Comm. Korean Math. Soc. **16** (2001), No. 2, pp. 277–285

A PROOF OF A CONVEX-VALUED SELECTION THEOREM WITH THE CODOMAIN OF A FRÉCHET SPACE

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ABSTRACT. The purpose of this paper is to give a proof of a generalized convex-valued selection theorem which is given by weakening a Banach space to a completely metrizable locally convex topological vector space, i.e., a Fréchet space. We also develop the properties of upper semi-continuous singlevalued mappings to those of upper semi-continuous multivalued mappings. These properties will be applied in our further considerations of selection theorems.

1. Introduction

Let X and Y be topological spaces, and 2^Y be the family of nonempty subsets of Y. A mapping $F : X \to 2^Y$ is called a *set-valued mapping.* A *selection* for $F : X \to 2^Y$ is a map $f : X \to Y$ such that $f(x) \in F(x)$ for every $x \in X$. Of course the axiom of choice (which in this paper is assumed as part of the axiomatics) guarantees that $F(x)$ admits a selection. However, if we look for selections that satisfy some regularity condition, like continuity, the problem of existence becomes more difficult.

In this paper we concentrate our attention to the question of existence of continuous selections. A set-valued mapping $F : X \to 2^Y$ is called *lower semi-continuous* (respectively, *upper semi-continuous*) or l.s.c. (respectively, u.s.c.) if for every open subset V of Y,

$$F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$$

Received December 27, 1999.

2000 Mathematics Subject Classification: 54B20, 54C60, 54C65.

Key words and phrases: Fréchet space, selection, selective, semi-continuous.

The first author's research was supported by Wonkwang University Research Grant, 2000.

$$(\text{respectively}, F^{\#}(V) = \{x \in X : F(x) \subset V\})$$

is an open subset of X.

The fundamental results in selection theory stemmed from the mid 1950's by E. Michael ([6], [7], [8], [9], [10]). Most of the classical Michael's selection theorems establish that existence of continuous selections for lower semicontinuous set-valued mapping $F : X \to 2^Y$ with non-empty convex values which is equivalent to some higher separation axioms (i.e., paracompactness, collectionwise normality, normality, etc) of X. As classical theorems, there are four main selection theorems: convex-valued, zero-dimensional, compact-valued, and finite-dimensional theorems. These theorems were obtained by different authors with various methods ([2], [4], [5], [11], [14]). Moreover, it was pointed out that the proof of some of these theorems were erroneous and this has not been recognized for a long time (see [12], [13]).

First of all, consider the following general question: Under what conditions on the topological spaces X and Y, on the family of subsets of Y where the multivalued map $F : X \to 2^Y$ takes its values, and on the type of continuity of the multivalued map F, does F have a continuous single-valued selection?

The following theorem is called a convex-valued selection theorem: Let X be a Hausdorff paracompact space, B a Banach space and $F : X \to 2^B$ a lower semicontinuous mapping with nonempty closed convex values. Then F admits a continuous singlevalued selection.

The purpose of this paper is to give a proof of a generalized convex-valued selection theorem which is given by weakening a Banach space to a completely metrizable locally convex topological vector space, i.e., a Fréchet space. We also develop properties of upper semi-continuous singlevalued mappings to those of upper semi-continuous multivalued mappings. These properties will be applied in our further considerations of selection theorems.

Throughout this paper, by a space and a selection we always mean a T_1 topological space and a continuous selection respectively. Also the upper-case-letters F, G, and H, etc., denote multivalued mappings and the lower-case-letters f, g, and h, etc., denote singlevalued mappings. As far as topological concepts are concerned, we follow [1] and [3].

2. Preliminaries

We first introduce some terminologies which will be used throughout the rest of this paper. A Hausdorff space X is said to be *paracompact* if every open covering of X admits a locally finite open refinement. A family $\{e_\alpha\}_{\alpha \in A}$ of nonnegative continuous functions on a topological space X is said to be a *locally finite partition of unity* if for every $x \in X$, there exists a neighborhood W of x and a finite subset $A(x) \subset A$ such that $\sum_{\alpha \in A(x)} e_\alpha(y) = 1$ for all $y \in W$ and $e_\alpha(y) = 0$ for $y \notin W$ and $\alpha \in A(x)$. A locally finite partition of unity $\{e_\alpha\}_{\alpha \in A}$ is said to be *inscribed* into an open covering $\{G_\gamma\}_{\gamma \in \Gamma}$ of a topological space X if for any $\alpha \in A$, there exists $\gamma \in \Gamma$ such that

$$\operatorname{supp}(e_\alpha) = \overline{\{x \in X | e_\alpha(x) > 0\}} \subset G_\gamma.$$

It is well-known([3]) that a Hausdorff space X is paracompact if and only if each open covering of X admits a locally finite partition of unity inscribed into this covering. A *topological vector space* is a pair $(E, \mathcal{T})$, where E is a vector space and $\mathcal{T}$ is a topology in E such that the vector operations $(x, y) \mapsto x + y$, $(\lambda, x) \mapsto \lambda x$ are continuous mappings with respect to the topology $\mathcal{T}$. A topological vector space E is called *locally convex* if there exists a local basis $\mathcal{B}_0$ at the origin consisting of convex subsets of E. The intersection of all convex subsets of E which contain a subset S is called the *convex hull* of S and is denoted by $\operatorname{conv} S$.

A prototype of an upper semicontinuous multivalued mapping is $F : \mathbb{R} \to 2^{\mathbb{R}}$ with non-empty convex closed values defined by

$$F(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{+1\} & \text{if } x > 0. \end{cases}$$

It is easy to see that this multivalued mapping cannot have a continuous selection. So the class of upper semicontinuous multivalued mappings does not seem to be the right one for the continuous selection problem. However, we have the sufficient condition for the lower semicontinuity of a given multivalued mapping(see [8], [15]).

THEOREM 2.1. ([8], [15]) *If $F : X \to 2^Y$ is lower semicontinuous, W is open in Y, and $F(x) \cap W \neq \emptyset$ for each $x \in X$, then the mapping $G : X \to 2^Y$ defined by $G(x) = F(x) \cap W$ is lower semicontinuous.*

In the following theorem, (Y, ρ) is a topological space with a metric ρ and $D(x, \epsilon)$ means an open ball with center x and radius ϵ.

THEOREM 2.2. ([15]) *Let* $F : X \to 2^Y$ *be a lower semicontinuous mapping of* X *into a metric space* (Y, ρ) *and let* $f : X \to Y$ *be a singlevalued continuous mapping such that for some* $\epsilon > 0$, $F(x) \cap D(f(x), \epsilon) \neq \emptyset$ *for each* $x \in X$. *Then the mapping* $G : X \to 2^Y$ *defined by* $G(x) = F(x) \cap D(f(x), \epsilon)$ *is lower semicontinuous.*

3. A Generalization of Convex-valued Selection Theorem

In this section, we will prove a generalized convex-valued selection theorem obtained by weakening a Banach space to a completely metrizable locally convex topological vector space. The main idea of a proof of the theorem is to consider a well-known method of outside approximations which gives another proof of the convex-valued selection theorem (see [15]).

DEFINITION 3.1. Let $F : X \to 2^B$ be a multivalued mapping of a topological space X into a locally convex topological vector space B. Then a singlevalued mapping $f : X \to B$ is said to be an V*-selection* of F if $F(x) \cap (f(x) + V) \neq \emptyset$ for all $x \in X$, where V is a convex neighborhood of the origin $O \in B$.

PROPOSITION 3.2. *Let* X *be a paracompact space,* B *a locally convex topological vector space, and* $F : X \to 2^B$ *a convex-valued lower semicontinuous map. Then for every convex neighborhood* V *of the origin* $O \in B$, *there exists a continuous singlevalued* V*-selection* $f_V : X \to B$ *of the map* F.

PROOF. Let V be a convex neighborhood of the origin $O \in B$. Then $V_y = y + V$ is a convex neighborhood of y for all $y \in B$. Let $U_y = F^{-1}(V_y) = \{x \in X | F(x) \cap V_y \neq \emptyset\}$. Then $\{U_y\}_{y \in B}$ is an open covering of X. Since X is paracompact, there exists a locally finite partition of unity $\{e_\alpha\}_{\alpha \in A}$ inscribed into $\{U_y\}_{y \in B}$. Let y_α be an arbitrary element of B such that $\text{supp}(e_\alpha) \subset U_{y_\alpha}$ and let $f_V(x) = \sum_{\alpha \in A} e_\alpha(x) \cdot y_\alpha$. Then f_V is a well-defined continuous mapping since $f_V(x)$ is a sum of a finite number of continuous mappings $e_\alpha(x) \cdot y_\alpha$ in some neighborhood of x.

Claim : $(f_V(x) + V) \cap F(x) \neq \emptyset$ for all $x \in X$.

For a given $x \in X$, let $\{\alpha \in A | x \in \text{supp}(e_\alpha)\} = \{\alpha_1, \cdots, \alpha_n\}$. Then $x \in \text{supp}(e_{\alpha_i}) \subset U_{y_{\alpha_i}}$, i.e., $F(x) \cap V_{y_{\alpha_i}} \neq \emptyset$. Let $z_i \in F(x) \cap V_{y_{\alpha_i}}$

for each $i = 1, 2, \cdots, n$ and let $z = \sum_{i=1}^{n} e_{\alpha_i}(x) \cdot z_i$. By the convexity of $F(x)$, $z \in F(x)$. Also $z_i \in y_{\alpha_i} + V$, i.e., $z_i - y_{\alpha_i} \in V$ for each $i = 1, 2, \cdots, n$. Since V is convex, $\sum_{i=1}^{n} e_{\alpha_i}(x)(z_i - y_{\alpha_i}) = z - f_V(x) \in V$. Hence $z \in f_V(x) + V$. Thus $(f_V(x) + V) \cap F(x) \neq \emptyset$. □

PROPOSITION 3.3. *Let X be a paracompact space, B a locally convex metrizable space and $F : X \to 2^B$ a convex-valued lower semicontinuous map. Then for every countable basis $\{V_n\}$ of convex neighborhoods of the origin $O \in B$, where $\operatorname{diam} V_n$ converges to zero, there exists a uniform Cauchy sequence $\{f_n\}$ of continuous singlevalued V_n-selections $f_n : X \to B$ of the map F.*

PROOF. Let ρ be a metric on B which induces the topology on B. We shall construct by induction a sequence of continuous lower semicontinuous mappings $\{F_n : X \to 2^B\}_{n \in \mathbb{N}}$ and a sequence of continuous singlevalued mappings $\{f_n : X \to B\}_{n \in \mathbb{N}}$ such that :

(i) $F(x) = F_0(x) \supset F_1(x) \supset \cdots \supset F_n(x) \supset F_{n+1}(x) \supset \cdots$, for all $x \in X$;

(ii) $\operatorname{diam} F_n(x) \leq \operatorname{diam} V_n$;

(iii) f_n is a V_n-selection of the mapping F_{n-1} for every $n \in \mathbb{N}$.

Base of induction : We apply Proposition 3.2 for the spaces X and B, the mapping $F = F_0$, and for $V = V_1$. Then there exists a V_1-selection f_1 of F_0. Let $F_1(x) = F_0(x) \cap (f_1(x) + V_1)$. Then $F_1(x)$ is a nonempty convex subset of $F_0(x)$, $\operatorname{diam} F_1(x) \leq \operatorname{diam}(f_1(x) + V_1) = \operatorname{diam} V_1$, and by Theorem 2.1, $F_1 : X \to 2^B$ is lower semicontinuous.

Inductive step : Suppose that $F_1, F_2, \cdots, F_{m-1}, f_1 \cdots, f_{m-1}$ have properties (i)-(iii). We apply Proposition 3.2 for spaces X and B, mapping F_{m-1} and for the convex neighborhood V_m. Let $F_m(x) = F_{m-1} \cap (f_m(x) + V_m)$. Then $F_m(x)$ is a nonempty convex subset of $F_{m-1}(x)$, $\operatorname{diam} F_m(x) \leq \operatorname{diam}(f_m(x) + V_m) = \operatorname{diam} V_m$, and by Theorem 2.1, $F_m : X \to 2^B$ is lower semicontinuous.

Claim : $\{f_n\}_{n \in \mathbb{N}}$ is a uniform Cauchy sequence of continuous singlevalued V_n-selections $f_n : X \to B$ of F.

Since $F_{m-1}(x) \subset F(x)$, f_n is a continuous V_n-selection of F. For every $n, p \in \mathbb{N}$ and $x \in X$, we can choose $y_1 \in F_{n-1}(x) \cap (f_n(x) + V_n)$ and $y_2 \in F_{n+p-1}(x) \cap (f_{n+p}(x) + V_{n+p})$. Note that $F_{n+p-1}(x) \subset F_{n-1}(x)$

and $\operatorname{diam} F_{n-1}(x) \leq \operatorname{diam} V_{n-1}$. Then

$$\begin{aligned}\rho(f_n(x), f_{n+p}(x)) &\leq \rho(f_n(x), y_1) + \rho(y_1, y_2) + \rho(y_2, f_{n+p}(x)) \\ &\leq \operatorname{diam} V_n + \operatorname{diam} F_{n-1}(x) + \operatorname{diam} V_{n+p} \\ &\leq \operatorname{diam} V_n + \operatorname{diam} V_{n-1} + \operatorname{diam} V_{n+p}.\end{aligned}$$

Since $\operatorname{diam} V_n$ converges to zero, $\{f_n\}_{n\in\mathbb{N}}$ is a uniform Cauchy sequence.□

THEOREM 3.4. *Let X be a paracompact space, B a completely metrizable locally convex space, i.e., a Fréchet space, and $F : X \to 2^B$ a lower semicontinuous map with closed convex values. Then F admits a continuous singlevalued selection.*

PROOF. Let ρ be a complete metric on B which induces the topology on B. Choose a countable basis $\{V_n\}$ of convex neighborhoods of the origin $O \in B$ where $\operatorname{diam} V_n$ converges to zero, and let $\{f_n\}_{n\in\mathbb{N}}$ be a uniform Cauchy sequence of continuous singlevalued V_n-selections $f_n : X \to B$ of F constructed in Proposition 3.3.

For $x \in X$, pick $\epsilon > 0$ and $N \in \mathbb{N}$ such that $\operatorname{diam} V_n < \frac{\epsilon}{3}$ and $\rho(f_n(x), f_{n+p}(x)) < \frac{\epsilon}{3}$ for all $n > N$ and $p \in \mathbb{N}$. For each $n \in \mathbb{N}$, we can find an element $z_n(x) \in F(x)$ such that $z_n(x) \in (f_n(x) + V_n)$. Hence

$$\begin{aligned}\rho(z_n(x), z_{n+p}(x)) &\leq \rho(z_n(x), f_n(x)) + \rho(f_n(x), f_{n+p}(x)) \\ &\quad + \rho(f_{n+p}(x), z_{n+p}(x)) \\ &< \operatorname{diam} V_n + \frac{\epsilon}{3} + \operatorname{diam} V_{n+p} < \epsilon.\end{aligned}$$

Therefore $\{z_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete subspace $F(x)$ of the metric space B and there exists $\lim_{n\to\infty} z_n(x) = z(x) \in F(x)$. Finally, $\lim_{n\to\infty} \rho(z_n(x), f_n(x)) = 0$ because $\operatorname{diam} V_n$ converges to zero. So there exists $\lim_{n\to\infty} f_n(x) = f(x)$ and $z(x) = f(x)$. Hence $f(x) \in F(x)$ and the map f is continuous as the pointwise limit of a uniform Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions. □

COROLLARY 3.5. (Convex-valued Selection Theorem) *Let X be a paracompact space, B a Banach space and $F : X \to 2^B$ a lower semicontinuous mapping with nonempty closed convex values. Then F admits a continuous singlevalued selection.*

PROOF. It follows directly from the fact that every Banach space is a completely metrizable locally convex topological vector space. □

4. More on Michael's Selection Theorems

Historically, the two kinds of semi-continuity, lower semi-continuity and upper semi-continuity, of a multivalued mapping were introduced independently by Kuratowski and Bouligrand in 1932. In general, the definitions given by different authors do not coincide whenever we deal with non-compact spaces (at least for upper semi-continuity, which is more important from the point of view of applications). The definitions adopted in section 4 are given by C. Berge, i.e., a multivalued mapping $F : X \to 2^Y$ is said to be upper semi-continuous on X if it is upper semi-continuous at every point in X and if also, $F(x)$ is compact for each $x \in X$.

In this section, we develop properties of upper semi-continuous singlevalued mappings to those of upper semi-continuous multivalued mappings. These properties will be applied in our further considerations of selection theorems. Of course a multivalued mapping is called continuous if it is both lower semi-continuous and upper semi-continuous.

The following theorem is well-known as a Maximum Theorem:

LEMMA 4.1. *If $f : Y \to \mathbb{R}$ is continuous and $F : X \to 2^Y$ is continuous, then the real-valued function m defined by $m(x) = \max\{f(y) \mid y \in F(x)\}$ is continuous on X and the mapping G defined by $G(x) = \{y \mid y \in F(x),\ f(y) = m(x)\}$ is an u.s.c. mapping of X into 2^Y.*

Let $f : Y \to \mathbb{R}$ be a continuous function defined on a topological space Y. A family $\mathcal{K} = \{K_\gamma \mid \gamma \in \Gamma\}$ of compact subsets of Y is called *selective* with respect to f if for each $\gamma \in \Gamma$ there exists one and only one $a_\gamma \in K_\gamma$ such that $f(a_\gamma) = \max\{f(y) \mid y \in K_\gamma\}$.

In other words, the maximum of f is attained at only one point of the set K_γ.

For example, every family of closed balls in $\mathbb{R}^n$ is selective with respect to $f(y) = pr_n(y)$, where pr_n is the projection map onto the nth coordinate space. In particular, every family of compact sets in $\mathbb{R}$ is selective with respect to $f(y) = y$.

THEOREM 4.2. *Let $F : X \to 2^Y$ be a continuous mapping. If the family $\{F(x) \mid x \in X\}$ is selective, then there exists a continuous selection f for F.*

PROOF. Let $g : Y \to \mathbb{R}$ be continuous for which $\{F(x) \mid x \in X\}$ is selective. If $G(x) = \{y \mid y \in F(x), g(y) = m(x)\}$, where m is the function defined in Lemma 4.1, then G is a single-valued mapping of X

into Y. Moreover, it is u.s.c. by Lemma 4.1 and so is continuous (since it is single-valued).

The mapping $f(x) = G(x)$ satisfies $f(x) \in F(x)$ for every $x \in X$, i.e., f is the continuous selection f for F. □

COROLLARY 4.3. *If $F : X \to 2^{\mathbb{R}}$ is continuous, then there exists a continuous selection f for F.*

PROOF. It is sufficient to take $f(y) = y$, whence $f(x) = \max F(x)$.□

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