol.2, V.A.Skibin and V.I.Solonin (ed.), Mashinostroenie, Moscow, 2000, p.211-291, [in Russian].

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