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ON p-GROUPS OF ORDER p^4

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ABSTRACT. In this paper we will determine Schur multipliers of some finite p-groups of order p^4.

1. Introduction

Let G be a finite group and let F be an algebraically closed field of characteristic zero with its multiplicative group $F^* = F - \{0\}$. A mapping $T : G \longrightarrow GL_n(F)$ of G into the general linear group $GL_n(F)$ is called a *projective representation of* G *of degree* n *over* F if

$$T(g)T(h) = \alpha(g, h)T(gh), \quad \alpha(g, h) \in F^*$$

holds for all $g, h \in G$. The function $\alpha : G \times G \longrightarrow F^*$ is called a *factor set* of G. Two factor sets α and β are called *equivalent* if there exists a function $c : G \longrightarrow F^*$ such that

$$\alpha(g, h) = \beta(g, h)c(g)c(h)c(gh)^{-1}$$

for all $g, h \in G$. This is an equivalence relation, and the equivalence class containing the factor set α will be denoted by $\{\alpha\}$. For any two factor sets α and β, let $\alpha\beta$ denote the function defined by

$$(\alpha\beta)(g, h) = \alpha(g, h)\beta(g, h), \quad g, h \in G.$$

Then $\alpha\beta$ is a factor set. If α^{-1} denotes the function for which

$$\alpha^{-1}(g, h) = \alpha(g, h)^{-1}, \quad g, h \in G,$$

then α^{-1} is also a factor set. The set $M(G)$ of all equivalence classes of factor sets forms an abelian group under the multiplication defined by

$$\{\alpha\}\{\beta\} = \{\alpha\beta\}.$$

The identity element in $M(G)$ is given by $\{1\}$ where 1 is the factor set $1(g, h) = 1$, $g, h \in G$; and for any $\{\alpha\} \in M(G)$, we have $\{\alpha\}^{-1} =$

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$\{\alpha^{-1}\}$. This group is called the *Schur multiplier* of G over F. In fact, $M(G)$ is the second cohomology group $H^2(G,\, F^*)$, where F^* is a trivial G-module.

The purpose of this paper is to explicitly determine Schur multipliers of some finite p-groups.

2. Main results

Let G be a nonabelian p-group of order 2^4. It is known that G is isomorphic to one of the nine groups(see [1], p. 145) and their Schur multipliers can be found in [6].

To show the main results, we begin with following(see [5], Theorem 4.6 and 4.8).

THEOREM 1. *Let G be a finite nonabelian group with $|G : Z(G)| = p^3$. Then one of the following holds.*

(1) *$G/Z(G)$ is an elementary abelian group of order p^3, and G' is an elementary abelian p-group with $\{1\} \neq G' \subseteq Z(G)$.*

(2) *$G/Z(G)$ is a nonabelian p-group of order p^3, and we have*

$$Z_2(G) = Z(G)G', \quad |Z_2(G) : Z(G)| = p, \quad |G : Z_2(G)| = p^2.$$

THEOREM 2. *Let G is a nonabelian p-group of order p^4. Then one of the following holds.*

(1) $Z(G) = p^2,\ |G'| = p,$ *and* $G' \subseteq Z(G)$.

(2) $Z(G) = p,\ |G'| = p^2,$ *and* $Z(G) \subseteq G'$.

We now determine Schur multipliers of nonabelian p-groups which satisty the conditions in Theorem 2 (1). Actually, it is well-known that $|G| = p^4, |G'| = p$ and $G' \subseteq Z(G)$, then G is isomorphic to one of the following six groups(see [3], p. 346).

$$G_1 = \langle x, y \,|\, x^{p^3} = y^p = 1,\ x^y = x^{1+p^2} \rangle$$
$$G_2 = \langle x, y, z \,|\, x^p = y^p = z^{p^2} = 1,\ [x,\, z] = [y,\, z] = 1,\ [x,\, y] = z^p \rangle$$
$$G_3 = \langle x, y \,|\, x^{p^2} = y^{p^2} = 1,\ x^y = x^{1+p} \rangle$$
$$G_4 = M_p \times \langle w \rangle,\ \text{ where}$$
$$M_p = \langle x, y, z \,|\, x^p = y^p = z^p = 1,\ [x,\, z] = [y,\, z] = 1,\ [x,\, y] = z \rangle,$$
$$\langle w \rangle = \langle w \,|\, w^p = 1 \rangle$$
$$G_5 = \langle x, y, z \,|\, x^{p^2} = y^p = z^p = 1,\ [x,\, z] = [y,\, z] = 1,\ x^y = x^{1+p} \rangle$$
$$G_6 = \langle x, y, z \,|\, x^{p^2} = y^p = z^p = 1,\ [x,\, y] = z,\ [x,\, z] = [y,\, z] = 1 \rangle$$

THEOREM 3. *Let p be an odd prime and let G be a nonabelian group of order p^4 such that $|G'| = p$, $G' \subseteq Z(G)$ and $Z(G)$ is cyclic of order p^2. And let $M(G)$ be a Schur multiplier of G. Then one of the following holds.*

(1) *$G = G_1$ and $M(G) \cong \{1\}$.*
(2) *$G = G_2$ and $M(G) \cong C_p \times C_p$.*

PROOF. (1) Since $\langle x \rangle \lhd G$ and $G/\langle x \rangle \cong C_p$, we have that G is metacyclic. Using the fact by reference[4, p. 289], we have

$$M(G) \cong C_n,$$

where $n = \frac{(1+p^2-1,\, p^3) \times l}{p^3} = 1$, $l = (1 + (1+p^2) + (1+p^2)^2 + \cdots + (1 + p^2)^{p-1},\, p^3) = p$. Thus it follows that

$$M(G) \cong \{1\}.$$

(2) Since $Z(G) = \langle z \rangle$ and $G/Z(G) \cong C_p \times C_p$, we have $M(G/Z(G)) \cong C_p$. And we also have $G' \cap Z(G) \cong C_p$. Note that

$$G/G' = \langle xG' \rangle \times \langle yG' \rangle \times \langle zG' \rangle$$
$$\cong C_p \times C_p \times C_p.$$

It follows that $Z(G) \otimes G \cong C_p \times C_p \times C_p$. Consider the exact sequence

$$Z(G) \otimes G \longrightarrow M(G) \longrightarrow M(G/Z(G)) \longrightarrow G' \cap Z(G) \longrightarrow 1.$$

Then $M(G/Z(G)) \cong C_p$, $G' \cap Z(G) \cong C_p$ and hence we obtain the following map

$$Z(G) \otimes G \longrightarrow M(G) \longrightarrow 1.$$

Since $Z(G) \otimes G$ is an elementary abelian group of order p^3, $M(G)$ is an elementary abelian group of order at most p^3. Since $[x,\, y] =$

$z^p \in Z(G)$, we have $z^{p^2} = [x, y]^p = [x^p, y] = [1, y] = 1$. The relations $x^p = 1$ and $[x, y] = z^p$ imply $z^{p^2} = 1$. Thus G is generated by three elements and five defining relations. Let $d(G)$ be the minimal number of generators of G. Then $5 \geq 3 + d(M(G))$, and hence $|M(G)| \leq p^2$. On the other hand, $d(G) = 3$ and we have

$$p^{\frac{3(3-1)}{2}} \leq |M(G)||G'| = |M(G)|p.$$

It follows that $|M(G)| \geq p^2$. Therefore $M(G) \cong C_p \times C_p$. □

THEOREM 4. *Let p be an odd prime and let G be a nonabelian group of order p^4 such that $|G'| = p$, $G' \subseteq Z(G)$ and $Z(G)$ is elementary abelian of order p^2. And let $M(G)$ be a Schur multiplier of G. Then one of the following holds.*

(1) *$G = G_3$ and $M(G) \cong C_p$.*
(2) *$G = G_4$ and $M(G) \cong C_p \times C_p \times C_p \times C_p$.*
(3) *$G = G_5$ and $M(G) \cong C_p \times C_p$.*
(4) *$G = G_6$ and $M(G) \cong C_p \times C_p$.*

PROOF. (1) It is similar to the proof of theorem 3 (1).

(2) Let $G = M_p \times \langle w \rangle$. It is easy to show that $M(M_p) \cong C_p \times C_p$. Since $M_p/M_p' = \langle xM_p' \rangle \times \langle yM_p' \rangle \cong C_p \times C_p$, we have

$$\begin{aligned} M_p \otimes \langle w \rangle &= M_p/M_p' \otimes \langle w \rangle \\ &\cong C_p \times C_p. \end{aligned}$$

Thus it follows that

$$\begin{aligned} M(G) &\cong M(M_p)) \times M(\langle w \rangle) \times (M_p \otimes \langle w \rangle) \\ &\cong C_p \times C_p \times C_p \times C_p. \end{aligned}$$

(3) Let

$$G = \langle x, y, z \,|\, x^{p^2} = y^p = z^p = 1,\ [x, z] = [y, z] = 1,\ x^y = x^{1+p} \rangle.$$

Then we have $G \cong K \times \langle z \rangle$, where $K = \langle x, y \,|\, x^{p^2} = y^p = 1,\ x^y = x^{1+p} \rangle$, and $\langle z \rangle = \langle z \,|\, z^p = 1 \rangle$. We can easily prove that $M(K) \cong \{1\}$. Since $K \otimes \langle z \rangle = K/K' \otimes \langle z \rangle \cong (C_p \times C_p) \otimes C_p \cong C_p \times C_p$, we have

$$\begin{aligned} M(G) &\cong M(K) \times M(\langle z \rangle) \times (K \otimes \langle z \rangle) \\ &\cong C_p \times C_p. \end{aligned}$$

(4) The proof can be found in the next theorem. □

We know that the schur multiplier of a group G is related to central extensions of G. Also schur showed that for any finite group G there exists a finite central extension whose kernel is isomorphic to $M(G)$(see [2], Theorem 25.5). Such an extension is called a *representation group* for G.

THEOREM 5. *Let p be an odd prime and let $G = G_6$. Then the following hold.*

(1) *G is a p-group of order p^4 and*

$$G' = \langle z\rangle \cong C_p, \quad Z(G) = \langle x^p\rangle \times \langle z\rangle \cong C_p \times C_p,$$
$$G/G' \cong C_p \times C_{p^2},\ G/Z(G) \cong C_p \times C_p.$$

Set $u_1 = x^p,\ u_2 = z^{-1},\ u_3 = y$. Then we have $G = \langle u_1, u_2, u_3, x\rangle$, where $\langle u_1, u_2, u_3\rangle$ is elementary abelian of order p^3 and

$$x^p = u_1,\ u_1^x = u_1,\ u_2^x = u_1u_2,\ u_3^x = u_3.$$

(2) *Let*

$$G^* = \langle a, b, c \,|\, a^{p^3} = b^{p^2} = c^p = 1,\ [a, c] = a^{p^2},\ [b, c] = b^p,\ [a, b] = c\rangle.$$

Then G^* is a representation group of G such that $G^*/\langle a^{p^2}, b^p\rangle \cong G$.

(3) *$M(G) \cong \langle a^{p^2}, b^p\rangle \cong C_p \times C_p$.*

PROOF. (1) It is easy to show that $z \in Z(G)$ and we have

$$(x^p)^y = (x^y)^p = (xz)^p = x^pz^p = x^p$$

and

$$(x^p)^z = (x^z)^p = x^p.$$

This implies that $x^p \in Z(G)$ and hence $Z(G) = \langle x^p, z\rangle = \langle x^p\rangle \times \langle z\rangle \lhd G$. It is easy to show that the subgroup $U = \langle x^p\rangle \times \langle z\rangle \times \langle y\rangle$ is an elementary abelian p-group of order p^3. Since $y^x = yz^{-1}$, we have $U \lhd G$. So $G = U\langle x\rangle$, $U \cap \langle x\rangle = \langle x^p\rangle$, and it follows that $|G| = \frac{|U||\langle x\rangle|}{|U\cap\langle x\rangle|} = p^4$. Next, we wish to show that $G = \langle x, y, z\rangle = \langle u_1, u_2, u_3, x\rangle$. In fact,

$$u_1^x = u_1,$$
$$u_2^x = (z^{-1})^x = z^{-1} = u_2,$$
$$u_3^x = y^x = yz^{-1} = z^{-1}y = u_2u_3.$$

(2) The relation $c^{-1}ac = a^{1+p^2}$ implies that $c^{-k}ac^k = a^{(1+p^2)^k} = a^{1+p^2k}$. Thus we have

$$\begin{aligned}(a^p)^b &= aa^{1+p^2(p-1)}\cdots a^{1+2p^2}a^{1+p^2}\\ &= a^p a^{p^2\frac{p(p-1)}{2}} = a^p\end{aligned}$$

and $(a^p)^c = (a^c)^p = (a^{1+p^2})^p = a^p$. It implies that $a^p \in Z(G^*)$. Similarly, we have $b^p \in Z(G^*)$ and therefore $\langle a^p, b^p\rangle \subseteq Z(G^*)$. Suppose that $Z(G^*) \neq \langle a^p, b^p\rangle$. Then $G^*/Z(G^*)$ is abelian and so $[G^*, G^*] \subseteq Z(G^*)$. But this is a contradition because $c \in [G^*, G^*]$, but $c \notin Z(G^*)$. Thus we show that $Z(G^*) = \langle a^p, b^p\rangle$. And we have

$$[G^*, G^*] = \langle c, a^{p^2}, b^p\rangle = \langle c\rangle \times \langle a^{p^2}\rangle \times \langle b^p\rangle.$$

Set $Z = \langle a^{p^2}, b^p\rangle$. Then $Z \subseteq Z(G^*) \cap [G^*, G^*]$ and cleary $G^*/Z \cong G$. We consider the map

$$f : \mathrm{Hom}(Z, \mathbb{C}^*) \longrightarrow M(G^*/Z) \cong M(G).$$

Then we have $\mathrm{im} f \cong [G^*, G^*] \cap Z = Z \cong C_p \times C_p$ and so $M(G)$ contains a subgroup isomorphic to Z.

(3) Since G is generated by 3 elements and 5 defining relations, we have

$$|M(G)| \leq p^2.$$

Hence $M(G) \cong C_p \times C_p$. □

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