

## APPLICATIONS OF A CERTAIN FAMILY OF HYPERGEOMETRIC SUMMATION FORMULAS ASSOCIATED WITH PSI AND ZETA FUNCTIONS

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ABSTRACT. The main object of this paper is first to give two contiguous analogues of a well-known hypergeometric summation formula for  ${}_2F_1(1/2)$ . We then apply each of these analogues with a view to evaluating the sums of several classes of series in terms of the Psi (or Digamma) and the Zeta functions. Relevant connections of the series identities presented here with those given elsewhere are also pointed out.

### 1. Introduction and Preliminaries

The generalized hypergeometric function with  $p$  numerator and  $q$  denominator parameters is defined by

$$(1.1) \quad {}_pF_q \left[ \begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n z^n}{(\beta_1)_n \dots (\beta_q)_n n!},$$

where  $(\alpha)_n$  denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(1.2) \quad (\alpha)_n := \begin{cases} 1 & (n = 0) \\ \alpha(\alpha + 1) \dots (\alpha + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \dots\}), \end{cases}$$

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which can also be rewritten in the form:

$$(1.3) \quad (\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)},$$

where  $\Gamma$  is the well-known Gamma function whose Weierstrass canonical product form is

$$(1.4) \quad \Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \right\},$$

$\gamma$  being the Euler-Mascheroni constant defined by

$$(1.5) \quad \gamma := \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \log n \right) \cong 0.577\,215\,664\,901\,532\dots$$

With the notation (1.1), the Gaussian hypergeometric series is  ${}_2F_1$ , which is also denoted simply by  $F$ .

The Psi (or Digamma) function is defined as the logarithmic derivative of the Gamma function:

$$(1.6) \quad \psi(z) := \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

We recall here some well-known properties of the  $\psi$ -function (see [8]): For a positive integer  $n$ ,

$$(1.7) \quad \begin{aligned} \psi(1) &= -\gamma; & \psi\left(\frac{1}{2}\right) &= -\gamma - 2 \log 2; \\ \psi(z+n) - \psi(z) &= \sum_{k=0}^{n-1} \frac{1}{z+k}. \end{aligned}$$

The Polygamma functions are defined by (see [8, p. 41])

$$(1.8) \quad \psi^{(n)}(z) := \begin{cases} \frac{d^{n+1}}{dz^{n+1}} \log \Gamma(z) & (n \in \mathbb{N}) \\ \psi(z) & (n = 0). \end{cases}$$

By definition it is easy to see that

$$(1.9) \quad \psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, z) \quad (n \in \mathbb{N}),$$

where  $\zeta(z, a)$  is the generalized (or Hurwitz) Zeta function defined by

$$(1.10) \quad \zeta(z, a) = \sum_{k=0}^{\infty} (k+a)^{-z} \quad (\Re(z) > 1; a \neq 0, -1, -2, \dots)$$

and  $\zeta(z, 1) = \zeta(z)$  is the Riemann Zeta function. It is not difficult to derive the following properties (see [18, pp. 265-275]):

$$(1.11) \quad \begin{aligned} \zeta(z) &= \frac{1}{1-2^{-z}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^z} = \frac{1}{2^z-1} \zeta\left(z, \frac{1}{2}\right) \quad (\Re(z) > 1); \\ \zeta(z, a) &= \zeta(z, n+a) + \sum_{k=0}^{n-1} (k+a)^{-z} \quad (n \in \mathbb{N}). \end{aligned}$$

There are four known main summation theorems for  ${}_2F_1$  with arguments 1, -1, and  $\frac{1}{2}$ , which play important roles in theory and applications. The following summation formula was given by Kummer [11]:

$$(1.12) \quad {}_2F_1 \left[ \begin{matrix} a, b; \\ \frac{1}{2}(a+b+1); \end{matrix} \frac{1}{2} \right] = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} + \frac{1}{2}a + \frac{1}{2}b)}{\Gamma(\frac{1}{2} + \frac{1}{2}a) \Gamma(\frac{1}{2} + \frac{1}{2}b)}.$$

In this paper we first give two summation formulas for  ${}_2F_1(\frac{1}{2})$  which are actually contiguous to Kummer's summation theorem (1.12). We also show how these formulas can be applied in order to evaluate the sums of several classes of series in terms of the Psi (or Digamma) and the Zeta functions.

## 2. Contiguous analogues of the summation formula (1.12)

The first question is to express

$${}_2F_1 \left[ \begin{matrix} a, b; \\ 1 + \frac{1}{2}(a+b); \end{matrix} \frac{1}{2} \right]$$

in a closed form like (1.12).

For a given  $F = F(A, B; C; z)$ , as usual, denote

$$\begin{aligned} F(A+) &:= F(A+1, B; C; z), \\ F(A+, B-) &:= F(A+1, B-1; C; z), \end{aligned}$$

and so on. Recall a known contiguous formula (see Rainville [14, p. 72]):

$$(2.1) \quad F = F(A-, B+) + \frac{1}{C}(B+1-A)zF(B+, C+).$$

Now, replacing  $a$  and  $b$  by  $a-1$  and  $b-1$ , respectively, we find from (1.12) that

$$(2.2) \quad {}_2F_1 \left[ \begin{matrix} a-1, b; 1 \\ \frac{1}{2}(a+b); \frac{1}{2} \end{matrix} \right] = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}a + \frac{1}{2}b)}{\Gamma(\frac{1}{2}a) \Gamma(\frac{1}{2}b + \frac{1}{2})}$$

and

$$(2.3) \quad {}_2F_1 \left[ \begin{matrix} a, b-1; 1 \\ \frac{1}{2}(a+b); \frac{1}{2} \end{matrix} \right] = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}a + \frac{1}{2}b)}{\Gamma(\frac{1}{2}a + \frac{1}{2}) \Gamma(\frac{1}{2}b)}.$$

Setting  $A = a$ ,  $B = b-1$ ,  $C = \frac{1}{2}(a+b)$ , and  $z = \frac{1}{2}$  in (2.1), and applying (2.2) and (2.3) to the resulting equation, we obtain the first desired summation formula:

$$(2.4) \quad {}_2F_1 \left[ \begin{matrix} a, b; 1 \\ 1 + \frac{1}{2}(a+b); \frac{1}{2} \end{matrix} \right] = \frac{a+b}{a-b} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b\right) \left\{ \frac{1}{\Gamma(\frac{1}{2}a) \Gamma(\frac{1}{2}b + \frac{1}{2})} - \frac{1}{\Gamma(\frac{1}{2}b) \Gamma(\frac{1}{2}a + \frac{1}{2})} \right\}.$$

Replacing  $a$  and  $b$  by  $2a$  and  $2b$ , respectively, and using  $\Gamma(z+1) = z\Gamma(z)$ , we obtain the following equivalent form of (2.4):

$$(2.5) \quad {}_2F_1 \left[ \begin{matrix} 2a, 2b; 1 \\ 1+a+b; \frac{1}{2} \end{matrix} \right] = \frac{\Gamma(\frac{1}{2}) \Gamma(a+b+1)}{a-b} \left\{ \frac{1}{\Gamma(a) \Gamma(b + \frac{1}{2})} - \frac{1}{\Gamma(b) \Gamma(a + \frac{1}{2})} \right\}.$$

Another summation formula contiguous to (1.12) and (2.4) is given by

$$(2.6) \quad {}_2F_1 \left[ \begin{matrix} a, b; \\ \frac{1}{2}(a+b); \end{matrix} \frac{1}{2} \right] = \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}a + \frac{1}{2}b\right) \left\{ \frac{1}{\Gamma\left(\frac{1}{2}a\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} + \frac{1}{\Gamma\left(\frac{1}{2}b\right) \Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)} \right\}.$$

Indeed, we first recall a known contiguous formula (see Rainville [14, p. 71]):

$$(2.7) \quad (1-z)F = F(A-) - \frac{C-B}{C} zF(C+).$$

Setting  $A = a$ ,  $B = b$ ,  $C = \frac{1}{2}(a+b)$ , and  $z = \frac{1}{2}$  in (2.7), and making use of (2.2) and (2.4) in the resulting equation, we have the desired formula (2.6).

If we replace  $a$  and  $b$  by  $2a$  and  $2b$ , respectively, we obtain the following equivalent form of (2.6):

$$(2.8) \quad {}_2F_1 \left[ \begin{matrix} 2a, 2b; \\ a+b; \end{matrix} \frac{1}{2} \right] = \Gamma\left(\frac{1}{2}\right) \Gamma(a+b) \left\{ \frac{1}{\Gamma(a) \Gamma\left(b + \frac{1}{2}\right)} + \frac{1}{\Gamma(b) \Gamma\left(a + \frac{1}{2}\right)} \right\}.$$

It may be remarked in passing that Lavoie *et al.* [12] established many contiguous extensions of a well-known  ${}_3F_2$  summation theorem, and deduced (2.4) and (2.6) in a markedly different way.

### 3. Product formulas for hypergeometric functions

The summation formulas (2.4) and (2.6) can readily be applied to derive some contiguous analogues of Kummer's second transformation [11]:

$$(3.1) \quad e^{-\frac{x}{2}} {}_1F_1(\alpha; 2\alpha; x) = {}_0F_1\left(-; \alpha + \frac{1}{2}; \frac{x^2}{16}\right).$$

Indeed, if we let

$$(3.2) \quad e^{-\frac{x}{2}} {}_1F_1(\alpha; 2\alpha + 1; x) := \sum_{n=0}^{\infty} a_n x^n$$

and use the Cauchy product rule for two series, we find that

$$(3.3) \quad a_n = \frac{(\alpha)_n}{(2\alpha + 1)_n \cdot n!} {}_2F_1 \left[ \begin{matrix} -n, -2\alpha - n; 1 \\ 1 - \alpha - n; \frac{1}{2} \end{matrix} \right].$$

Setting  $a = -n$  and  $b = -2\alpha - n$  in (2.4) in order to evaluate the  ${}_2F_1(\frac{1}{2})$  in (3.3), we get

$$(3.4) \quad \begin{aligned} a_n &= \Gamma\left(\frac{1}{2}\right) \frac{(\alpha)_n}{(2\alpha + 1)_n \cdot n!} \frac{\alpha + n}{\alpha} \frac{\Gamma(-n - \alpha)}{\Gamma(-\alpha - \frac{n}{2}) \Gamma(\frac{1-n}{2})} \\ &\quad - \Gamma\left(\frac{1}{2}\right) \frac{(\alpha)_n}{(2\alpha + 1)_n \cdot n!} \frac{\alpha + n}{\alpha} \frac{\Gamma(-n - \alpha)}{\Gamma(-\alpha - \frac{n}{2} + \frac{1}{2}) \Gamma(-\frac{n}{2})} \\ &\quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \end{aligned}$$

so that, finally,

$$(3.5) \quad e^{-\frac{x}{2}} {}_1F_1(\alpha; 2\alpha + 1; x) = \sum_{n=0}^{\infty} b_n x^{2n} + \sum_{n=0}^{\infty} c_n x^{2n+1},$$

where

$$(3.6) \quad b_n = \frac{1}{(\alpha + \frac{1}{2})_n \cdot n! \cdot 2^{4n}}$$

and

$$(3.7) \quad c_n = -\frac{1}{2\alpha + 1} \frac{1}{(\alpha + \frac{3}{2})_n \cdot n! \cdot 2^{4n}}.$$

From (3.5), (3.6), and (3.7), we obtain

$$(3.8) \quad \begin{aligned} &e^{-\frac{x}{2}} {}_1F_1(\alpha; 2\alpha + 1; x) \\ &= {}_0F_1\left(-; \alpha + \frac{1}{2}; \frac{x^2}{16}\right) - \frac{x}{2(2\alpha + 1)} {}_0F_1\left(-; \alpha + \frac{3}{2}; \frac{x^2}{16}\right). \end{aligned}$$

Similarly, by letting

$$(3.9) \quad e^{-\frac{x}{2}} {}_1F_1(\alpha; 2\alpha - 1; x) := \sum_{n=0}^{\infty} a_n x^n,$$

we have

$$(3.10) \quad a_n = \frac{(\alpha)_n}{(2\alpha - 1)_n \cdot n!} {}_2F_1 \left[ \begin{matrix} -n, & 2 - 2\alpha - n; & 1 \\ & 1 - \alpha - n; & 2 \end{matrix} \right].$$

Now, applying the formula (2.6) to (3.10), we easily obtain another product formula:

$$(3.11) \quad \begin{aligned} & e^{-\frac{x}{2}} {}_1F_1(\alpha; 2\alpha - 1; x) \\ &= {}_0F_1 \left( -; \alpha - \frac{1}{2}; \frac{x^2}{16} \right) + \frac{x}{2(2\alpha - 1)} {}_0F_1 \left( -; \alpha + \frac{1}{2}; \frac{x^2}{16} \right). \end{aligned}$$

Both (3.8) and (3.11) were derived elsewhere by Rathie and Nagar [15].

It should be observed that, in view of Kummer's first transformation [11]:

$$(3.12) \quad {}_1F_1(\alpha; \gamma; z) = e^z {}_1F_1(\gamma - \alpha; \gamma; -z),$$

both (3.8) and (3.11) are the same result: For instance, we just replace  $\alpha$  by  $\alpha - 1$  and  $x$  by  $-x$  in (3.8), and apply (3.12) to obtain (3.11) immediately from (3.8).

#### 4. Certain classes of infinite series associated with Psi and Zeta functions

Many infinite series have been evaluated in terms of the Psi and Zeta functions (see [9], [10]). Al-Saqabi *et al.* [1] presented a systematic account of several interesting infinite series expressed in terms of the Psi (or Digamma) functions. Aular de Durán *et al.* [2] examined rather systematically the sums of numerous interesting families of infinite series with or without the use of fractional calculus. Shen [16] investigated the connections between the Stirling numbers  $s(n, k)$  of the first kind and the Riemann Zeta function  $\zeta(n)$  by means of the Gauss summation formula for  ${}_2F_1$ . Various other classes of infinite series have also been evaluated

by making use of known summation formulas for  ${}_2F_1$  and  ${}_3F_2$  (see [4], [5], and [6]). In this section we evaluate the sums of certain families of infinite series by using our formulas (2.5) and (2.8).

We introduce the Stirling numbers  $s(n, k)$  of the first kind defined by the following equation (see [7, pp. 204-218], [13, p. 43], and [17, p. 396, Problem 25]):

$$z(z-1)\cdots(z-n+1) = \sum_{k=0}^n s(n, k)z^k.$$

From the above definition of  $s(n, k)$ , the Pochhammer symbol (or the shifted factorial) can be written in the form:

$$(4.1) \quad (z)_n = z(z+1)\cdots(z+n-1) = \sum_{k=0}^n (-1)^{n+k} s(n, k)z^k.$$

It is not difficult to see also that

$$(4.2) \quad \begin{aligned} (-1)^{n+1} s(n, 1) &= (n-1)!; & (-1)^n s(n, 2) &= (n-1)! \sum_{k=1}^{n-1} \frac{1}{k}; \\ (-1)^{n+1} s(n, 3) &= \frac{(n-1)!}{2} \left\{ \left( \sum_{k=1}^{n-1} \frac{1}{k} \right)^2 - \sum_{k=1}^{n-1} \frac{1}{k^2} \right\}. \end{aligned}$$

Considering (1.3) and differentiating each side of (4.1) with respect to  $z$ , successively, we obtain the following finite sums involving  $s(n, k)$ :

$$(4.3) \quad \begin{aligned} & \sum_{k=1}^n (-1)^{n+k} k s(n, k) z^{k-1} = (z)_n [\psi(z+n) - \psi(z)]; \\ & \sum_{k=2}^n (-1)^{n+k} k(k-1) s(n, k) z^{k-2} \\ & = (z)_n \left[ \{\psi(z+n) - \psi(z)\}^2 + \psi^{(1)}(z+n) - \psi^{(1)}(z) \right]; \\ & \sum_{k=3}^n (-1)^{n+k} k(k-1)(k-2) s(n, k) z^{k-3} \\ & = (z)_n \left[ \{\psi(z+n) - \psi(z)\}^3 + 3\{\psi(z+n) - \psi(z)\} \right. \\ & \quad \left. \times \{\psi^{(1)}(z+n) - \psi^{(1)}(z)\} + \psi^{(2)}(z+n) - \psi^{(2)}(z) \right]. \end{aligned}$$



Replace  $a$  by  $z$  in (2.5) and set

$$\begin{aligned}
 f(z) &:= 1 + \sum_{n=1}^{\infty} \frac{(2b)_n}{2^n n!} \frac{(2z)_n}{(1+b+z)_n} \\
 (4.4) \quad &= \frac{\Gamma(\frac{1}{2}) \Gamma(z+b+1)}{z-b} \left\{ \frac{1}{\Gamma(z) \Gamma(b+\frac{1}{2})} - \frac{1}{\Gamma(b) \Gamma(z+\frac{1}{2})} \right\} \\
 &:= \sum_{n=0}^{\infty} a_n z^n.
 \end{aligned}$$

Setting  $f(z) := g(z)h(z)$ , where

$$(4.5) \quad g(z) := \frac{\Gamma(\frac{1}{2}) \Gamma(z+b+1)}{z-b}$$

and

$$(4.6) \quad h(z) := \frac{1}{\Gamma(z) \Gamma(b+\frac{1}{2})} - \frac{1}{\Gamma(b) \Gamma(z+\frac{1}{2})},$$

and making use of the aforementioned properties of Gamma, generalized Zeta, and Psi functions, we readily obtain

$$\begin{aligned}
 (4.7) \quad g(0) &= -\Gamma\left(\frac{1}{2}\right) \Gamma(b), \quad h(0) = -\frac{1}{\Gamma\left(\frac{1}{2}\right) \Gamma(b)}; \\
 g'(0) &= -\Gamma\left(\frac{1}{2}\right) \Gamma(b+1) \frac{b\psi(b+1)+1}{b^2}, \\
 h'(0) &= \frac{1}{\Gamma\left(b+\frac{1}{2}\right)} + \frac{\psi\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma(b)}; \\
 g''(0) &= -\Gamma\left(\frac{1}{2}\right) \Gamma(b) \left[ \{\psi(b+1)\}^2 + \frac{2\psi(b+1)}{b} + \zeta(2, b+1) + \frac{2}{b^2} \right], \\
 h''(0) &= \frac{2\gamma}{\Gamma\left(b+\frac{1}{2}\right)} + \frac{1}{\Gamma(b) \Gamma\left(\frac{1}{2}\right)} \left[ 3\zeta(2) - \left\{ \psi\left(\frac{1}{2}\right) \right\}^2 \right].
 \end{aligned}$$

Now, by using Leibniz's rule for differentiation, we find that

$$\begin{aligned}
 (4.8) \quad a_0 &= f(0) = 1, \\
 a_1 &= f'(0) = \psi(b+1) - \psi\left(\frac{1}{2}\right) + \frac{1}{b} - \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)}{\Gamma\left(b+\frac{1}{2}\right)}, \\
 a_2 &= \frac{1}{2} \left[ \left\{ \psi(b+1) - \psi\left(\frac{1}{2}\right) \right\}^2 + \frac{2}{b} \left\{ \psi(b+1) - \psi\left(\frac{1}{2}\right) \right\} \right. \\
 &\quad \left. - 2 \left\{ \psi(b+1) + \frac{1}{b} + \gamma \right\} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)}{\Gamma\left(b+\frac{1}{2}\right)} + \zeta(2, b+1) - 3\zeta(2) + \frac{2}{b^2} \right],
 \end{aligned}$$

and so on. On the other hand, consider the first equality of (4.4). From (4.1), we have

$$(4.9) \quad (2z)_n := \sum_{k=1}^n A_k z^k,$$

where

$$(4.10) \quad A_k = (-1)^{n+k} s(n, k) \cdot 2^k \quad (k = 1, \dots, n).$$

In view of the following series identity:

$$\sum_{k=0}^n \sum_{\ell=0}^k B_{k,\ell} = \sum_{\ell=0}^n \sum_{k=\ell}^n B_{k,\ell},$$

we obtain

$$\begin{aligned}
 (4.11) \quad (1+b+z)_n &= \sum_{k=0}^n (-1)^{n+k} s(n, k) (1+b+z)^k \\
 &= \sum_{k=0}^n (-1)^{n+k} s(n, k) \left\{ \sum_{\ell=0}^k \binom{k}{\ell} (1+b)^{k-\ell} \cdot z^\ell \right\} \\
 &:= \sum_{\ell=0}^n B_\ell z^\ell,
 \end{aligned}$$

where

$$(4.12) \quad B_\ell = \sum_{k=\ell}^n (-1)^{n+k} s(n, k) \binom{k}{\ell} (1+b)^{k-\ell}.$$

It follows from (4.2), (4.3), (4.10), and (4.12) that

$$(4.13) \quad f(z) = 1 + \sum_{n=1}^{\infty} \frac{(2b)_n}{2^n n!} (\alpha_1 z + \alpha_2 z^2 + \dots),$$

where

$$(4.14) \quad \begin{aligned} \alpha_1 &= \frac{A_1}{B_0} = \frac{2(n-1)!}{(1+b)_n}, \\ \alpha_2 &= \frac{1}{B_0} \left( A_2 - \frac{A_1 B_1}{B_0} \right) \\ &= \frac{2(n-1)!}{(1+b)_n} \left[ 2 \sum_{k=1}^{n-1} \frac{1}{k} + \psi(1+b) - \psi(1+b+n) \right], \end{aligned}$$

and so on. Considering (4.4), (4.8), (4.13), and (4.14), we finally obtain the following formulas:

$$(4.15) \quad \sum_{n=1}^{\infty} \frac{(2b)_n}{2^{n-1} n (1+b)_n} = \psi(b+1) - \psi\left(\frac{1}{2}\right) + \frac{1}{b} - \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)}{\Gamma\left(b+\frac{1}{2}\right)};$$

$$(4.16) \quad \begin{aligned} &\sum_{n=1}^{\infty} \frac{(2b)_n}{2^{n-1} n (1+b)_n} \left[ 2 \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+b} \right] \\ &= \frac{1}{2} \left[ \left\{ \psi(b+1) - \psi\left(\frac{1}{2}\right) \right\}^2 + \frac{2}{b} \left\{ \psi(b+1) - \psi\left(\frac{1}{2}\right) \right\} \right. \\ &\quad \left. - 2 \left\{ \psi(b+1) + \frac{1}{b} + \gamma \right\} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)}{\Gamma\left(b+\frac{1}{2}\right)} + \zeta(2, b+1) - 3\zeta(2) + \frac{2}{b^2} \right]. \end{aligned}$$

Formula (4.15) is equivalently written in the form:

$$(4.17) \quad \begin{aligned} &{}_3F_2 \left[ \begin{matrix} 1, 1, 2b+1; \\ 2, 2+b; \end{matrix} \middle| \frac{1}{2} \right] \\ &= \frac{1+b}{2b} \left[ \psi(b+1) - \psi\left(\frac{1}{2}\right) + \frac{1}{b} - \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)}{\Gamma\left(b+\frac{1}{2}\right)} \right]. \end{aligned}$$

Applying the same procedure as above to (2.8), we readily obtain the following formulas:

$$(4.18) \quad \sum_{n=1}^{\infty} \frac{(2b)_n}{2^{n-1} n (b)_n} = \psi(b) - \psi\left(\frac{1}{2}\right) + \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(b)}{\Gamma\left(b + \frac{1}{2}\right)};$$

$$(4.19) \quad \sum_{n=1}^{\infty} \frac{(2b)_n}{2^{n-1} n (b)_n} \left[ 2 \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+b-1} \right] \\ = \left[ \frac{1}{2} \left\{ \psi(b) - \psi\left(\frac{1}{2}\right) \right\}^2 + \frac{2\Gamma\left(\frac{1}{2}\right) \psi(b) \Gamma(b)}{\Gamma\left(b + \frac{1}{2}\right)} + \zeta(2, b) - 3\zeta(2) \right].$$

Just as in the case of (4.15), (4.18) can also be written in the equivalent form:

$$(4.20) \quad {}_3F_2 \left[ \begin{matrix} 1, 1, 2b+1; \\ 2, b+1; \end{matrix} \frac{1}{2} \right] \\ = \frac{1}{2} \left[ \psi(b) - \psi\left(\frac{1}{2}\right) + \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(b)}{\Gamma\left(b + \frac{1}{2}\right)} \right].$$

Setting  $b = 1$  in (4.16), and using (1.7), (1.11), and the following known formula (see [5]):

$$(4.21) \quad \sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 2^n} = \frac{1}{2} [\zeta(2) - (\log 2)^2],$$

we readily obtain

$$(4.22) \quad \sum_{n=1}^{\infty} \frac{H_n}{n \cdot 2^n} = \frac{1}{2} \zeta(2),$$

where  $H_n$  denotes the harmonic numbers defined by

$$(4.23) \quad H_n := \sum_{k=1}^n \frac{1}{k}.$$

Numerous other identities involving the harmonic numbers  $H_n$ , analogous to (4.22), can be found in (for example) [3] and [16].

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