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NEAR ROTATIONAL DIRECTED TRIPLE SYSTEMS

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Abstract. A directed triple system of order v, denoted by $DTS(v)$, is said to be k–near rotational if it admits an automorphism consisting of exactly three fixed elements and k cycles of length $\frac{v-3}{k}$. In this paper, we obtain necessary and sufficient conditions for the existence of k–near rotational $DTS(v)$s for every positive integer k.

1. Introduction

A *directed triple* is a set of three ordered pairs of the forms (x, y), (y, z) and (x, z). A *directed triple system of order* v, denoted by $DTS(v)$, is a v-set X of elements together with a set $\mathfrak{B}$ of directed triples of distinct elements of X, called *blocks*, such that every ordered pair of distinct elements of X occurs in exactly one block of $\mathfrak{B}$. The notation $[x, y, z]$ will be used for the block containing the three ordered pairs (x, y), (y, z) and (x, z). It is well-known [3] that there exists a $DTS(v)$ if and only if $v \equiv 0$ or $1 \pmod 3$. An *automorphism* of a $DTS(v)$, $(X, \mathfrak{B})$, is a permutation α of X such that $\{\alpha(B) \,|B \in \mathfrak{B}\} = \mathfrak{B}$.

A permutation α of degree v is said to be of *type* $[\alpha] = [\alpha_1, \alpha_2, \ldots, \alpha_v]$ if its disjoint cyclic decomposition contains α_i cycles of length i for $i = 1, 2, \ldots, v$. A set of blocks β is said to be a *set of starter blocks* for a $DTS(v)$ under the automorphism α if the orbits of the blocks of β under α produce the $DTS(v)$ and each orbit contains exactly one block of β. An interesting problem is the following: Given a permutation α of degree v, for what orders v does there exist a $DTS(v)$ with α as an automorphism? If a $DTS(v)$ admits an automorphism α, it is denoted

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by $DTS_\alpha(v)$. The problem has been studied for the following types of automorphisms:

(i) If α is of type $[0,0,\ldots,0,1]$, a $DTS_\alpha(v)$ is said to be *cyclic*. There exists a cyclic $DTS(v)$ if and only if $v \equiv 1$, 4 or 7 (mod 12) [2].

(ii) If α is of type $[1,0,\ldots,0,k,0,\ldots,0]$, a $DTS_\alpha(v)$ is called *k–rotational*. There exists a 1–rotational $DTS(v)$ if and only if $v \equiv 0$ (mod 3) [1].

(iii) There exists a 3–rotational $DTS(v)$ if and only if $v \equiv 1$ (mod 3) [1].

The spectrum of rotational directed triple systems is derived from the existence of 1– and 3–rotational systems: There exists a k–rotational $DTS(v)$ if and only if

$$\begin{cases} \mathrm{k} \equiv 1, 2 \pmod 3,\ v \equiv 0 \pmod 3 \text{ and } v \equiv 1 \pmod k \text{ or} \\ \mathrm{k} \equiv 0 \pmod 3 \text{ and } v \equiv 1 \pmod k. \end{cases}$$

We are concerned with automorphisms of the type $[f, 0, \ldots, 0, k, 0, \ldots, 0]$, $f \geq 3$. A $DTS_\alpha(v)$ with type $[\alpha] = [f,0,\ldots,0,k,0,\ldots,0]$ as an automorphism is said to be *(f,k)–near rotational* and is denoted by (f,k)–near $RDTS(v)$. A $(3,k)$–near $RDTS(v)$ is called a *k–near* $RDTS(v)$. In this paper, we give a necessary and sufficient condition for the existence of k–near $RDTS(v)$s.

2. One–near rotational directed triple systems

1–near rotational $DTS(v)$s admit an automorphism of the type [3, 0, $\ldots$, 0, 1, 0, 0, 0]. We will construct these on the set $X = \{\infty_1, \infty_2, \infty_3\} \cup Z_{v-3}$ with the automorphism $\alpha = (\infty_1)(\infty_2)(\infty_3)(0, 1, \ldots, v-4)$. It is easy to see the following necessary condition.

LEMMA 2.1. *If there exist a (f,k)–rotational $DTS(v)$, then*
(i) $v \equiv 0$ *or* 1 (mod 3),
(ii) $f \equiv 0$ *or* 1 (mod 3),
(iii) $v \equiv f \pmod k$,
(iv) $k[v(v-1) - f(f-1)] \equiv 0 \pmod{3(v-f)}$.

REMARK 2.2. If there exists a 1–near $RDTS(v)$, then $v \equiv 1, 4 \pmod 6$ or $v = 3$.

We now show that the above necessary condition is sufficient. We first require the use of the following four structures. An (A,k)*–system* is a set of ordered pairs $\{(a_r, b_r)|r = 1, 2, \ldots, k\}$ that partition the set $\{1, 2, \ldots, 2k\}$ with the property that $b_r - a_r = r$ for each $r = 1, 2, \ldots, k$. There exists an (A, k)–system if and only if $k \equiv 0$ or 1 (mod 4) [5, 6]. A *(B, k)–system* is a set of ordered pairs $\{(a_r, b_r)|r = 1, 2, \ldots, k\}$ that partition the set $\{1, 2, \ldots, 2k-1, 2k+1\}$ with the property that $b_r - a_r = r$ for each $r = 1, 2, \ldots, k$. There exists a (B, k)–system if and only if $k \equiv 2$ or 3 (mod 4) [4, 5]. A *(C, k)–system* is a set of ordered pairs $\{(a_r, b_r)|r = 1, 2, \ldots, k\}$ that partition the set $\{1, 2, \ldots, k, k+2, \ldots, 2k+1\}$ with the property that $b_r - a_r = r$ for each $r = 1, 2, \ldots, k$. There exists a (C, k)–system if and only if $k \equiv 0$ or 3 (mod 4) [5]. A *(D, k)–system* is a set of ordered pairs $\{(a_r, b_r)|r = 1, 2, \ldots, k\}$ that partition the set $\{1, 2, \ldots, k, k+2, \ldots, 2k, 2k+2\}$ with the property that $b_r - a_r = r$ for each $r = 1, 2, \ldots, k$. There exists a (D, k)–system if and only if $k \equiv 1$ or 2 (mod 4) and $k \neq 1$ [5].

It is trivial that there exists a 1–near $RDTS(3)$.

THEOREM 2.3 [see 5]. *There exists a cyclic Steiner triple system of order v if and only if $v \equiv 1$ or 3(mod 6) and $v \neq 9$.*

LEMMA 2.4. *If $v \equiv 4$ (mod 6), then there exists a 1–near $RDTS(v)$.*

PROOF. If $v = 6k + 4$, then there exists a cyclic Steiner triple system of order $6k + 1$ with k starter blocks. So let $\{0, a_r, b_r\}$, $r = 1, 2, \ldots, k$, be the starter blocks for a cyclic Steiner triple system of order $6k+1$ under the cyclic automorphism $(0, 1, \ldots, 6k)$. Then the following directed triples

$$\begin{array}{lll} [\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & [0, a_k, b_k] \\ [a_k, \infty_1, 0], & [b_k, \infty_2, 0], & [b_k - a_k, \infty_3, 0], \\ [0, a_r, b_r], & [b_r, a_r, 0], & r = 1, 2, \ldots, k-1, \end{array}$$

form a set of starter blocks for a 1–near $RDTS(v)$ under the automorphism α. □

LEMMA 2.5. *If $v \equiv 7$ or 13(mod 24), then there exists a 1–near $RDTS(v)$.*

PROOF. Let $v = 6k + 7$ and let $k \equiv 0$ or 1 (mod 4). Let $\{(a_r, b_r)|r = 1, 2, \ldots, k\}$ be an (A, k)–system. Then the following directed triples

$$\begin{array}{lll} [\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & \\ [0, \infty_1, 3k+2], & [0, \infty_2, 3k+1], & [0, \infty_3, 3k+3], \\ [0, r, b_r + k], & [b_r + k, r, 0], & r = 1, 2, \ldots, k \end{array}$$

form a set of starter blocks for a 1–near $RDTS(v)$ under the automorphism α. □

LEMMA 2.6. *If $v \equiv 1$ or $19 \pmod{24}$, then there exists a 1–near $RDTS(v)$.*

PROOF. Let $v = 6k+7$ and let $k \equiv 2$ or $3 \pmod 4$ and let $\{(a_r, b_r) | r = 1, 2, \ldots, k-1\}$ be a (B, k)–system. Then the following cyclic triples

$$\begin{array}{lll} [\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & \\ [0, \infty_1, 3k+2], & [0, \infty_2, 3k], & [0, \infty_3, 3k+4], \\ [0, r, b_r + k], & [b_r + k, r, 0], & r = 1, 2, \ldots, k \end{array}$$

form a set of starter blocks for a 1–near $RDTS(v)$ under the automorphism α. □

Remark 2.2 and Lemmas 2.4, 2.5 and 2.6 together yield the following theorem.

THEOREM 2.7. *Then there exists a 1–near $RDTS(v)$ if and only if $v \equiv 1, 4 \pmod 6$ or $v = 3$.*

3. Three–near rotational directed triple systems

It is easy to see that if there exists a k–near $RDTS(v)$, then $v \equiv 3 \pmod k$. Let us consider the existence of 3–near rotational directed triple systems. 3-near $RDTS(v)$s admit an automorphism α of the type $[\alpha] = [3, 0, \ldots, 0, 3, 0, \ldots, 0]$. We will construct these on the set $\{\infty_1, \infty_2, \infty_3\} \cup (Z_{\frac{v-3}{3}} \times \{1,2,3\})$ with the automorphism $\alpha = (\infty_1)(\infty_2)(\infty_3)(0_1, 1_1, \ldots, (\frac{v-3}{3} - 1)_1)(0_2, 1_2, \ldots, (\frac{v-3}{3} - 1)_2)(0_3, 1_3, \ldots, (\frac{v-3}{3} - 1)_3)$ where we write for brevity x_i instead of (x, i). First of all, we have immediately the following necessary condition for the existence of a 3–near $RDTS(v)$.

THEOREM 3.1. *If there exists a 3–near $RDTS(v)$, then $v \equiv 0 \pmod 3$.*

PROOF. Since $v \equiv 0$ or $1 \pmod 3$ and $v - 3$ is divisible by 3, we have $v \equiv 0 \pmod 3$. □

LEMMA 3.2. *There exists a 3–near $RDTS(30)$.*

PROOF. The following directed triples

$$
\begin{array}{lll}
[\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & [0_3, 0_2, 0_1], \\
[0_1, \infty_1, 2_1], & [2_1, \infty_2, 0_1], & [0_2, \infty_1, 2_2], \\
[2_2, \infty_2, 0_2], & [0_3, \infty_1, 2_3], & [2_3, \infty_2, 0_3], \\
[0_1, \infty_3, 0_2], & [0_2, 0_3, \infty_3], & [\infty_3, 0_1, 0_3], \\
[0_i, 1_i, 4_i], & [4_i, 1_i, 0_i], & i = 1, 2, 3, \\
[0_1, x_2, (2x)_3], & [(2x)_3, x_2, 0_1], & x = 1, 2, \dots, 8
\end{array}
$$

form a set of starter blocks for a 3–near $RDTS(30)$ under the automorphism α. □

LEMMA 3.3. *If* $v \equiv 12 \pmod{18}$, *then there exists a* 3*–near* $RDTS(v)$.

PROOF. Let $v = 18t + 12$. The case $t = 1$ has been treated in Lemma 3.2. Let $t > 1$ and let $\{(a_r, b_r) | r = 1, 2, \dots, t\}$ be a (C, t)–system if $t \equiv 0$, $3 \pmod 4$ or a (D, t)–system if $t \equiv 1$, $2 \pmod 4$. Then the following directed triples

$$
\begin{array}{lll}
[\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & [0_3, 0_2, 0_1], \\
[0_1, \infty_1, (2t+1)_1], & [(2t+1)_1, \infty_2, 0_1], & [0_2, \infty_1, (2t+1)_2], \\
[(2t+1)_2, \infty_2, 0_2], & [0_3, \infty_1, (2t+1)_3], & [(2t+1)_3, \infty_2, 0_3], \\
[0_1, \infty_3, 0_2], & [0_2, 0_3, \infty_3], & [\infty_3, 0_1, 0_3], \\
[0_i, r_i, (b_r + t)_i], & [(b_r + t)_i, r_i, 0_i], & i = 1, 2, 3, \quad r = 1, 2, \dots, t, \\
[0_1, x_2, (2x)_3], & [(2x)_3, x_2, 0_1], & x = 1, 2, \dots, 6t + 2
\end{array}
$$

form a set of starter blocks for a 3-near $RDTS(v)$ under the automorphism α. □

LEMMA 3.4. *If* $v \equiv 6 \pmod{18}$, *then there exists a* 3*–near* $RDTS(v)$.

PROOF. Let $v = 18t + 6$ and let $\{(a_r, b_r) | r = 1, 2, \dots, t\}$ be an (A, t)–system if $t \equiv 0$, $1 \pmod 4$ or a (B, t)–system if $t \equiv 2, 3 \pmod 4$. Then the following directed triples

$$
\begin{array}{lll}
[\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & [2_3, 1_2, 0_1], \\
[0_1, \infty_1, 0_2], & [0_3, \infty_2, 0_2], & [0_1, \infty_3, 1_2], \\
[0_2, 0_3, \infty_1], & [0_2, 0_1, \infty_2], & [1_2, 2_3, \infty_3], \\
[\infty_1, 0_1, 0_3], & [\infty_2, 0_3, 0_1], & [\infty_3, 0_1, 2_3], \\
[0_i, r_i, (b_r + t)_i], & [(b_r + t)_i, r_i, 0_i], & i = 1, 2, 3, \quad r = 1, 2, \ldots, t, \\
[0_1, x_2, (2x)_3], & [(2x)_3, x_2, 0_1], & x = 2, 3, \ldots, 6t
\end{array}
$$

form a set of starter blocks for a 3–near $RDTS(v)$ under the automorphism α. □

LEMMA 3.5. *If* $v \equiv 18 \pmod{36}$, *then there exists a* 3*–near* $RDTS(v)$.

PROOF. Let $v = 18t + 18$ and $t \equiv 0 \pmod 2$. Then $2t + 1 \equiv 1$ (mod 4); so let $\{(a_r, b_r) | r = 1, 2, \ldots, 2t + 1\}$ be an $(A, 2t + 1)$–system. Then the following directed triples

$$
\begin{array}{lll}
[\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & \\
[(4t+3)_i, \infty_1, 0_1], & i = 1, 2, 3, & \\
[0_i, \infty_2, (4t+3)_i], & i = 1, 2, 3, & \\
[0_1, \infty_3, 0_2], & [\infty_3, 0_1, 0_3], & [0_2, 0_3, \infty_3], \\
[0_3, 0_2, 0_1], & & \\
[0_1, x_2, (2x)_3], & [(2x)_3, x_2, 0_1], & x = 1, 2, \ldots, 6t + 4, \\
[r_i, 0_i, (b_r)_i], & i = 1, 2, 3, & r = 1, 2, \ldots, 2t + 1
\end{array}
$$

form a set of starter blocks for a 3–near $RDTS(v)$ under the automorphism α. □

LEMMA 3.6. *If* $v \equiv 0 \pmod{36}$, *then there exists a* 3*–near* $RDTS(v)$.

PROOF. Let $v = 18t + 18$ and $t \equiv 1 \pmod 2$. Then $2t + 1 \equiv 3$ (mod 4); so let $\{(a_r, b_r) | r = 1, 2, \ldots, 2t + 1\}$ be an $(B, 2t + 1)$–system. Then the following directed triples

$$\begin{array}{lll}
[\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & \\
[(4t+2)_i, \infty_1, 0_1], & i = 1,2,3, & \\
[0_i, \infty_2, (4t+2)_i], & i = 1,2,3, & \\
[0_1, \infty_3, 0_2], & [\infty_3, 0_1, 0_3], & [0_2, 0_3, \infty_3], \\
[0_3, 0_2, 0_1], & & \\
[0_1, x_2, (2x)_3], & [(2x)_3, x_2, 0_1], & x = 1,2,\ldots,6t+4, \\
[r_i, 0_i, (b_r)_i], & i = 1,2,3, & r = 1,2,\ldots,2t+1
\end{array}$$

form a set of starter blocks for a 3–near $RDTS(v)$ under the automorphism α. □

LEMMA 3.7. *If $v \equiv 3$ or $9 (\mathrm{mod}\ 24)$, then there exists a 3–near $RDTS(v)$.*

PROOF. Let $v = 6k+3$ and $k \equiv 0$ or 1 (mod 4), and let $\{(a_r, b_r) | k = 1, 2, \ldots, k-1\}$ be a $(C, k-1)$–system. Then the following directed triples

$$\begin{array}{llll}
[\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & & \\
[0_i, \infty_1, k_i], & i = 1,2,3, & & \\
[0_1, \infty_2, k_2], & [0_2, \infty_2, k_3], & [0_3, \infty_2, k_1], & \\
[k_2, \infty_3, 0_1], & [k_3, \infty_3, 0_2], & [k_1, \infty_3, 0_3], & \\
[0_1, 0_2, 0_3], & [0_3, 0_2, 0_1], & & \\
[0_i, r_i, (b_r)_{i+1}], & [(b_r)_{i+1}, r_i, 0_i], & i = 1,2,3, & r = 1,2,\ldots,k-1
\end{array}$$

form a set of starter blocks for a 3–near $RDTS(v)$ under the automorphism α. □

LEMMA 3.8. *If $v \equiv 15$ (mod 24), then there exists a 3–near $RDTS(v)$.*

PROOF. Let $v = 6k+3$ and $k \equiv 2 (\mathrm{mod}\ 4)$, and let $\{(a_r, b_r) | k = 1, 2, \ldots, k-1\}$ be a $(A, k-1)$–system. Then the following directed triples

$$
\begin{array}{lll}
[\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & \\
[0_i, \infty_1, k_i], & i = 1, 2, 3, & \\
[0_1, \infty_2, (2k-1)_2], & [0_2, \infty_2, (2k-1)_3], & [0_3, \infty_2, (2k-1)_1], \\
[(2k-1)_2, \infty_3, 0_1], & [(2k-1)_3, \infty_3, 0_2], & [(2k-1)_1, \infty_3, 0_3], \\
[0_1, 0_2, 0_3], & [0_3, 0_2, 0_1], & \\
[0_i, r_i, (b_r)_{i+1}], & [(b_r)_{i+1}, r_i, 0_i], & i = 1, 2, 3, \quad r = 1, 2, \ldots, k-1
\end{array}
$$

form a set of starter blocks for a 3–near $RDTS(v)$ under the automorphism α. □

LEMMA 3.9. *If* $v \equiv 21 \pmod{24}$, *then there exists a* 3*–near* $RDTS(v)$.

PROOF. Let $v = 6k + 3$ and $k \equiv 3 \pmod 4$, and let $\{(a_r, b_r) | k = 1, 2, \ldots, k-1\}$ be a $(B, k-1)$-system. Then the following directed triples

$$
\begin{array}{lll}
[\infty_1, \infty_2, \infty_3], & [\infty_3, \infty_2, \infty_1], & \\
[0_i, \infty_1, k_i], & i = 1, 2, 3, & \\
[0_1, \infty_2, (2k-2)_2], & [0_2, \infty_2, (2k-2)_3], & [0_3, \infty_2, (2k-2)_1], \\
[(2k-2)_2, \infty_3, 0_1], & [(2k-2)_3, \infty_3, 0_2], & [(2k-2)_1, \infty_3, 0_3], \\
[0_1, 0_2, 0_3], & [0_3, 0_2, 0_1], & \\
[0_i, r_i, (b_r)_{i+1}], & [(b_r)_{i+1}, r_i, 0_i], & i = 1, 2, 3, \quad r = 1, 2, \ldots, k-1
\end{array}
$$

form a set of starter blocks for a 3–near $RDTS(v)$ under the automorphism α. □

We can now conclude the following theorem.

THEOREM 3.10. *There exist a* 3*–near* $RDTS(v)$ *if and only if* $v \equiv 0 \pmod 3$ *or* $v = 4$.

4. Concluding remark

REMARK 4.1. If a permutation α of degree v is of type [3, 0, 0, $\cdots$, 0, 1, 0, 0, 0] and $v \equiv 3 \pmod k$, then α^k is of type [3, 0, 0, $\cdots$, 0, k, 0, $\cdots$, 0] which is a permutation consisting of exactly three fixed elements and k cycles of length $\frac{v-3}{k}$. If $k \equiv 1$ or $2 \pmod 3$ and there exists a k–near $RDTS(v)$, then $v \equiv 1 \pmod 3$ since $k(v+2) \equiv 0 \pmod 3$; and if $k \equiv 0 \pmod 3$ and there exists a k–near $RDTS(v)$, then $v \equiv 0 \pmod 3$ since $v \equiv 3 \pmod k$. Thus , the existence of a 1–near $RDTS(v)$ implies the existence of a k–near $RDTS(v)$, provided that $v \equiv 1 \pmod 3$, $v \equiv 3 \pmod k$, and $k(v+2) \equiv 0 \pmod 3$ for $k \equiv 1$ or $2 \pmod 3$, and the existence of a 3–near $RDTS(v)$ implies the existence of a k–near $RDTS(v)$, provided that $v \equiv 0 \pmod 3$ and $v \equiv 3 \pmod k$ for $k \equiv 0 \pmod 3$.

Therefore, we can conclude the following theorem.

THEOREM 4.2. *There exists a k–near $RDTS(v)$ if and only if*
(i) $v \equiv 0$ *or* $1 \pmod 3$,
(ii) $v \equiv 3 \pmod k$, $k > 1$,
(iii) $k(v+2) \equiv 0 \pmod 3$.

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