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On the Numerical Inversion of the Laplace Transform by the Use of an Optimized Legendre Polynomial

Abdulaziz Al-Shuaibi

Abstract

A method for inverting the Laplace transform is presented, using a finite series of the classical Legendre polynomials. The method recovers a real-valued function $f(t)$ in a finite interval of the positive real axis when $f(t)$ belongs to a certain class $\mathcal{W}_\beta$ and requires the knowledge of its Laplace transform $F(s)$ only at a finite number of discrete points on the real axis $s > 0$. The choice of these points will be carefully considered so as to improve the approximation error as well as to minimize the number of steps needed in the evaluations. The method is tested on few examples, with particular emphasis on the estimation of the error bounds involved.

Introduction:

The Laplace transform $F(s)$ of a real-valued function $f(t)$ with $f(t) = 0$ for $t < 0$, if it exists, is defined by the integral

$$F(s) = \int_0^\infty e^{-st} f(t) dt. \tag{1}$$

The significance of numerical Laplace inversion is obvious from the big range of applications. Unfortunately, the problem of inverting the Laplace transform is severely ill-posed and highly instable. A number of numerical inversion methods have been developed during the last few decades, and discussed in the mathematical literature [1, 2, 3]. We shall confine ourselves to a class of functions $\mathcal{W}_\beta$, to be defined later, for which our proposed method can be shown to be accurate and of rapid convergence.

The Class $\mathcal{W}_\beta$:

Let $f(t)$ be defined at each point of the positive real axis. Then for $\lambda > 0$, the following change of variable will map the interval $[0, \infty)$ to the interval $[-1, 1]$.

$$\begin{aligned}
&x : [0, \infty) \to [-1, 1] \\
&x(t) = 1 - 2e^{-\lambda t} \\
&t : [-1, 1] \to [0, \infty) \\
&t(x) = -\frac{\ln\left(\frac{1-x}{2}\right)}{\lambda}.
\end{aligned}$$

Given $f(t)$, define

$$h(t) = e^{-\beta t} f(t), \text{ for arbitrary } \beta > \lambda > 0$$
$$g(x) = h\left(-\frac{\ln\left(\frac{1-x}{2}\right)}{\lambda}\right).$$

In other words $g(x) = \left\{\frac{1-x}{2}\right\}^{\frac{\beta}{\lambda}} f\left(-\frac{1}{\lambda}\ln\left(\frac{1-x}{2}\right)\right)$. Define the class $\mathcal{W}_\beta$ to be the set of all real-valued functions $f(t)$ with $f(t) = 0$ for $t < 0$ for which $g(x)$ as defined above has a derivative of bounded variation, i.e.

$$\mathcal{W}_\beta = \{f / V_{-1}^{1} g'(x) < \infty\},$$

where $V_{-1}^{1} g'(x)$ is the total variation of g' over $[-1, 1]$.

Remark 1: If $f(t) \in \mathcal{W}_\beta$, then the corresponding $g(x)$ is bounded and absolutely continuous.

At the first glance, one may think that our hypothesis above is too restrictive and yields a relatively small class of functions for practical purposes. On the contrary, the following lemma will show that our class $\mathcal{W}_\beta$ is a large one indeed.

The Class D_β:

Define the class $\mathcal{D}_\alpha$ to be the set of all real-valued functions defined at each point of the positive real axis and satisfy

(i) $f(t)$ is twice continuously differentiable in $[0, \infty)$,

(ii) for every $f(t)$, $\exists\, c > 0$ and $\alpha \in \mathcal{R}$ such that $|f''(t)| < ce^{\alpha t}$ for $t \geq 0$.

$$\mathcal{D}_\alpha \equiv \{f(t) \in C^2[0, \infty), \sup_{t \geq 0} |f''(t) e^{-\alpha t}| < \infty\}.$$

Lemma: Let $\beta > \max\{\lambda, \alpha + \lambda\}$, where $\lambda > 0$, then $\mathcal{D}_\alpha \subset \mathcal{W}_\beta$.

Proof: The differentiability of the associated $g(x)$ defined above follows immediately from part (i), and we only prove that $g'(x)$ is of bounded variation. For this, it suffices to show that $g''(x)$ is absolutely integrable in $[-1, 1]$. Now, changing variables again yields

$$\int_{-1}^{1} |g''(x)| dx = \frac{1}{2\lambda} \int_0^\infty |\{\beta(\beta - \lambda) f(t) + (\lambda - 2\beta) f'(t) + f''(t)\} e^{(\lambda - \beta)t}| dt. \qquad (*)$$

This equation requires bounds for $|f'|$ and $|f|$, and to obtain such estimates, we observed that part (i) implies the existence of $f(0^+)$ and $f'(0^+)$.
Define

$$y(t) = f(t) - t f'(0^+) - f(0^+). \qquad (2)$$

Hence,

$$
\begin{aligned}
&y''(t) = f''(t)\\
&y'(t) = f'(t) - f'(0^+)\\
&y'(0^+) = y(0^+) = 0.
\end{aligned}
$$

Now, consider the following cases:

Case (i): $\alpha = 0$; i.e., $|f''(t)| = |y''(t)| \leq c$.

Integrating both sides of the inequality $-c \leq y''(t) \leq c$ yields,

$$
\left.\begin{aligned}
&|y'(t)| \leq ct\\
&|y(t)| \leq \tfrac{c}{2}t^2
\end{aligned}\right\} \tag{3}
$$

Using equation $(*)$ together with (2) and (3) implies

$$
\begin{aligned}
\int_{-1}^{1} |g''(x)|dx \;\leq\; & \frac{1}{2\lambda(\beta-\lambda)^2}\left[(\beta-\lambda)\left\{\left|(\lambda-2\beta)f'(0^+) + \beta(\beta-\lambda)f(0^+)\right| + c\right\}\right.\\
+\; & \left. c\beta + c(2\beta-\lambda) + \beta(\beta-\lambda)|f'(0^+)|\right]
\end{aligned} \tag{4}
$$

Case (ii): $|\alpha| > 0$, i.e., $|y''(t)| \leq ce^{\alpha t}$.

Again, interchanging both sides of the inequality $-ce^{\alpha t} \leq y''(t) \leq ce^{\alpha t}$

$$
\left.\begin{aligned}
&|y'(t)| \leq \frac{c}{\alpha}\left[e^{\alpha t} - 1\right]\\
&|y(t)| \leq \tfrac{c}{\alpha^2}\left[e^{\alpha t} - \alpha t - 1\right]
\end{aligned}\right\} \tag{5}
$$

Putting (2) and (5) in equation $(*)$ yields

$$
\begin{aligned}
\int_{-1}^{1} |g''(x)|dx \;\leq\; & \frac{\alpha^2|\beta(\beta-\lambda)f(0^+) + (\lambda-2\beta)f'(0^+)| - c\beta(\beta-\lambda) - c\alpha(2\beta-\lambda)}{2\lambda\alpha^2(\beta-\lambda)}\\
+\; & \frac{\alpha\beta(\beta-\lambda)|f'(0^+)| - c\beta(\beta-\lambda)}{2\lambda\alpha(\beta-\lambda)^2}\\
+\; & \frac{c\beta(\beta-\lambda) + \alpha c(2\beta-\lambda) + \alpha^2 c}{2\lambda\alpha^2(\beta-\lambda-\alpha)}.
\end{aligned} \tag{6}
$$

Since we know that the total variation of $g'(x)$ in $[-1,1]$ cannot exceed $\int_{-1}^{1} |g''(x)|dx$, the lemma is proved.

Before we state our proposed method for inverting the Laplace transform of a given function belonging to the class $\mathcal{W}_\beta$, we find the following discussion quite helpful.

Series and Asymptotics of Legendre Polynomials

For any $g(x) \in \mathcal{L}_2[-1,1]$, we can have the following Legendre polynomials expansion

$$g(x) \cong \sum_{n\geq 0} a_n P_n(x) \quad \text{for} \quad x \in [-1,1] \tag{7}$$

where

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} g(x)P_n(x)dx$$
$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{(n-k)!(k!)^2}\left(\frac{1-x}{2}\right)^k . \tag{8}$$

The series $\sum_{n\geq 0} a_n P_n(x)$ is called the Legendre series of $g(x)$ and we can assert that if $g(x)$ is square integrable in $[-1,1]$, then this series converges in the mean in $[-1,1]$ to $g(x)$.

We can also affirm that if $g(x)$ is continuous in $[-1,1]$ and its Legendre series is uniformly convergent there, then

$$g(x) \equiv \sum_{n\geq 0} a_n P_n(x), \quad \text{for} \quad x \in [-1,1]. \tag{9}$$

We list below an important approximation formula, to be used later for computational purposes, with the help of which we can significantly minimize the number of steps needed in our computation as well as the error involved.

Jackson's Theorem [4; p. 205]:

Let $w(x)$ be of bounded variation in $[-1,1]$, and let U and V denote respectively the least upper bound of $|w(x)|$ and the total variation of $w(x)$ in $[-1,1]$. Given the function

$$g(x) = g(-1) + \int_{-1}^{x} w(x)dx, \tag{10}$$

then the coefficient a_n,

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} g(x)P(x)dx$$

of its Legendre series satisfy the inequality

$$|a_n| < \frac{4}{\sqrt{\pi}}(U+V)\frac{1}{n^{3/2}} , \quad \text{for} \quad n \geq 2. \tag{11}$$

Moreover, the Legendre series of $g(x)$ converges uniformly and absolutely to $g(x)$ in $[-1,1]$. The remainder of the series beginning with the $(n+1)$-st term satisfies the inequalities

$$|R_{n+1}(x)| < \frac{8}{\sqrt{\pi}}(U+V)\frac{1}{\sqrt{n}}\,, \quad \text{for } |x| \leq 1, \ n \geq 1 \tag{12}$$

$$|R_{n+1}(x)| < \frac{16\sqrt{2}}{\pi} \cdot \frac{U+V}{\sqrt[4]{1-\delta^2}} \cdot \frac{1}{n}\,, \quad \text{for } |x| \leq \delta < 1, \ n \geq 1. \tag{13}$$

The above discussions have furnished a good survey of the Legendre polynomials we intend to use as tools for our approximation as well as full description of the class of functions $\mathcal{W}_\beta$ to which the approximated functions belong.

Now, we are in a position for departure for our main problem, namely, the construction of the technique to be used for inverting the Laplace transform.

The Inversion of the Laplace Transform:

The Inversion Theorem: *Given the Laplace transform $F(s)$ for a real-valued function $f(t) \in \mathcal{W}_\beta$, and given $\epsilon > 0$, there exists an integer N such that*

$$f_a(t) = \sum_{n=0}^{N} a_n \tilde{P}_n(t), \quad \textit{for} \ \ 0 \leq t_0 \leq t \leq T < \infty$$

satisfies $= \sup_{t_0 \leq t \leq T} |f(t) - f_a(t) < \epsilon$ *where,*

$$a_n = \lambda(2n+1) \sum_{k=0}^{n} \frac{(-1)^n \Gamma(1+n+k)}{(n-k)!(k!)^2} \, F(\beta + \lambda + \lambda k)$$

$$\tilde{P}_n(t) = \sum_{k=0}^{n} \frac{(-1)^n \Gamma(1+n+k)}{(n-k)!(k!)^2} \, e^{-(k\lambda - \beta)t}$$

and N can be chosen such that

$$N \geq \left[\frac{16e^{\beta T}(U+V)}{\epsilon\sqrt{\pi}}\right]^2 \,, \quad \textit{for} \ \ t_0 = 0 \tag{14}$$

or,

$$N \geq \frac{32\sqrt{2}e^{\beta T}(U+V)}{\epsilon\pi\sqrt[4]{1-\delta^2}}\,, \quad \textit{for} \ \ t_0 > 0. \tag{15}$$

The second estimate for N can be used if the function $f(t)$ is to be recovered in an interval interior to $[0,\infty)$, i.e., for $t \in [t_0, T]$ with $t_0 > 0$ and $T < \infty$. Then, $\lambda = \max\left\{|1-2e^{-\lambda t_0}|, \ |1-2e^{-\lambda T}|\right\}$. U and V represent respectively the least upper bound of $|g'(x)|$ and total variation of $g'(x)$ in $[-1,1]$.

Proof: We may assume without loss of generality that $F(s)$ is defined for Re $s > 0$; a simple translation in the imaginary axis can be done if this is not the case.

Now, let us follow the same notations and change of variables introduced earlier.

Put,

$$h(t) = e^{-\beta t} f(t), \qquad \text{for } \beta > \lambda > 0$$

$$x = 1 - 2e^{-\lambda t}, \qquad \text{for } t \geq 0$$

$$g(x) = h\left(\frac{-\ln\left(\frac{1-x}{2}\right)}{\lambda}\right), \qquad \text{for } x \in [-1, 1].$$

Since $f(t) \in \mathcal{W}_\beta$, Remark 1 implies that $g(x)$ is the indefinite integral of its derivative

$$g(x) = g(-1) + \int_{-1}^{x} g'(x)dx.$$

Now, with U and V being the least upper bound of $|g'(x)|$ and the total variation of $g'(x)$ respectively, Jackson's theorem states that $g(x)$ can be approximated by the first N terms of its Legendre series in equation (8)

$$g(x) \cong \sum_{n=1}^{N} a_n P_n(x), \quad \text{for } x \in [-1, 1]$$

where

$$\begin{aligned}
P_n(x) &= \sum_{k=0}^{n} \frac{(-1)^k \Gamma(1+n+k)}{(n-k)!(k!)^2} \left(\frac{1-x}{2}\right)^k \\
a_n &= \frac{2n+1}{2} \int_{-1}^{1} g(x) P_n(x) dx \\
&= \frac{2n+1}{2} \int_{0}^{\infty} e^{-\beta t} f(t) P_n\left(1 - 2e^{-\lambda t}\right)\left(2\lambda e^{-\lambda t}\right) dt. \\
&= \lambda(2n+1) \sum_{k=0}^{n} \frac{(-1)^k \Gamma(1+n+k)}{(n-k)!(k!)^2} \int_{0}^{\infty} f(t) e^{-[\beta+\lambda k]t} dt \\
&= \lambda(2n+1) \sum_{k=0}^{n} \frac{(-1)^k \Gamma(1+n+k)}{(n-k)!(k!)^2} F(\beta + \lambda + \lambda k).
\end{aligned}$$

Now, to show that our uniform error in approximating the original function $f(t)$ by the function $f_a(t)$ cannot exceed ϵ in magnitude, we consider the following two cases:

Case (i): $t_0 = 0$.

Then, inequalities (13) and (15) give the remainder

$$|R_{N+1}(x)| < e^{-\beta T} \cdot \frac{\epsilon}{2}\ , \text{ for } \ x \in \left[-1, 1-2e^{-\lambda T}\right].$$

Hence,

$$g(x) = \sum_{n=1}^{N} a_n P_n(x) + R_{N+1}(x), \quad \text{for } \ x \in [-1,1].$$

Putting $x = 1 - 2e^{-\lambda t}$ we have

$$\begin{aligned} e^{-\beta t} f(t) &= \sum_{n=0}^{N} a_n P_n \left(1-2e^{-\lambda t}\right) + R_{N+1}\left(1-2e^{-\lambda t}\right) \\ f(t) &= \sum_{n=0}^{N} a_n \tilde{P}_n(t) + e^{\beta t} R_{N+1}\left(1-2e^{-\lambda t}\right). \end{aligned}$$

If we put $f_a(t) = \sum_{n=0}^{N} a_n \tilde{P}_n(t)$. Then,

$$\max_{0 \le t \le T} |f_a(t) - f(t)| = e^{\beta t} R_{N+1}(1-2e^{-\lambda t}) < \frac{\epsilon}{2}.$$

Case (ii): $t_0 > 0$:

Similarly inequalities (14) and (16) give the remainder

$$|R_{N+1}(x)| < e^{-\beta T} \cdot \frac{\epsilon}{2} \quad \text{for } \ x \in [-\delta, \delta]$$

where, $\delta = \max\left\{|1-2e^{-\lambda t_0}|,\ |1-2e^{\lambda T}|\right\}$. Also, $g(x) = \sum_{n=0}^{N} a_n P_n(x) - R_{N+1}(x)$, for $x \in [-\delta, \delta]$. Putting $x = 1-2e^{-\lambda t}$, for $t \in [t_0, T]$, we get

$$\begin{aligned} e^{-\beta t} f(t) &= \sum_{n=0}^{N} a_n P_n(1-2e^{-\lambda t}) + R_{N+1}(1-2e^{-\lambda t}) \\ f(t) &= \sum_{n=0}^{N} a_n \tilde{P}_n(t) + e^{\beta t} R_{N+1}(1-2e^{-\lambda t}). \end{aligned}$$

If we put $f_a(t) = \sum_{n=0}^{N} a_n \tilde{P}_n(t)$, then

$$\max_{t_0 \le t \le T} |f(t) - f_a(t)| = e^{\beta t} R_{N+1}(1-2e^{-\lambda t}) < \frac{\epsilon}{2}.$$

This completes the proof.

The Choice of β and λ:

In our numerical computations for functions belonging to the class $\mathcal{D}_\alpha$, it is always desirable to minimize the time and effort needed in the computation to achieve the accuracy within the pre-assigned tolerance ϵ. For this, let us denote the right sides of the inequalities (4) and (6), standing for the total variation by $V_1(\beta,\lambda)$ and $V_2(\beta,\lambda)$ respectively, and observe that for the least upper bound of $g'(x)$ in $[-1, 1-2^{-\lambda T}]$, we have by inequalities (3) and (5) with $\beta > \max\{\lambda, \lambda+\alpha\}$

$$|g'(x)| \le U_1(\beta,\lambda) = \frac{cT}{4\lambda}[\beta T + 2], \qquad \text{for } \alpha = 0$$

$$|g'(x)| \le U_2(\beta,\lambda) = \frac{\beta c}{2\lambda\alpha}\left[e^{\alpha T} - 1\right] + \frac{c}{2\lambda\alpha^2}\left[e^{\alpha T} - \alpha T - 1\right], \qquad \text{for } |\alpha| > 0.$$

Thus, we may take U and V in inequalities (14) and (15)

$$U = U_i(\beta,\lambda) \qquad i = 1 \text{ or } 2.$$
$$V = V_i(\beta,\lambda) \qquad i = 1 \text{ or } 2.$$

$i = 1$ or 2, depend on whether α in our class $\mathcal{D}_\alpha$ is zero or not respectively.

Now, we can pose the following optimization problem, responsible for minimizing our integer N that determines the number of polynomials needed to achieve the desired accuracy. The minimization is taken over β, λ, and adopt the powerful computational algorithm SUMT [6].

Minimize

$$e^{\beta T}[U_i(\beta,\lambda) + V_i(\beta,\lambda)] \qquad i = 1 \text{ for } 2$$

subject to

$$\lambda > 0$$
$$\beta > \max\{\lambda, \lambda+\alpha\}.$$

When this minimum is achieved, say at $\lambda = \lambda_{opt}$ and $\beta = \beta_{opt}$, then we may take $U = U_i(\beta_{opt}, \lambda_{opt})$ and $V = V_i(\beta_{opt}, \lambda_{opt})$ in our bounds (14) and (15). This gives an optimal choice of $N = N_{opt}$, with which we can advance in our calculations.

Determination of c and α:

The problem arising in the optimization recommended above is the determination of the best constants c and α needed there, when we only have the given function $F(s)$ in hand. For this, we recall the following theorem.

Tauberian Theorem [5, p. 185]: *If the function $f(t)$ satisfies the inequality $|f(t) < Me^{\alpha t}$ for all $t > 0$, M being a positive constant, then*

$$\lim_{s\to\infty} sF(s) = f(0^+).$$

Clearly, our function $f(t)$ as it belongs to the class $\mathcal{D}_\alpha$ satisfies the hypothesis of Tauberian theorem. This can be shown with the help of the inequalities (3) and (5).

Therefore, given $F(s)$ we calculate the following limits involved in our bounds and known to exist by the hypothesis of our class

$$f(0^+) = \lim_{s\to\infty} sF(s)$$
$$f(0^+) = \lim_{s\to\infty} s[sF(s) - f(0^+)]$$
$$f(0^+) = \lim_{s\to\infty} s[s^2F(s) - sf(0^+) - f'(0)].$$

Now, we can use these limits to estimate lower bounds for c and α.

Since, $|f''(t)| \le ce^{\alpha t}$, it is immediate to see that

$$|f''(0)| \le c$$
$$|s^2F(s) - sf(0^+) - f'(0)| \le \frac{c}{s-\alpha}, \quad \text{for} \ \ s > \alpha. \tag{16}$$

If the left side of the above inequality, which is the Laplace transform of $f''(t)$ is different from zero, we have, for $s > \alpha$

$$\frac{s|s^2F(s) - sf(0^+) - f'(0)| - c}{|s^2F(s) - sf(0^+) - f'(0^+)|} \le \alpha. \tag{17}$$

In most cases, inequality (18) will provide a good estimate for α directly, otherwise we need to estimate the maximum of the left side over all $s > 0$, and use it as a lower bound for α.

If the procedure of determining lower bounds for α and c is too difficult, depending on the nature of the function $F(s)$, then the following theorems are recommended.

Definition: *An operator $L_{k,t}[F(s)]$ is defined by the equation*

$$L_{k,t}[F(s)] = \frac{(-1)^k}{k!} F^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1}$$

for any real positive number t and any positive integer k.

Condition: A function $F(s)$ satisfies condition A if it has derivatives of all orders in $(0 < s < \infty)$ and if there exists a constant M such that for $(0 < s < \infty)$

$$L_{k,t}[F(s)] < M \qquad (k = 1, 2, \dots)$$
$$|sF(s)| < M.$$

Result: [5; page 315]:

Condition A is necessary and sufficient that

$$F(s) = \int_0^\infty e^{-st} f(t)dt,$$

where $f(t)$ is bounded in $(0 < t < \infty)$.

Result: [5; page 316]:

If $F(s)$ is the Laplace transform of a function $f(t)$ with $f(t)$ bounded in $(0 < t < \infty)$, then

$$\lim_{k\to\infty} \left\{ \sup_{0<t<\infty} |L_{k,t}[F(s)]| \right\} = \operatorname*{ess\ sup}_{0<t<\infty} |f(t)|.$$

Now suppose we are given $F(s)$ for a function $f(t)$ which is known to satisfy part (i) of the definition of our class $\mathcal{D}_\alpha$, and we want to know if condition (ii) is also satisfied, i.e., the existence of c and α such that $|f(t)| \leq ce^{\alpha t}$. Moreover, we want to estimate the least c and α. Then the results of Widder [5] suggest the following.

Let $G(s,\eta) = F(s-\eta)$
where η is a real number to be fixed later. Then, the original function of $G(s,\eta)$ is simply $e^{\eta t} f(t)$. We try to find the least possible value η that makes the following limit exist

$$\lim_{k\to\infty} \left\{ \sup_{0<t<\infty} |L_{k,t}[G(s,\eta)]| \right\} = \operatorname*{ess\ sup}_{0<t<\infty} \left| e^{\eta t} f(t) \right|.$$

If we succeed in doing so, then we claim that

$$\left| e^{\eta t} f(t) \right| \leq c$$

and our c and α are on hand, with $\alpha = -\eta$.

Example:

$$F(s) = \frac{1}{s - \frac{1}{2}}$$

$$f(t) = e^{\frac{1}{2}t}$$

$$G(s,\alpha) = F(s-\eta) = \frac{1}{s - \eta - \frac{1}{2}}$$

$$g(t) = e^{\left(\frac{1}{2}+\eta\right)t}$$

$$L_{k,t}[G(s,\alpha)] = \frac{(-1)^k}{k!} \frac{(-1)^k k!}{\left(\frac{k}{t} - \eta - \frac{1}{2}\right)^{k+1}} \left(\frac{k}{t}\right)^{k+1} = \left(\frac{k}{k - t\left(\eta + \frac{1}{2}\right)}\right)^{k+1}.$$

Clearly,

$$\lim_{k\to\infty}\left\{\sup_{0<t<\infty}|L_{k,t}[G(s,\eta)]|\right\} = \begin{cases} 1 & \eta = -\frac{1}{2} \\ \infty & \eta \neq -\frac{1}{2} \end{cases}$$

hence, we choose $\eta = -\frac{1}{2}$ and the corresponding limit $c = 1$. Then, $|e^{\eta t} f(t)| \leq c = 1$, and $|f(t)| \leq ce^{-\eta t} = e^{\frac{1}{2}t}$. Giving $\alpha = \frac{1}{2}$ as expected.

Numerical Implementation:

We use the software Mathematica to implement our technique. The following input statements provide the desired results.

Algorithm

$K :=$ Machine precision (to be specified)

$L := N[\lambda, K] = \lambda_{opt}$ (to be specified)

$B := N[\beta, K] = \beta_{opt}$ (to be specified)

$m :=$ number of polynomials needed $= N_{opt}$ (to be specified)

$f[s\_] := f[s] = N[F(s), K] =$ The Laplace Transform (to be specified).

$a[n\_] := a[n] = N[(L + 0.)(2n + 1)\text{Sum}[(((-1)\hat{}i)((n + i)!)f[B + L + 0.)$

$+(L + 0.)i])/((n - i)!((i!)\hat{}2)), \{i, 0, n\}\, K] =$ (the coefficient a_n)

$f_a[x\_] := N[\text{Exp}[(B + 0.)x]\text{Sum}[a[n]\text{Legendre } P[n, 1 - 2\text{Exp}[-2(L + 0.)]],$

$\{n, 0, m\}], K] =$ (the approximation function).

$g(x\_] := N[f(x), K] =$ the exact function (to be specified).

$h[x\_] : N[\text{Abs}[f_a[x] - g[x]], K] =$ the error function.

Table $[\{N[x, 1]N[f_a[x], 8]N[g[x], 8]$, Number Form$[h[x], 2]\}, \{x, t_0, T, t_1\}], [t_0, T]$ is the interval of approximation and t_1 is the increment size of the calculation (to be specified)

Examples:

Example 1:

$$F(s) = \frac{1}{s}$$
$$f(t) = 1.$$

$m = 2$			
t	$f(t)$ Exact	$f_a(t)$ Appr.	Error
0.0	1.0	1.0	0.0
20.0	1.0	1.0	0.0
40.0	1.0	1.0	0.0
60.0	1.0	1.0	0.0
80.0	1.0	1.0	0.0
100.0	1.0	1.0	0.0
120.0	1.0	1.0	0.0
140.0	1.0	1.0	$3.1\ 10^{-14}$
160.0	1.0	1.0	$6.1\ 10^{-4}$
180.0	1.0	1.0	$2.0\ 10^{-14}$
200.0	1.0	1.0	$7.0\ 10^{-12}$

Example 2:

$$F(s) = \frac{1}{(s+1)^2+1}$$
$$f(t) = e^{-t} \sin t$$

(a)

$m = 10$			
t	$f(t)$ Exact	$f_a(t)$ Appr.	Error
0.0	−0.00000962189	0.0	$9.6\ 10^{-6}$
0.8	0.32239106	0.32239694	$5.9\ 10^{-6}$
2.0	0.20787082	0.20787958	$8.8\ 10^{-6}$
2.0	0.067052245	0.06701974	$3.3\ 10^{-5}$
3.0	−0.00049518539	0.0	$5.0\ 10^{-5}$
4.0	−0.01406353	−0.013932035	$5.0\ 10^{-4}$
5.0	−0.0083793352	−0.008983291	$6.10\ 10^{-4}$
5.0	−0.001941255	−0.0028961856	$9.5\ 10^{-4}$
6.0	−0.0019625595	0.0	$2.0\ 10^{-3}$

(b)

$m=20$			
t	$f(t)$ Exact	$f_a(t)$ Appr.	Error
0.0	2.5354036	0.0	$2.5\ 10^{-8}$
0.8	0.32239695	0.32239694	$1.1\ 10^{-8}$
2.0	0.20787955	0.20787958	$3.0\ 10^{-8}$
2.0	0.067019656	0.06701974	$8.4\ 10^{-8}$
3.0	−2.934823	0.0	$2.9\ 10^{-8}$
4.0	−0.013931553	−0.013932035	$4.8\ 10^{-7}$
5.0	−0.0089845545	−0.008983291	$1.3\ 10^{-6}$
5.0	−0.002895472	−0.0028961856	$7.1\ 10^{-7}$
6.0	0.000010587337	0.0	$1.1\ 10^{-4}$

Example 3:

$$F(s) = \frac{1}{(s+2)^{3/2}(s+1)}$$

$$f(t) = e^{-x} Er\, f(\sqrt{t}) - 2\sqrt{\frac{t}{\pi}} e^{-2t}$$

(a)

$m=5$			
t	$f(t)$ Exact	$f_a(t)$ Appr.	Error
0.0	−0.0091284955	0.0	$9.1\ 10^{-3}$
0.2	0.051823691	0.048925307	$2.9\ 10^{-3}$
0.4	0.099266734	0.10090527	$1.6\ 10^{-3}$
0.6	0.13233794	0.13555411	$3.2\ 10^{-3}$
0.8	0.15181664	0.15304602	$1.2\ 10^{-3}$
1.0	0.1594026	0.15730278	$2.1\ 10^{-3}$
1.0	0.15723974	0.15251432	$4.7\ 10^{-3}$
1.0	0.14761049	0.14216317	$5.4\ 10^{-3}$
2.0	0.13274715	0.12884934	$3.9\ 10^{-3}$
2.0	0.11472268	0.11438317	$3.4\ 10^{-4}$
2.0	0.095393769	0.099949961	$4.6\ 10^{-3}$

(b)

$m = 25$			
t	$f(t)$ Exact	$f_a(t)$ Appr.	Error
0.0	−0.000046726633	0.0	$4.7\ 10^{-5}$
0.2	0.048926224	0.048925307	$9.2\ 10^{-7}$
0.4	0.10090542	0.10090527	$1.5\ 10^{-7}$
0.6	0.1355571	0.13555411	$1.6\ 10^{-6}$
0.8	0.15304517	0.15304602	$8.5\ 10^{-7}$
1.0	0.15730314	0.15730278	$3.6\ 10^{-7}$
1.0	0.15251373	0.15251432	$5.9\ 10^{-7}$
1.0	0.14216473	0.14216317	$1.6\ 10^{-6}$
2.0	0.12884655	0.12884934	$2.8\ 10^{-6}$
2.0	0.11438579	0.11438317	$2.6\ 10^{-6}$
2.0	0.09995069	0.099949961	$7.3\ 10^{-7}$

Example 4:

$$F(s) = \frac{\sqrt{s+2}}{(s+1)^{3/2}}$$
$$f(t) = te^{-\frac{3}{2}t} I_1\left(\frac{t}{2}\right) + (t+1)I_0\left(\frac{t}{2}\right)$$

where I_1 and I_0 are the modified Bessel functions of degree 1 and zero respectively.

(a)

$m = 5$			
t	$f(t)$ Exact	$f_a(t)$ Appr.	Error
0.0	1.0000323	0.0	$3.2\ 10^{-5}$
0.2	0.8986087	0.89862316	$1.4\ 10^{-5}$
0.4	0.79809951	0.79810129	$1.8\ 10^{-6}$
0.6	0.70224679	0.70223497	$1.2\ 10^{-5}$
0.8	0.61322768	0.61321473	$1.3\ 10^{-5}$
1.0	0.5321391	0.53213443	$4.7\ 10^{-6}$
1.2	0.45935012	0.45935646	$6.3\ 10^{-6}$
1.4	0.39475522	0.39476999	$1.5\ 10^{-5}$
1.6	0.33795381	0.33797177	$1.8\ 10^{-5}$
1.8	0.28837582	0.28839136	$1.6\ 10^{-5}$
2.0	0.24536722	0.24537636	$8.6\ 10^{-6}$

(b)

$m = 10$			
t	$f(t)$ Exact	$f_a(t)$ Appr.	Error
0.0	0.99999984	1.0	$1.6\ 10^{-7}$
0.2	0.89862312	0.89862316	$3.7\ 10^{-8}$
0.4	0.79810128	0.79810129	$1.6\ 10^{-8}$
0.6	0.702235	0.70223497	$3.4\ 10^{-8}$
0.8	0.6132147	0.61321473	$3.6\ 10^{-8}$
1.0	0.053213446	0.53213443	$3.4\ 10^{-8}$
1.2	0.45935644	0.45935646	$1.7\ 10^{-8}$
1.4	0.39476997	0.39476999	$2.1\ 10^{-8}$
1.6	0.33797182	0.33797177	$4.7\ 10^{-8}$
1.8	0.28839135	0.28839136	$8.7\ 10^{-9}$
2.0	0.2453763	0.24537636	$5.2\ 10^{-8}$

Example 5:

$$
\begin{aligned}
F(s) &= \frac{1}{\sqrt{s^2+1}} \\
f(t) &= J_0(t) \\
&= \text{Bessel function of degree zero}
\end{aligned}
$$

(a)

$m = 10$			
t	$f(t)$ Exact	$f_a(t)$ Appr.	Error
0.0	1.0001498	1.0	$1.5\ 10^{-4}$
0.2	0.99008446	0.99002497	$5.9\ 10^{-5}$
0.4	0.96033677	0.96039823	$6.1\ 10^{-5}$
0.6	0.91201646	0.91200486	$1.2\ 10^{-5}$
0.8	0.84638125	0.8462873	$9.4\ 10^{-5}$
1.0	0.76516841	0.76519769	$2.9\ 10^{-5}$
1.0	0.6709762	0.67113274	$1.6\ 10^{-4}$
1.0	0.56677705	0.56685512	$7.8\ 10^{-5}$
2.0	0.4555604	0.45540217	$1.6\ 10^{-4}$
2.0	0.34031201	0.33998641	$3.3\ 10^{-4}$
2.0	0.2241224	0.22389078	$2.3\ 10^{-4}$

(b)

$m = 20$			
t	$f(t)$ Exact	$f_a(t)$ Appr.	Error
0.0	0.99999486	1.0	$5.1\ 10^{-6}$
0.2	0.99002349	0.99002497	$1.5\ 10^{-6}$
0.4	0.96039749	0.96039823	$7.4\ 10^{-7}$
0.6	0.091200681	0.91200486	$1.9\ 10^{-6}$
0.8	0.84628488	0.84628735	$2.5\ 10^{-6}$
1.0	0.765200466	0.76519769	$2.8\ 10^{-6}$
1.0	0.067113118	0.67113274	$1.6\ 10^{-6}$
1.0	0.56685251	0.56685512	$2.6\ 10^{-6}$
2.0	0.45540876	0.45540217	$6.6\ 10^{-6}$
2.0	0.3399851	0.33998641	$1.3\ 10^{-6}$
2.0	0.22388007	0.22389078	$1.1\ 10^{-5}$

Conclusion:

Two functions that are equal almost everywhere, have the same Laplace transform if it exists. For this, it is clear that we can not recover the original function uniquely, i.e., we claim that our approximation function can only differ from the original function on a set of at most measure zero. Therefore, it is quite difficult, if not impossible, to predict from the given transform if our original function satisfies the hypothesis of part (i) of our class $\mathcal{W}_\beta$. This strongly suggests that we must rely on the nature of the problem being under transformation and the physical interpretation of the function in question.

The method presented demonstrates the dependence of the number of polynomials needed to achieve a certain desirable accuracy on the bounds and total variation of the original function; hence it was necessary to restrict our class to the class $\mathcal{D}_\alpha$ for which the maximum bounds and the total variation can be estimated.

Finally, the introduction of the parameters β and λ in the scheme has proven to be quite efficient and helpful in allowing us to enlarge our class $\mathcal{W}_\beta$ as well as minimize the number of steps needed in the evaluation when the optimal choice of β and λ is used. The dependence of β and λ on the transform $F(s)$ and the growth of the original function $f(t)$ demonstrates their significant importance.

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King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia