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L_p error estimates and superconvergence for finite element approximations for nonlinear parabolic problems

QIAN LI AND HONGWEI DU

Abstract

In this paper we consider finite element mathods for nonlinear parabolic problems defined in $\Omega \subset R^d$ $(d \leq 4)$. A new initial approximation is taken. Optimal order error estimates in L_p for $2 \leq p \leq \infty$ are established for arbitrary order finite element. One order superconvergence in $W^{1,p}$ for $2 \leq q \leq \infty$ are demonstrated as well.

1 Introduction

Consider the following initial boundary value problem for the nonlinear parabolic problem:

$$
\begin{array}{llcllcl}
\text{(a)} & u_t - \nabla \cdot (a(x,u)\nabla u) & = & f(x,u), & (x,t) & \in & \Omega \times J, \\
\text{(b)} & a(x,0) & = & u_0(x), & x & \in & \Omega, \\
\text{(c)} & u(x,t) & = & 0, & x & \in & \partial\Omega \times J,
\end{array}
\qquad (1.1)
$$

where Ω is a bounded in R^d $(d \leq 4)$ with smooth boundary, $J = [0,T]$. We assume data a, f, u_0 together with their derivatives to be bounded on $\Omega \times R$ and

$$0 < a_* \leq a(x,s), \qquad (x,s) \in \Omega \times R.$$

The global nature of those assumptions is not restrictive, as we shall show below that the approximate solutions are uniformly close to the exact solution u of (1.1).

The object of this paper is to demonstrate optimal error estimates of finite element approximation in L_p for $2 \leq p \leq \infty$ and to derive the superconvergence in $W^{1,p}$ for $2 \leq p \leq \infty$ between the numerical solution and the Ritz projection of the exact solution of (1.1). In actual application, superconvergence estimates can be used to improve the approximation accuracy of the numerical solution to u by certain postprocessing technique as in [8–10].

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For this purpose, let $\{S_h\}_{0<h\leq 1}$ be a family of finite-dimensional subspace of $H_0^1(\Omega)\cap W^{1,\infty}(\Omega)$, with the following approximation properties: for some $r\geq 2$, $1\leq s\leq r$, $2\leq p\leq\infty$ and C a positive constant

$$\begin{array}{rcl}\inf\limits_{\chi\in S_h}\{\|\chi-w\|_{0,p}+h\|\chi-w\|_{1,p}\} & \leq & Ch^s\|w\|_{s,p},\\ w & \in & W^{s,p}(\Omega)\cap H_0^1(\Omega),\end{array}\tag{1.2}$$

where $\|\cdot\|_{l,p}$ denotes the norm in the Sobolev space $W^{l,p}(\Omega)$. In the sequal we also use $\|\cdot\|_l$ for $\|\cdot\|_{l,2}$ and $\|\cdot\|$ for $\|\cdot\|_{0,2}$. In addition, we assume that $\{S_h\}$ satisfies the standard inverse properties in finite element spaces[2,6].

The semidiscrete finite-element approximation to the solution u of (1.1) is defined to be a map $U(t):J\to S_h$ such that

$$(U_t,\chi)+(a(U)\nabla U,\nabla\chi)=(f(U),\chi),\ \chi\in S_h,\tag{1.3}$$

and $U(0)\in S_h$ is the approximation to u_0 which will be given in (1.5) below.

We now define the Ritz projection operator $R_h:H_0^1(\Omega)\to S_h$ by

$$(a(u)\nabla(w-R_hw),\nabla\chi)=0,\qquad \chi\in S_h,\tag{1.4}$$

where u is the solution of (1.1).

Let the error $U-u=(U-R_hu)+(R_hu-u)=\xi+\eta$. Then we choose the initial approximation $U(0)$ to satisfy

$$\begin{array}{rcl}A(\xi(0),\chi) & \equiv & (a(u_0)\nabla\xi(0),\nabla\chi)+(a_u(u_0)\xi(0)\nabla u_0,\nabla\chi)\\ & & +\lambda(\xi(0),\chi)\\ & = & -(a_u(u_0)\eta(0)\nabla u_0,\nabla\chi),\qquad \chi\in S_h,\end{array}\tag{1.5}$$

where λ is selected large enough to ensure the coerciveness of the bilinear form $A(\cdot,\cdot)$ over H^1.

Finite element methods to (1.1) have been studied by several authors. For example, Douglas and Dupont [7] and Wheeler [15] initiated the analysis of the standard Galerkin finite element approximation and demonstrated optimal order convergence in the H^1 and L_2 norms. Optimal maximum norm estimates for the one-dimensional case was optimal in [13,15]. For the higher dimensional case Chen and Huang [3] get the almost optimal L_∞ error estimates for linear elements. In [12], maximum-norm superconvergence of the gradient in linear finite element approximation is derived. These results above are in consistent with those for the linear problem $(a(x,u)=a(x))$. Recently, the Chou and author [5] obtained optimal L_∞ error estimates of (1.1) with zero initial value $u_0=0$. Finite element methods for linear parabolic and hyperbolic integrodifferential equations, Sobolev's equations and the equations of visco-elasticity have been discussed by Lin, Thomee and Wahlbin [11], in which optimal L_p error estimates are shown for $2\leq p<\infty$. The standard references on the subject of superconvergence are

[3,14,17]. The reader is referred to [14] for the issue of carring superconvergence results over from linear problems to their nonlinear counterparts.

This paper is organized in the following way. In §2, some nesessary lemmas will be proved which are essential in the analysis. In §3, optimal L_p error estimates and superconvergence in $W^{1,p}$ for $2 \le p < \infty$ will be presented, while maximum norm error estimates and superconvergence of the gradients will be demonstrated in §4.

2 Lemmas

In this section we shall given the error estimates of Ritz projection and prove the estimates for the initial value error. In addition, we shall also establish L_2 estimates for ξ_t and $\nabla\xi$.

The following lemma is contained in [6,17].

Lemma 2.1 For $t \in J$, $0 \le l \le 2$, and $1 \le s \le r$ we have

$$\begin{aligned}
&\text{(a)} \quad \|D_t^l(w - R_h w)\|_{0,p} + h\|D_t^l(w - R_h w)\|_{1,p} \\
&\qquad\qquad \le Ch^s \sum_{j=0}^{l} \|D_t^j w\|_{s,p}, \qquad 2 \le p < \infty, \quad r \ge 2, \\
&\text{(b)} \quad \|w - R_h w\|_{0,\infty} \le Ch^s \log h^{-1} \cdot \|u\|_{s,\infty}, \qquad r = 2, \\
&\qquad\qquad \le Ch^s \|u\|_{s,\infty}, \qquad r > 2, \\
&\text{(c)} \quad \|w - R_h w\|_{1,\infty} \le Ch^{s-1} \|w\|_{s,\infty}, \qquad r \ge 2.
\end{aligned} \tag{2.1}$$

This lemma together with the inverse property derives the following conclusion.

Corollary 2.1 If $u \in L_\infty(0,t;W^{2,d}(\Omega) \cap W^{1,\infty}(\Omega))$, $u_t \in L_\infty(0,t;W^{1,\infty}(\Omega))$, then $\nabla R_h u$ and $\nabla (R_h u)_t$ are uniformly bounded on $[0,t]$.

Proof. Obviously by (2.1c)

$$\|\nabla R_h u\|_{0,\infty} \le \|R_h u - u\|_{1,\infty} + \|u\|_{1,\infty} \le C\|u\|_{1,\infty}.$$

On the other hand, we know that[4]

$$\|(R_h u)_t - R_h u_t\|_{1,p} \le C\|R_h u - u\|_{1,p}, \quad 1 < p < \infty.$$

This together with inverse properties and (2.1) obtain

$$\begin{aligned}
\|\nabla (R_h u)_t\|_{0,\infty} &\le \|(R_h u)_t - R_h u_t\|_{1,\infty} + \|R_h u_t - u_t\|_{0,\infty} + \|u_t\|_{0,\infty} \\
&\le Ch^{-1}\|(R_h u)_t - R_h u_t\|_{1,d} + \|u_t\|_{1,\infty} \\
&\le Ch^{-1}\|R_h u - u\|_{1,d} + \|u\|_{1,\infty} \\
&\le C(\|u\|_{2,d} + \|u_t\|_{1,\infty}).
\end{aligned}$$

The proof is completed.

Now, let us establish the estimates for $\nabla\xi(0)$ and ξ_t.

Lemma 2.2 If $u_0 \in W^{s,4}(\Omega)$, $u_t(0) \in H^s(\Omega)$, for $2 \le s \le r$ and $r \ge 2$, then

$$\begin{array}{lll} \text{(a)} & \|(U - R_h u)(0)\|_1 & \le & Ch^s\|u_0\|_s, \\ \text{(b)} & \|(U - R_h u)_t(0)\| & \le & Ch^s\{\|u_0\|_s + \|u_0\|_{s,4}^2 + \|u_t(0)\|_s\}, \end{array} \tag{2.2}$$

where $u_t(0) = \nabla \cdot (a(x, u_0)\nabla u_0) + f(x, u_0)$ which comes from (1.1a) and (1.1b).

Proof. We first take $\chi = \xi(0)$ in (1.5) to get

$$\|\xi(0)\|_1^2 \le C\|\eta(0)\|\|\nabla\xi(0)\|,$$

which, by (2.1a), implies (2.2a).

We mext show (2.2b). For this, combine (1.1a), (1.3) and (1.4) to yield the error equation

$$\begin{array}{ll} & (\xi_t, \chi) + (a(U)\nabla\xi, \nabla\chi) \\ = & (f(U) - f(u) - \eta_t, \chi) - ((a(U) - a(u))\nabla R_h u, \chi), \quad \chi \in S_h. \end{array} \tag{2.3}$$

Now subtract (1.5) from (2.3) with $t = 0$ and set $\chi = \xi_t(0)$ to derive (in the sequal, $t = 0$ will be omitted)

$$\begin{array}{lll} \|\xi_t\|^2 & = & (f(U) - f(u) - \eta_t + \lambda\xi, \xi_t) \\ & & +(a_u(u)(U - u)\nabla u - (a(U) - a(u))\nabla u, \nabla\xi_t) \\ & & +((a(U) - a(u))\nabla(u - U), \nabla\xi_t) \\ & = & I_1 + I_2 + I_3. \end{array} \tag{2.4}$$

Then it follows from (2.1a), (2.2a), imbedding inequalities[1] and inverse properties that

$$\begin{array}{lll} I_1 & \le & C(\|\xi\| + \|\eta\| + \|\eta_t\|)\|\xi_t\| \\ & \le & Ch^s(\|u_0\|_s + \|u_t(0)\|_s)\|\xi_t\|, \\ I_2 & = & (\int_0^1 [a_u(u) - a_u(u + s(U - u))]ds(U - u)\nabla u, \nabla\xi_t) \\ & = & (\int_0^1 [\int_0^1 a_{uu}(u + s(1 - \tau)(U - u))d\tau](-s)ds(U - u)^2\nabla u, \nabla\xi_t) \\ & \le & C(\|\xi\|_{0,4}^2 + \|\eta\|_{0,4}^2)\|\xi_t\|_1 \\ & \le & C(\|\xi\|_1^2 + \|\eta\|_{0,4}^2)h^{-1}\|\xi_t\| \\ & \le & Ch^{2s-1}\|u_0\|_{s,4}^2\|\xi_t\|, \\ I_3 & \le & C(\|\xi\|_{0,4} + \|\eta\|_{0,4})(\|\xi\|_{1,4} + \|\eta\|_{1,4})\|\xi_t\|_1 \\ & \le & C(\|\xi\|_1 + \|\eta\|_{0,4})(h^{-1}\|\xi\|_1 + \|\eta\|_{1,4})h^{-1}\|\xi_t\| \\ & \le & Ch^{2s-2}\|u_0\|_{s,4}^2\|\xi_t\|. \end{array}$$

Collecting the estimates of I_1–I_3 with (2.4) completes the proof.

Our next aim is to derive estimates for ξ_t and $\nabla\xi$.

Lemma 2.3 Assume that $u_0 \in W^{s,4}(\Omega)$, $u_t(0) \in H^s(\Omega)$, u, u_t, $u_{tt} \in L_2(0,t;H^s(\Omega))$, $u \in L_\infty(0,t;W^{2,d}(\Omega) \cap W^{1,\infty}(\Omega))$ and $u_t \in L_\infty(0,t;W^{1,\infty}(\Omega))$. Then for $t \in J$

$$\|(U - R_h u)_t\| + \|U - R_h u\|_1 \le Ch^s, \quad 2 \le s \le r \quad \text{and} \quad r \ge 2. \tag{2.5}$$

Proof. By differentiating (2.3) with respect to t we obtain

$$\begin{array}{rl} & (\xi_{tt}, \chi) + (a(U)\nabla\xi_t, \nabla\chi) \\ = & -(a_u(U)U_t\nabla\xi, \nabla\chi) + ((f(U) - f(u))_t - \eta_{tt}, \chi) \\ & -((a(U) - a(u))_t\nabla R_h u, \nabla\chi) - ((a(U) - a(u))\nabla(R_h u)_t, \nabla\chi), \\ & \qquad\qquad\qquad\qquad\qquad\qquad \chi \in S_h. \end{array} \tag{2.6}$$

Setting $\chi = \xi_t$, by using ε-inequality and Corollary 2.1, we have

$$\begin{array}{rl} & \frac{1}{2}\frac{d}{dt}\|\xi_t\| + a_*\|\nabla\xi_t\|^2 \\ \le & C\{(\|\xi_t\|_{0,\infty}^2 + 1)\|\nabla\xi\|^2 + \|\xi\|^2 + \|\xi_t\|^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2\} \\ & +\varepsilon\|\nabla\xi_t\|^2. \end{array}$$

Hence integration now implies

$$\begin{array}{rl} & \|\xi_t\|^2 + \int_0^t \|\xi_t\|_1^2 d\tau \\ \le & C\{\|\xi_t(0)\|^2 + \int_0^t [(\|\xi_t\|_{0,\infty}^2 + 1)\|\xi\|_1^2 + \|\xi\|^2 + \|\xi_t\|^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2]d\tau\}. \end{array}$$

If we assume that

$$\|\xi\|_{L_\infty(J;H^1(\Omega))}\|\xi_t\|_{L_2(J;L_\infty(\Omega))} \le Ch^s, \tag{2.7}$$

then it follows from (2.1) and (2.2b) that

$$\|\xi_t\|^2 + \int_0^t \|\xi_t\|_1^2 d\tau \le C\{h^{2s} + \int_0^t (\|\xi_t\|^2 + \|\xi\|^2)d\tau\}. \tag{2.8}$$

Now the inequality

$$\begin{array}{rcl} \|\xi(t)\|_1^2 & = & \|\xi(0)\|_1^2 + \int_0^t \frac{d}{dt}\|\xi\|_1^2 d\tau \\ & \le & \|\xi(0)\|_1^2 + C\int_0^t \|\xi\|_1^2 d\tau + \varepsilon\int_0^t \|\xi_t\|_1^2 d\tau, \end{array}$$

can be combined with (2.8), by (2.2a), to show that

$$\|\xi_t\|^2 + \|\xi\|_1^2 + \int_0^t \|\xi_t\|_1^2 d\tau \le C\{h^{2s} + \int_0^t (\|\xi_t\|^2 + \|\xi\|_1^2)d\tau\}. \tag{2.9}$$

Gronwall Lemma now yields

$$\|\xi_t\| + \|\xi\|_1 + (\int_0^t \|\xi_t\|_1^2 d\tau)^{\frac{1}{2}} \leq Ch^s. \tag{2.10}$$

Finally it remains to vertify the induction hypothesis (2.7). First note that (2.7), by (2.2), holds for $t = 0$. We then need discuss the diffetent cases. If $d = 1$ or 2, it follows from imbedding inequalities and (2.10) that

$$\begin{aligned} & \|\xi(t)\|_1 (\int_0^t \|\xi_t\|_{0,\infty}^2 d\tau)^{\frac{1}{2}} \\ \leq\ & C(\log h^{-1})^{\frac{1}{2}} \|\xi(t)\|_1 (\int_0^t \|\xi_t\|_1^2 d\tau)^{\frac{1}{2}} \\ \leq\ & C(\log h^{-1})^{\frac{1}{2}} h^{2s} = \circ(h^s), \end{aligned}$$

which implies that (2.7) is volid.

For $s = 3$ or 4, applying the inverse property and imbedding inequality we conclude that

$$\begin{aligned} & \|\xi(t)\|_1 (\int_0^t \|\xi_t\|_{0,\infty}^2 d\tau)^{\frac{1}{2}} \\ \leq\ & Ch^{-\frac{d}{2}+1} \|\xi(t)\|_1 (\int_0^t \|\xi_t\|_{0,q}^2 d\tau)^{\frac{1}{2}}, \quad \text{with} \quad q = \frac{2d}{d-2} \\ \leq\ & Ch^{-\frac{d}{2}+1} \|\xi(t)\|_1 (\int_0^t \|\xi_t\|_1^2 d\tau)^{\frac{1}{2}} \\ \leq\ & Ch^{2s+1-\frac{d}{2}} = \circ(h^s). \end{aligned}$$

Therefore the proof has been completed.

3 Error estimates and superconvergence for $2 \leq p < \infty$

In this section the optimal L_p error estimates for the semidiscrete finite element approximation for $2 \leq p < \infty$ will be proved in Theorem 3.1. In addition, superconvergence results in $W^{1,p}$ $(2 \leq p < \infty)$ between the approximate solution and Ritz projection of the exact solution of (1.1) will be derived in Theorem 3.3.

Theorem 3.1 Let u and U be the solution of (1.1) and (1.3), respectively. If, in addition to the hypotheses of Lemma 2.2, $u \in W^{s,d}(\Omega)$ for $2 \leq s \leq r$ and $r \geq 2$, then for $t \in J$

$$\|U - u\|_{0,p} \leq Ch^s, \quad 2 \leq p < \infty. \tag{3.1}$$

Proof. Write the error $U - u = (U - R_h u) + (R_h u - u) = \xi + \eta$ as before. To prove (3.1) we define a auxiliary the problem. For $\phi \in W^{1,p'}(\Omega)$, $p^{-1} + p'^{-1} = 1$, let Φ be the solution of

$$(a(u)\nabla v, \nabla \Phi) = (v, \phi), \quad v \in H_0^1(\Omega). \tag{3.2}$$

Thus

$$\|\Phi\|_{2,p'} \leq C\|\phi\|_{0,p'}. \tag{3.3}$$

By (2.3) we then have

$$\begin{aligned}(\xi,\phi) &= (a(u)\nabla\xi,\nabla\Phi)\\ &= ((a(u)-a(U))\nabla\xi,\nabla R_h\Phi)+(a(U)\nabla\xi,\nabla R_h\Phi)\\ &= ((a(u)-a(U))\nabla U,\nabla R_h\Phi)\\ &\quad +(f(U)-f(u)-\xi_t-\eta_t,R_h\Phi).\end{aligned} \tag{3.4}$$

The first term on the right-hand side can be bounded as follows. From the inverse property, Corollary 2.1 and (2.5).

$$\begin{aligned}\|\nabla U\|_{0,\infty} &\leq \|\nabla\xi\|_{0,\infty}+\|\nabla R_h u\|_{0,\infty}\\ &\leq C(h^{-\frac{d}{2}}\|\nabla\xi\|+1)\\ &\leq C(h^{s-\frac{d}{2}}+1)\\ &\leq C,\end{aligned}$$

and it follows from imbedding inequalities and the stability of R_h in $W^{1,d'}$, $d^{-1}+d'^{-1}=1$, that

$$\begin{aligned}&((a(U)-a(u))\nabla\xi,\nabla R_h\Phi)\\ \leq\ & C(\|\xi\|_{0,d}+\|\eta\|_{0,d})\|R_h\Phi\|_{1,d'}\\ \leq\ & C(\|\xi\|_1+\|\eta\|_{0,d})\|\Phi\|_{1,d'}\\ \leq\ & Ch^s\|\Phi\|_{2,p'},\end{aligned}$$

where (2.1a) and (2.5) has been applied at the last step.

In order to estimate the remaining term, by using the same way as in [11, Theorem 3.2] we can select $\sigma > 1$ such that

$$\|R_h\Phi\| \leq C\|R_h\Phi\|_{1,\sigma} \leq C\|\Phi\|_{1,\sigma} \leq C\|\Phi\|2,p'$$

Hence, by (2.1b) and (2.5),

$$\begin{aligned}&(f(U)-f(u)-\xi_t-\eta_t,R_h\Phi)\\ \leq\ & C(\|\xi\|+\|\eta\|+\|\xi_t\|+\|\eta_t\|)\|R_h\Phi\|\\ \leq\ & Ch^s\|\Phi\|_{2,p}.\end{aligned}$$

Combining our estimates with (3.4) and noting (3.3), the inequality (3.1) follows.

Theorem 3.2 Let u and U be the solutions of (1.1) and (1.3), respectively.If, in addition the hypotheses of Lemma 2.2, $u \in W^{3,p}(\Omega)$ for $2 \leq s \leq r$ and $r \geq 2$, then for $t \in J$

$$\|U-R_h u\|_{1,p} \leq Ch^s, \tag{3.5}$$

where $2 \leq p < \infty$ when $d=1$ or 2, $2 \leq p \leq \frac{2d}{d-2}$ when $d=3$ or 4.

Proof. we first introduce the other auxiliary problem. Denote ψ_x to be an arbitrary component of $\nabla\psi$ and let Ψ be the solution of

$$(a(u)\nabla v, \nabla\Psi) = -(v, \psi_x),\ v \in H_0^1(\Omega), \tag{3.6}$$

Thus

$$\|\Psi\|_{1,p'} \leq C\|\psi\|_{0,p'},\ p^{-1} + p'^{-1} = 1. \tag{3.7}$$

We then have by the analogue of (3.4) that

$$\begin{aligned}(\xi_x, \psi) \ &= \ ((a(u) - a(U))\nabla U, \nabla R_h\Psi) \\ &\quad +(f(U) - f(u) - \xi_t - \eta_t, R_h\Psi).\end{aligned} \tag{3.8}$$

For the first term on the right-hand side we have

$$\begin{aligned} & ((a(u) - a(U))\nabla U, \nabla R_h\Psi) \\ \leq\ & C\|\nabla\xi\|_{0,\infty}(\|\xi\|_{0,p} + \|\eta\|_{0,p})\|R_h\Psi\|_{1,p'} \\ \leq\ & C(\|\xi\|_1 + \|\eta\|_{0,p})\|\Psi\|_{1,p'} \\ \leq\ & Ch^s\|\Psi\|_{1,p'}.\end{aligned}$$

The second term on the right-hand side is easily treated as before.

$$\begin{aligned} & ((f(U) - f(u)) - \xi_t - \eta_t, R_h\Psi) \\ \leq\ & C(\|\xi\| + \|\eta\| + \|\xi_t\| + \|\eta_t\|)\|R_h\Phi\| \\ \leq\ & Ch^s\|\Phi\|_{1,p'}.\end{aligned}$$

Together our estimates with (3.8) and (3.7) implies that the desired results (3.5) holds.

4 Error estimates and superconvergence for $p = \infty$

In this section we only consider two-dimensional space R^2. The optimal maximum norm error estimates and superconvergence of gradients will be established.

We shall first show the following L_∞ norm error estimates.

Theorem 4.1 Let u and U be the solutions of (1.1) and (1.3), respectively. Assume that the hypotheses of Lemma 2.2 are satisfied. Moreover, assume that $u \in W^{s,\infty}(\Omega)$ for $2 \leq s \leq r$ and $r \geq 2$. Then for $t \in J$

$$\begin{aligned}\|U - u\|_{0,\infty} \ &\leq \ Ch^s \log h^{-1}, \qquad r = 2, \\ &\leq \ Ch^s, \qquad\qquad r > 2.\end{aligned} \tag{4.1}$$

Proof. From (2.1b) we need to bound ξ only. Let $G_z^h \in S_h$ be the discrete Green function associated with the bilinear form $(a(u)\nabla\cdot, \nabla\cdot)$[17]. Hence the definition of G_z^h and (3.4) now imply that for $z \in \Omega$ and $t \in J$

$$\begin{aligned}\xi(z,t) \ &= \ (a(u)\nabla\xi, \nabla G_z^h) \\ &= \ ((a(u) - a(U))\nabla U, \nabla G_z^h) + (f(U) - f(u) - \xi_t - \eta_t, G_z^h) \\ &\leq \ C\{(\|\xi\|_1 + \|\eta\|_{0,p})\|G_z^h\|_{1,p'} + (\|\xi\| + \|\eta\| + \|\xi_t\| + \|\eta_t\|)\|G_z^h\|\} \\ &\leq \ Ch^s(\|G_z^h\|_{1,p'} + \|G_z^h\|).\end{aligned}$$

Recalling that[17]

$$\|G_z^h\|_{1,p'} + \|G_z^h\| \leq C,\ p^{-1} + p'^{-1} = 1,\ p > 2,$$

the conclusion of the theorem is now concluded.

Corollary 4.1 Under the hypotheses of Lemma 2.2, assume that $u \in W^{s,p}(\Omega)$ for $2 \leq s \leq r$, $r \geq 2$ and $p > 2$. Then for $t \in J$

$$\|U - R_h u\|_{0,\infty} \leq Ch^s. \tag{4.2}$$

We finally show $W^{1,\infty}$ superconvergence for $U - R_h u$.

Theorem 4.2 Under the hypotheses of Theorem 4.1, we have for $2 \leq s \leq r$ and $t \in J$

$$\begin{aligned} \|U - R_h u\|_{1,\infty} &\leq Ch^s(\log h^{-1})^2, \qquad r = 2 \\ &\leq Ch^s \log h^{-1}, \qquad r > 2. \end{aligned} \tag{4.3}$$

Proof. Let $g_z^h \in S_h$ be the finite element approximation of the derivative type regularized Green function, which is associated with the bilinear form $(a(u)\nabla\cdot, \nabla\cdot)$ (see [3]). Thus the following estimates hold[3]:

$$\|g_z^h\|^2 + \|g_z^h\|_{1,1} \leq C\log h^{-1}. \tag{4.4}$$

Now the definition of g_z^h implies that for $z \in \overline{\Omega}$ and $t \in J$

$$\begin{aligned} \partial_z \xi(z,t) &= (a(u)\nabla\xi, \nabla g_z^h) \\ &= ((a(u) - a(U))\nabla U, \nabla g_z^h) + (f(U) - f(u) - \xi_t - \eta_t, g_z^h) \\ &\leq C\{(\|\xi\|_{0,\infty} + \|\eta\|_{0,\infty})\|g_z^h\|_{1,1} + (\|\xi\| + \|\eta\| + \|\xi_t\| + \|\eta_t\|)\|g_z^h\|\}. \end{aligned} \tag{4.5}$$

Recall that

$$\|\xi\|_{0,\infty} \leq C(\log h^{-1})^{\frac{1}{2}}\|\xi\|_1,$$

This together with (4.4) and (4.5) complete the proof.

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Department of Mathematics
Shandong Normal University
Jinan, Shandong, 250014, P. R. China
e-mail: li_qian@163.net

College of Business Administration
Midwetern State University
Wichta Falls, TX 76308, U.S.A.
e-mail: fduh@nexus.mwsu.edu