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THE ENERGY INEQUALITY OF A QUASILINEAR HYPERBOLIC MIXED PROBLEM

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ABSTRACT. In this paper, we establish the energy inequalities for second order quasilinear hyperbolic mixed problems in the domain $\mathbf{R}^n_-$.

1. Introduction

Let u be a smooth enough solution to a quasilinear hyperbolic mixed problem:

$$
(1)\qquad \begin{aligned} &P_2(t,x,u,Du,D)u = f(t,x,u,Du) && \text{in} \quad \Omega,\\ &u(t,x',0) = g(t,x') && \text{on} \quad \partial\Omega,\\ &u(0,x) = u_0, \quad u_t(0,x) = u_1, \end{aligned}
$$

where $\Omega = \mathbf{R}^{n+1}_+ = \{(t,x',x_n)|(t,x') \in \mathbf{R}^n, x_n > 0\}$ and $\partial\Omega = \Omega \cap \{(t,x',x_n)|x_n = 0\}$ is the boundary of Ω, and

$$
(2)\qquad P_2(t,x,u,Du,D) \equiv (\partial_t^2 - \sum_{\substack{(i,j)\neq(0,0)\\ i,j=0}}^{n} a_{ij}(t,x,u,Du)\partial_{x_i}\partial_{x_j})
$$

is strictly hyperbolic with respect to $\{t = constant\}$, f and g are smooth functions of its arguments and (a_{ij}) is symmetric. In this paper, we consider appropriate energy inequalities for equation (1). This result will be effectively used in the proof of the regularity of conormal solutions to quasilinear hyperbolic mixed problems.

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2. Preliminaries

Before we prove the energy inequalities we need the following remark and lemmas.

We note that strictly hyperbolicity of $P_2(t,x,u,Du,D)$ implies the following inequality :

$$|\xi|^2 \le C(\xi_0^2 + \sum_{i,j=1}^{n} a_{ij}(t,x,u,Du)\xi_i\xi_j), \tag{3}$$

where C is a positive constant and $\xi = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbf{R}^{n+1}$.

Throughout this thesis, we use

$$\sum_{(i,j)\neq(0,0)}^{n} \quad \text{instead of} \quad \sum_{\substack{(i,j)\neq(0,0)\\ i,j=0}}^{n} \quad \text{for notational convenience.}$$

For the proof of Schauder's lemma, see Rauch [9], and for the Gagliardo-Nirenberg inequalities, see Nirenberg [8].

LEMMA 2.1. (Schauder's lemma) *If* $u, v \in H^s(\mathbf{R}^n)$ *and* $s > \frac{n}{2}$, *then* $uv \in H^s(\mathbf{R}^n)$ *and* $\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s}$.

LEMMA 2.2. (Gagliardo-Nirenberg inequalities) *Let* $u \in L^q(\mathbf{R}^n)$ *and its derivative of order* m, $D^m u$, *belong to* $L^r(\mathbf{R}^n)$, $1 \le q, r \le \infty$. *For the derivatives* $D^j u$, $0 \le j < m$, *the following inequalities hold*

$$\|D^j u\|_p \le constant \|D^m u\|_r^a \|u\|_q^{1-a}, \tag{4}$$

where $1/p = j/n + a(1/r - m/n) + (1-a)1/q$, *for all* a *in the interval* $j/m \le a \le 1$ *(the constant depending only on* n, m, j, q, r *and* a), *with the following exceptional cases:*

1. *If* $j = 0$, $rm < n$, $q = \infty$, *then we make the additional assumption that either* u *tends to 0 at infinity or* $u \in L^{\tilde{q}}(\mathbf{R}^n)$ *for some finite* $\tilde{q} > 0$.

2. *If* $1 < r < \infty$, *and* $m - j - n/r$ *is a nonnegative integer, then (4) holds only for* a *satisfying* $j/m \le a < 1$.

Throughout this paper we treat the case in which the regularity indices s and s' are integers; analogous results hold in the general case.

LEMMA 2.3. *Let* $0 \leq s' \leq s$ *and suppose that* $w \in L^\infty(\mathbf{R}^n) \cap H^s(\mathbf{R}^n)$.

Then for $|\alpha| = s'$, *it follows that* $D^\alpha w \in L^{2p}(\mathbf{R}^n)$ *and*

$$\|D^\alpha w\|_{L^{2p}} \leq C(\|w\|_{L^\infty})^{1-\frac{1}{p}}(\|w\|_{H^s})^{\frac{1}{p}},$$

where $p = s/s'$ *and* C *is a constant depending only on* s, s' *and* n.

COROLLARY 2.4. *Suppose that* $w \in L^\infty_{loc}(\mathbf{R}^n) \cap H^s_{loc}(\mathbf{R}^n)$ *for* $s \geq 0$. *If* $f \in C^\infty(\mathbf{R})$, *then* $f(w) \in H^s_{loc}(\mathbf{R}^n)$.

3. Energy inequality

Since the equation (1) satisfies the uniform Lopatinski condition, the equation is well-posed. See Chazarain-Piriou [5], or Sakamoto [10] for the definition of the uniform Lopatinski condition and the existence of solutions of the equation (1). Now we state and prove the main theorem for equation (1).

THEOREM 1. (Energy Inequality) *Let* $P_2(t,x,u,Du,D)$ *be a partial differential operator of order* 2 *on* Ω, *as in (2), strictly hyperbolic with respect to the planes* $\{t = constant\}$ *and let* $u(t,x) \in C(\mathbf{R}; H^s_{loc}(\Omega)) \cap C^1(\mathbf{R}; H^{s-1}_{loc}(\Omega))$, $s > \frac{n}{2} + 2$, *satisfy the equation*

$$(5) \qquad \begin{aligned} &P_2(t,x,u,Du,D)u = f(t,x,u,Du) && \text{in } \Omega, \\ &u(t,x',0) = g(t,x') && \text{on } \partial\Omega, \end{aligned}$$

where f *is smooth function of its arguments. If* $u(0,x) \in H^s_{loc}(\Omega)$ *and* $u_t(0,x) \in H^{s-1}_{loc}(\Omega)$ *and the boundary* $\partial\Omega$ *of* Ω *is not characteristic hypersurface of* P_2, *then, for all* t, $u(t,x) \in H^s_{loc}(\Omega)$. *Moreover, if* u *has compact support in* x *for each time* t, *then we have*

$$(6) \qquad \|u(t,x)\|_{H^s(\Omega)} \leq C_t\left(\|u(0,x)\|_{H^s(\Omega)} + \|u_t(0,x)\|_{H^{s-1}(\Omega)}\right)$$

for all t, *where* $C_t = C(t)$ *is independent of* u *and* f.

PROOF. By finite propagation speed and an analysis local in time, it may be assumed that u has compact support in x for each t. Let $|\alpha| = s-1$. We apply $D^\alpha \equiv \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$ to equation (1) obtaining an equation, by the Leibniz formula,

$$(7) \qquad P_2(D^\alpha u) = D^\alpha f + \sum_{\beta<\alpha} \sum_{(i,j)\neq(0,0)}^{n} C_\beta (D^{\alpha-\beta} a_{ij})(D^\beta u_{x_i x_j}) \equiv h,$$

where C_β are constants depending only on multiindices β. The use of the Leibniz formula is justified below, using Lemma 2.3. Let $w(t,x) = D^\alpha u(t,x)$. Then (7) can be written as

$$(8) \qquad P_2 w \equiv w_{tt} - \sum_{(i,j)\neq(0,0)}^{n} a_{ij}(Du) w_{x_i x_j} = h.$$

Let $E(t)$ be energy for equation (8) defined by

$$(9) \quad E^2(t) = \int_\Omega \big(w^2(t,x) + w_t^2(t,x) + \sum_{i,j=1}^{n} a_{ij}(Du) w_{x_i}(t,x) w_{x_j}(t,x)\big)\, dx.$$

By differentiating (9) with respect to t and integrating by parts, we have

$$\begin{aligned} 2E(t)\frac{dE(t)}{dt} &= \int_\Omega \big(2ww_t + 2w_t w_{tt} + \sum_{i,j=1}^{n} \Big(\frac{\partial a_{ij}}{\partial t}\Big) w_{x_i} w_{x_j} \\ &\quad + 2\sum_{i,j=1}^{n} a_{ij} w_{x_i t} w_{x_j}\big)\, dx \\ &= 2\int_\Omega w_t \big(w + w_{tt} - \sum_{(i,j)\neq(0,0)}^{n} a_{ij} w_{x_i x_j}\big)\, dx \\ &\quad + 4\int_\Omega \sum_{j=1}^{n} a_{0j} w_{tx_j} w_t\, dx - 2\sum_{i,j=1}^{n} \int_\Omega \Big(\frac{\partial a_{ij}}{\partial x_i}\Big) w_{x_j} w_t\, dx \\ &\quad + \sum_{i,j=1}^{n} \int_\Omega \Big(\frac{\partial a_{ij}}{\partial t}\Big) w_{x_i} w_{x_j}\, dx - \sum_{j=1}^{n} \int_{\partial\Omega} a_{nj} w_{x_j} w_t\, dx'. \end{aligned}$$

We note that

$$\begin{aligned} 4\int_\Omega \sum_{j=1}^{n} a_{0j} w_{tx_j} w_t dx &= 4\int_\Omega \sum_{j=1}^{n-1} a_{0j} w_{tx_j} w_t dx + 4\int_\Omega a_{0n} w_{tx_n} w_t dx \\ &= -4\int_\Omega \sum_{j=1}^{n} \left(\frac{\partial a_{0j}}{\partial x_j}\right) w_t w_t dx \\ &\quad - 4\int_\Omega \sum_{j=1}^{n} a_{0j} w_{tx_j} w_t dx \\ &\quad - 4\int_{\partial\Omega} a_{0n} w_t w_t dx' \end{aligned}$$

so that

$$4\int_\Omega \sum_{j=1}^{n} a_{0j} w_{tx_j} w_t dx = -2\Big[\int_\Omega \sum_{j=1}^{n} (\frac{\partial a_{0j}}{\partial x_j}) w_t w_t dx + \int_{\partial\Omega} a_{0n} w_t w_t dx'\Big].$$

It follows from (8) that

$$\begin{aligned} E(t)\frac{dE(t)}{dt} &= \int_\Omega w_t(w+h)dx - \sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial a_{ij}}{\partial x_j}\right) w_{x_j} w_t dx \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} \left(\frac{\partial a_{ij}}{\partial t}\right) w_{x_i} w_{x_j} dx \\ &\quad - \int_\Omega \sum_{j=1}^{n} \left(\frac{\partial a_{0j}}{\partial x_j}\right) w_t w_t dx \\ &\quad - \int_{\partial\Omega} \sum_{j=1}^{n} a_{nj} w_{x_j} w_t dx' - \int_{\partial\Omega} a_{n0} w_t w_t dx'. \end{aligned} \tag{10}$$

By letting $\xi = \nabla w = (w_t, w_{x_1}, \cdots, w_{x_n})$ in (3), we have

$$w_t^2 + w_{x_1}^2 + \cdots + w_{x_n}^2 \le C\big(w_t^2 + \sum_{i,j=1}^{n} a_{ij}(Du) w_{x_i} w_{x_j}\big)$$

and so, by (9),

$$\int_\Omega \sum_{i=1}^{n} w_{x_i}^2 dx \le C\, E^2(t).$$

Now we estimate the third term in (10). Hölder's and Schwarz's inequality imply that

$$\begin{aligned}
\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial a_{ij}(Du)}{\partial t}\right)w_{x_i}w_{x_j}dx &\le \sum_{i,j=1}^{n}\int_{\Omega}\left|\left(\frac{\partial a_{ij}(Du)}{\partial t}\right)w_{x_i}w_{x_j}\right|dx \\
&\le \sum_{i,j=1}^{n}\left(\left\|\frac{\partial a_{ij}(Du)}{\partial t}\right\|_{L^\infty}\int_{\Omega}|w_{x_i}w_{x_j}|dx\right) \\
&\le C(t)\sum_{i,j=1}^{n}\left(\int_{\Omega}w_{x_i}^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega}w_{x_j}^2dx\right)^{\frac{1}{2}} \\
&\le C(t)E^2(t), \qquad (11)
\end{aligned}$$

since $D^2u(t,x)\in H^{s-2}(\Omega)\subset L^\infty(\Omega)$, for all t, and $s-2>\frac{n}{2}$. Similarly, for the second and fourth terms, we have

$$\begin{aligned}
\sum_{i,j=1}^{n}\int_{\Omega}\left(\frac{\partial a_{ij}(Du)}{\partial x_j}\right)w_{x_j}w_tdx &\le \sum_{i,j=1}^{n}\left(\left\|\frac{\partial a_{ij}(Du)}{\partial x_j}\right\|_{L^\infty(\Omega)}\int_{\Omega}\left|w_{x_i}w_t\right|dx\right) \\
&\le C(t)\sum_{i=1}^{n}\left(\int_{\Omega}w_{x_i}^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega}w_t^2dx\right)^{\frac{1}{2}} \\
&\le C(t)E^2(t), \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{n}\int_{\Omega}\left(\frac{\partial a_{0j}(Du)}{\partial x_j}\right)w_tw_tdx &\le \sum_{j=1}^{n}\left(\left\|\frac{\partial a_{0j}(Du)}{\partial x_j}\right\|_{L^\infty(\Omega)}\int_{\Omega}w_t^2dx\right) \\
&\le C(t)E^2(t). \qquad (13)
\end{aligned}$$

For the fifth and sixth terms, by Schwarz's inequality and trace theorem, we have

$$\begin{aligned}
\sum_{j=1}^{n}\int_{\partial\Omega}a_{nj}(Du)w_{x_j}w_tdx' &\le \sum_{j=1}^{n}\left(\|a_{nj}(Du)\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}|w_{x_j}w_t|dx'\right) \\
&\le C(t)\sum_{j=1}^{n}\left(\int_{\partial\Omega}w_{x_j}^2dx'\right)^{\frac{1}{2}}\left(\int_{\partial\Omega}w_t^2dx'\right)^{\frac{1}{2}} \\
&\le C(t)E^2(t), \qquad (14)
\end{aligned}$$

and

$$\int_{\partial\Omega} a_{n0}(Du) w_t w_t dx' \leq \|a_{n0}(Du)\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} w_t^2 dx'$$
$$\leq C(t)E^2(t), \tag{15}$$

since $Du(t,x) \in H^{s-1}(\partial\Omega) \subset L^\infty(\partial\Omega)$, for all t, and $s-2 > \frac{n}{2}$.

For the first term, Schwarz's inequality yields that

$$\int_\Omega w_t(w+h)dx = \int_\Omega w_t w dx + \int_\Omega (w_t h) dx$$
$$\leq C(t)E^2(t) + \int_\Omega |w_t h| dx. \tag{16}$$

It remains to estimate $\int_\Omega |w_t h| dx$.

We will show that for $1 \leq l \leq s-1$, $\big(D^l a_{ij}(Du)\big)\big(D^{s+1-l}u\big) \in L^2(\Omega)$. By the chain rule and the Leibniz formula, $D^l a_{ij}(Du)$ may be written as a sum of terms of the form $b_\gamma(Du)\,(D^{\alpha_1}(Du)) \cdots (D^{\alpha_m}(Du))$ with b_γ smooth and $|\alpha_1| + \cdots + |\alpha_m| = l$, where $|\alpha_k| \geq 1$ for $k = 1, \cdots, m$. Let $D^2u(t,x) = v(t,x)$. Then $v \in L^\infty(\Omega) \cap H^{s-2}(\Omega)$ since $s-2 > n/2$, and $D^l a_{ij}(Du)$ is written as a sum of terms of the form $b_\gamma(Du)(D^{\beta_1}v) \cdots (D^{\beta_m}v)$ with $|\beta_1| + \cdots + |\beta_m| \leq l-1$, where $|\beta_k| \geq 0$ for $k = 1, \cdots, m$. As we know $b_\gamma(Du) \in L^\infty(\Omega)$ and $D^{\beta_k}v \in L^{\frac{2(s-2)}{|\beta_k|}}(\Omega)$ by Lemma 2.3. Thus Hölder's inequality implies that the use of the chain rule was justified, and since $(|\beta_1| + \cdots + |\beta_m|)/2(s-2) \leq (l-1)/2(s-2)$, $b_\gamma(Du)(D^{\beta_1}v) \cdots (D^{\beta_m}v) \in L^{\frac{2(s-2)}{l-1}}(\mathbf{R}^n)$. Since $D^{s+1-l}u = D^{s-1-l}v$, $D^{s+1-l}u \in L^{\frac{2(s-2)}{s-1-l}}(\Omega)$ by Lemma 2.3. Therefore, by Hölder's inequality and Lemma 2.3, $\big(D^l a_{ij}(Du)\big)\big(D^{s+1-l}u\big) \in L^2(\Omega)$ and

$$\big\|\big(D^l a_{ij}(Du)\big)\big(D^{s+1-l}u\big)\big\|_{L^2} \leq \big\|D^l a_{ij}(Du)\big\|_{L^p} \big\|D^{s+1-l}u\big\|_{L^q}$$
$$\leq C(t)\|v\|_{L^\infty}\|v\|_{H^{s-2}}$$

since $1/p + 1/q = 1/2$, where $p = 2(s-2)/(l-1)$ and $q = 2(s-2)/(s-1-l)$. $D^\alpha(f(Du))$ may be written as a sum of terms of the form

$$f_\alpha(Du)\,(D^{\alpha_1}(Du)) \cdots (D^{\alpha_m}(Du))$$

with f_α smooth, $\alpha_1 + \cdots + \alpha_m = \alpha$ and $f_\alpha(Du)\,(D^{\alpha_1}(Du)) \cdots (D^{\alpha_m}(Du)) \in L^2(\Omega)$. Moreover, by Lemma 2.3, $\|D^\alpha(f(Du))\|_{L^2} \leq C(t)\|Du\|_{H^{s-1}} \leq C(t)\|u\|_{H^s}$.

By Schwarz's inequality, Minkowski's inequality and the facts above,

$$\begin{aligned}\int_\Omega (w_t h)dx &= \int_\Omega w_t\big(D^\alpha f + \sum_{\beta<\alpha}\sum_{(i,j)\neq(0,0)}^{n} C_\beta (D^{\alpha-\beta}a_{ij})(D^\beta u_{x_i x_j})\big)dx \\ &\leq \left(\int_\Omega w_t^2 dx\right)^{\frac{1}{2}} \left\{ \left(\int_\Omega (D^\alpha f)^2 dx\right)^{\frac{1}{2}} + \right. \\ &\left. \sum_{\beta<\alpha}\sum_{(i,j)\neq(0,0)}^{n} C_\beta \left(\int_\Omega (D^{\alpha-\beta}a_{ij})^2 (D^\beta u_{x_i x_j})^2 dx\right)^{\frac{1}{2}} \right\} \\ &\leq C(t)E(t)\left(\|D^\alpha f\|_{L^2} + \|v\|_{L^\infty}\|v\|_{H^{s-2}}\right) \\ &\leq C(t)E(t)\left(\|u\|_{H^s}\right) \quad \leq \quad C(t)E^2(t),\end{aligned}$$

and so

$$\int_\Omega w_t(w+h)dx \leq C(t)E^2(t). \tag{17}$$

From (11) – (15), (17) and by dividing (10) by $E(t)$, we have

$$\frac{dE(t)}{dt} \leq C(t)E(t). \tag{18}$$

By applying Gronwall's inequality to (18),

$$E(t) \leq \tilde{C}(t)E(0), \quad \text{where} \quad \tilde{C}(t) = e^{\int_0^t C(\lambda)d\lambda} \tag{19}$$

Thus, for given t,

$$\|u(t,x)\|_{H^s} \leq C(t)\left(\|u(0,x)\|_{H^s} + \|u_t(0,x)\|_{H^{s-1}}\right). \qquad \square$$

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