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Duality for Nonsmooth Multiobjective Fractional Programming with V-ρ-Invexity

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Abstract

We obtain some duality results for nonsmooth multiobjective fractional programming problem under generalized invexity assumptions on the objective and constraint functions.

1. Introduction

Duality in fractional programming involving the optimization of a single ratio has been of much interest in the past (see e.g. Schaible [13]). Recenlty there has been of growing interest in studying duality theorems for multiobjective fractional programming problem involving generalized convex functions (see e.g. Chandra, Craven and Mond [1], Egudo [4], Mukherjee and Rao [11] and Weir [14]).

Kuk *et al.* [8] have introduced the concept of V-ρ-invexity for vector-valued functions, which is a generalization of the V-invex function, and they proved the weak and strong duality for nonsmooth multiobjective programs under the V-ρ-invexity assumptions.

In this paper, we formulate nonsmooth multiobjective fractional programming problem (FP) with V-ρ-invexity and prove the Weir type duality theorems and Schaible type duality theorems for (FP) under the V-ρ-invexity assumptions. The concept of efficiency is used to formulate duality for multiobjective fractional programming problems.

2. Definitions and Preliminaries

Let R^n be the n-dimensional Euclidean space. Throughout the paper, the following convention for vectors in R^n will be adopted:

$$\begin{aligned} x > y &\Leftrightarrow x_i > y_i && \text{for all } i = 1, \cdots, n, \\ x \geqq y &\Leftrightarrow x_i \geqq y_i && \text{for all } i = 1, \cdots, n, \\ x \geq y &\Leftrightarrow x_i \geqq y_i && \text{for all } i = 1, \cdots, n, \quad \text{but } x \neq y, \end{aligned}$$

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and $x \not> y$ is the negation of $x > y$.

The real-valued function $f : R^n \to R$ is said to be locally Lipschitz if for any $z \in R^n$ there exists a positive constant K and a neighborhood N of z such that, for each $x,\ y \in N$,

$$|f(x) - f(y)| \leqq K\|x - y\|.$$

In this paper, we consider the following multiobjective fractional programming problem:

$$\text{(FP)} \qquad \text{minimize} \quad \left(\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\right)$$
$$\text{subject to} \quad x \in X = \{x \in R^n | h_j(x) \leqq 0, \text{ for } j = 1, \cdots, m\}$$

where $f_i : R^n \to R,\ g_i : R^n \to R$ for $i = 1, \cdots, p$ and $h_j : R^n \to R$ for $j = 1, \cdots, m$ are locally Lipschitz functions. We assume that $f_i(x) \geqq 0$ and $g_i(x) > 0$ on R^n for $i = 1, \cdots, p$.

The Clarke generalized directional derivative of a locally Lipschitz function f at x in the direction d denoted by $f^0(x; d)$ is as follows:

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1}(f(y + td) - f(y)).$$

The Clarke generalized subgradient of f at x is denoted by

$$\partial f(x) = \{\xi | f^0(x; d) \geqq \xi^t d \text{ for all } d \in R^n\}.$$

Now we have the following definition:

Definition 2.1. A feasible solution $\bar{x}$ for (FP) is said to be an efficient solution for (FP) if there exist no $x \in X$ such that

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(\bar{x})}{g_i(\bar{x})} \qquad \text{for all } i = 1, \cdots, p,$$

and

$$\frac{f_k(x)}{g_k(x)} < \frac{f_k(\bar{x})}{g_k(\bar{x})} \qquad \text{for some } k.$$

The problem (FP) is said to be a V-ρ-invex fractional problem if the locally Lipschitz functions $f,\ g$ and h satisfy that there exist $\alpha_i,\ \beta_j : R^n \times R^n \to R_+ \setminus \{0\},\ \rho_i,\ \sigma_j \in R$ such that for all $x,\ u \in R^n$

$$\begin{aligned}
\alpha_i(x,u)[f_i(x) - f_i(u)] &\geqq \xi_i \eta(x,u) + \rho_i \|\theta(x,u)\|^2 \text{ for each } \xi_i \in \partial f_i(u), \\
\alpha_i(x,u)[g_i(x) - g_i(u)] &\leqq \zeta_i \eta(x,u) - \rho_i \|\theta(x,u)\|^2 \text{ for each } \zeta_i \in \partial g_i(u), \\
\beta_j(x,u)[h_j(x) - h_j(u)] &\geqq \mu_j \eta(x,u) + \sigma_j \|\theta(x,u)\|^2 \text{ for each } \mu_j \in \partial h_j(u),
\end{aligned}$$

with $\eta,\ \theta : R^n \times R^n \to R^n$.

We need the following proposition from Clarke [3] in order to prove the theorems of the next section.

Proposition 2.1. (Clarke [3]). Let $p_1,\ p_2$ be Lipschitz near x, and suppose $p_2(x) \neq 0$. Then p_1/p_2 is Lipschitz near x, and

$$\partial \left(\frac{p_1}{p_2}\right)(x) \subset \frac{p_2(x)\partial p_1(x) - p_1(x)\partial p_2(x)}{(p_2(x))^2}.$$

If in addition $p_1(x) \geqq 0,\ p_2(x) > 0$ and if p_1 and $-p_2$ are regular at x, then equality holds and p_1/p_2 is regular at x.

3. Duality Theorems

For the problem (FP), we consider the following Weir type dual problem:

$$\text{(FD1)} \qquad \text{maximize} \quad \left(\frac{f_1(u)}{g_1(u)}, \cdots, \frac{f_p(u)}{g_p(u)}\right)$$

$$\text{subject to} \quad 0 \in \sum_{i=1}^{p} \tau_i \partial \left(\frac{f_i}{g_i}\right)(u) + \sum_{j=1}^{m} \lambda_j \partial h_j(u),$$

$$\lambda_j h_j(u) \geqq 0, \quad j = 1, \cdots, m,$$

$$\lambda_j \geqq 0, \quad j = 1, \cdots, m,$$

$$\tau_i \geqq 0, \quad i = 1, \cdots, p, \quad \sum_{i=1}^{p} \tau_i = 1.$$

The following result will be required in the proofs of strong duality results.

Lemma 3.1 (Chankong and Haimes [2]). $\bar{x}$ is an efficient solution for (FP) if and only if $\bar{x}$ solves

$$(\text{FP}_k) \qquad \text{minimize} \quad \frac{f_k(x)}{g_k(x)}$$

$$\text{subject to} \quad \frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(\bar{x})}{g_i(\bar{x})} \quad \text{for all } i \neq k,$$

$$h_j(x) \leqq 0, \quad j = 1, \cdots, m$$

for each $k = 1, \cdots, p$.

We prove weak and strong duality results between (FP) and (FD1).

Theorem 3.1. (Weak duality). Let x be a feasible for V-ρ-invex fractional programming problem (FP) and (u, τ, λ) a feasible for (FD1). If either of the following is satisfied:

(a) $\tau > 0$ and

$$\sum_{i=1}^{p} \frac{\tau_i}{g_i(u)} \rho_i [1 + \frac{f_i(u)}{g_i(u)}] + \sum_{j=1}^{m} \lambda_j \sigma_j \geqq 0,$$

(b)

$$\sum_{i=1}^{p} \frac{\tau_i}{g_i(u)} \rho_i [1 + \frac{f_i(u)}{g_i(u)}] + \sum_{j=1}^{m} \lambda_j \sigma_j > 0,$$

then the following cannot hold:

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(u)}{g_i(u)} \qquad \text{for all } i = 1, \cdots, p, \tag{1}$$

and

$$\frac{f_k(x)}{g_k(x)} < \frac{f_k(u)}{g_k(u)} \qquad \text{for some } k. \tag{2}$$

Proof. (a) From the feasibility conditions and $\beta_j(x, u) > 0$, we have

$$\beta_j(x, u)\lambda_j h_j(x) \leqq \beta_j(x, u)\lambda_j h_j(u) \quad \text{for } j = 1, \cdots, m.$$

Then, by the V-ρ-invexity of h, we have

$$\lambda_j \mu_j \eta(x, u) + \lambda_j \sigma_j \|\theta(x, u)\|^2 \leqq 0 \qquad \text{for each } \mu_j \in \partial h_j(u).$$

Hence we have

$$\sum_{j=1}^{m} \lambda_j \mu_j \eta(x, u) + \sum_{j=1}^{m} \lambda_j \sigma_j \|\theta(x, u)\|^2 \leqq 0 \qquad \text{for each } \mu_j \in \partial h_j(u). \tag{3}$$

Now, suppose contrary to the result of the theorem that for some feasible x for (FP) and (u, τ, λ) for (FD), (1) and (2) hold. If we let $\frac{f_i(u)}{g_i(u)} = \gamma_i$ for $i = 1, \cdots, p$, then, from the assumption $\tau > 0$, we have

$$\sum_{i=1}^{p} \tau_i [f_i(x) - \gamma_i g_i(x)] < \sum_{i=1}^{p} \tau_i [f_i(u) - \gamma_i g_i(u)].$$

Then, from the V-ρ-invexity of f and g, we have

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x, u) + \sum_{i=1}^{p} \tau_i \rho_i \|\theta(x, u)\|^2 < \sum_{i=1}^{p} \tau_i \gamma_i \zeta_i \eta(x, u) - \sum_{i=1}^{p} \tau_i \gamma_i \rho_i \|\theta(x, u)\|^2 \tag{4}$$

for each $\xi_i \in \partial f_i(u)$ and each $\zeta_i \in \partial g_i(u)$. Hence, from the first condition in constraints of (FD1) and the assumption

$$\sum_{i=1}^{p} \frac{\tau_i}{g_i(u)} \rho_i [1 + \frac{f_i(u)}{g_i(u)}] + \sum_{j=1}^{m} \lambda_j \sigma_j \geqq 0,$$

we obtain

$$\sum_{j=1}^{m} \lambda_j \mu_j \eta(x,u) + \sum_{i=1}^{p} \lambda_j \sigma_j \|\theta(x,u)\|^2 > 0,$$

which contradicts (3).

(b) Since $\tau \geq 0$, (4) holds for the inequality $\leqq$. Hence, from the assumption

$$\sum_{i=1}^{p} \frac{\tau_i}{g_i(u)} \rho_i [1 + \frac{f_i(u)}{g_i(u)}] + \sum_{j=1}^{m} \lambda_j \sigma_j > 0,$$

we obtain

$$\sum_{j=1}^{m} \lambda_j \mu_j \eta(x,u) + \sum_{i=1}^{p} \lambda_j \sigma_j \|\theta(x,u)\|^2 > 0,$$

which contradicts (3).

Remark 3.1. If we assume that either f and g are strictly V-ρ-invex functions (*i.e.*, the strict inequalities $>$ and $<$ hold instead of inequalities $\geqq$ and $\leqq$ for the definition of V-ρ-invexity of f and g, respectively) or $\sum_{j=1}^{m} \lambda_j h_j(\cdot)$ is strictly V-ρ-invex function, and the condition

$$\sum_{i=1}^{p} \frac{\tau_i}{g_i(u)} \rho_i [1 + \frac{f_i(u)}{g_i(u)}] + \sum_{j=1}^{m} \mu_j \sigma_j \geqq 0$$

holds, then we can also obtain the result of the above theorem.

Corollary 3.1. (Egudo [4]). Let the conditions of weak duality (Theorem 3.1) hold. Then if $(\bar{u}, \bar{\tau}, \bar{\lambda})$ is a feasible solution for (FD1) such that $\bar{u}$ is also feasible for (FP), then $\bar{u}$ is efficient for (FP) and $(\bar{u}, \bar{\tau}, \bar{\lambda})$ is efficient for (FD1).

Theorem 3.2. (Strong duality). Let $\bar{x}$ be an efficient solution for (FP) and assume that $\bar{x}$ satisfies a constraint qualification for (FP_k) for at least one $k = 1, \cdots, p$. Then there exist $\bar{\tau} \in R^p$ and $\bar{\lambda} \in R^m$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a feasible solution for (FD1). If also weak duality (Theorem 3.1) holds between (FP) and (FD1), then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is an efficient solution for (FD1).

Proof. Since $\bar{x}$ is efficient solution for (FP), from Lemma 3.1, $\bar{x}$ solves (FP_k) for each $k = 1, \cdots, p$. By hypothesis there exists a k such that $\bar{x}$ satisfies a constraint qualification for (FP_k). From the generalized Karush-Kuhn-Tucker necessary conditions there exist $\tau \in R^p$ and $\lambda \in R^m$ such that

$$0 \in \partial\left(\frac{f_k}{g_k}\right)(\bar{x}) + \sum_{i\neq k}\tau_i\partial\left(\frac{f_i}{g_i}\right)(\bar{x}) + \sum_{j=1}^{m}\lambda_j\partial h_j(\bar{x}), \tag{5}$$

$$\lambda_j h_j(\bar{x}) = 0, \quad j = 1,\cdots,m, \tag{6}$$

$$\tau_i \geqq 0, \quad \text{for all } i \neq k, \tag{7}$$

$$\lambda_j \geqq 0, \quad j = 1,\cdots,m. \tag{8}$$

Dividing all terms in (5) and (6) by $1+\sum_{i\neq k}\tau_i$ and setting $\bar{\tau}_k = \dfrac{1}{1+\sum_{i\neq k}\tau_i} > 0$, $\bar{\tau}_i = \dfrac{\tau_i}{1+\sum_{i\neq k}\tau_i} \geqq 0$, and $\bar{\lambda}_j = \dfrac{\lambda_j}{1+\sum_{i\neq k}\tau_i} \geqq 0$, we conclude $(\bar{x},\bar{\tau},\bar{\lambda})$ is a feasible solution for (FD1). Since weak duality (Theorem 3.1) holds between (FP) and (FD1), efficiency of $(\bar{x},\bar{\tau},\bar{\lambda})$ for (FD1) follows from Corollary 3.1.

Now we consider the following Schaible type dual problem for (FP).

(FD2) maximize $(v_1,\cdots,v_p)$

subject to

$$0 \in \sum_{i=1}^{p}\tau_i[\partial f_i(u) - v_i\partial g_i(u)] + \sum_{j=1}^{m}\lambda_j\partial h_j(u),$$

$$\sum_{i=1}^{p}\tau_i[f_i(u) - v_i g_i(u)] \geqq 0,$$

$$\lambda_j h_j(u) \geqq 0, \quad j = 1,\cdots,m,$$

$$\lambda_j \geqq 0, \quad j = 1,\cdots,m,$$

$$\tau_i \geqq 0, \quad i = 1,\cdots,p, \quad \sum_{i=1}^{p}\tau_i = 1.$$

We establish the weak and strong duality theorems between (FP) and (FD2) under assumptions of V-ρ-invexity.

¿From Lemma 3.1, we can prove the following Kuhn-Tucker type necessary optimality theorem for (FP) by the method similar to the proof in Theorem 3.4 of [7].

Theorem 3.3. Let $\bar{x}$ be an efficient solution of (FP) and assume that $\bar{x}$ satisfies a constraint qualification for (FP_k), $k = 1,\cdots,p$. Then there exist $\bar{\tau} \in R^p$, $\bar{\lambda} \in R^m$ and $\bar{v} \in R^p$ such that

$$0 \in \sum_{i=1}^{p}\bar{\tau}_i[\partial f_i(\bar{x}) - \bar{v}_i\partial g_i(\bar{x})] + \sum_{j=1}^{m}\bar{\lambda}_j\partial h_j(\bar{x}),$$

$$f_i(\bar{x}) - \bar{v}_i g_i(\bar{x}) = 0, \quad i = 1, \cdots, p,$$
$$\lambda_j h_j(\bar{x}) = 0, \quad j = 1, \cdots, m,$$
$$\bar{\tau} > 0, \quad \bar{\lambda}_j \geqq 0, \quad j = 1, \cdots, m.$$

Theorem 3.4. (Weak duality). Let x be a feasible for V-ρ-invex fractional programming problem (FP) and (u, τ, λ, v) a feasible for (FD2). If either of the following is satisfied:

(a) $\tau > 0$ and

$$\sum_{i=1}^{p} \tau_i \rho_i [1 + \frac{f_i(u)}{g_i(u)}] + \sum_{j=1}^{m} \lambda_j \sigma_j \geqq 0,$$

(b)

$$\sum_{i=1}^{p} \tau_i \rho_i [1 + \frac{f_i(u)}{g_i(u)}] + \sum_{j=1}^{m} \lambda_j \sigma_j > 0,$$

then the following cannot hold:

$$\frac{f_i(x)}{g_i(x)} \leqq v_i, \qquad \text{for all } i = 1, \cdots, p, \tag{9}$$

and

$$\frac{f_k(x)}{g_k(x)} < v_k, \qquad \text{for some } k. \tag{10}$$

Proof. (a) From the feasibility conditions and $\beta_j(x, u) > 0$, we have

$$\beta_j(x, u)\lambda_j h_j(x) \leqq \beta_j(x, u)\lambda_j h_j(u).$$

Then, by the V-ρ-invexity of h, we have

$$\lambda_j \mu_j \eta(x, u) + \lambda_j \sigma_j \|\theta(x, u)\|^2 \leqq 0 \qquad \text{for each } \mu_j \in \partial h_j(u).$$

Hence we have

$$\sum_{j=1}^{m} \lambda_j \mu_j \eta(x, u) + \sum_{j=1}^{m} \lambda_j \sigma_j \|\theta(x, u)\|^2 \leqq 0 \qquad \text{for each } \mu_j \in \partial h_j(u). \tag{11}$$

Now, suppose contrary to the result of the theorem that for some feasible x for (FP) and (u, τ, λ, v) for (FD2), such that

$$\frac{f_i(x)}{g_i(x)} \leqq v_i \text{ for all } i \text{ and } \frac{f_k(x)}{g_k(x)} < v_k \text{ for some } k.$$

Then, we have

$$f_i(x) - v_i g_i(x) \leqq 0 \text{ for all } i \text{ and } f_k(x) - v_k g_k(x) < 0 \text{ for some } k.$$

Since $\tau > 0$, we have

$$\sum_{i=1}^{p} \tau_i[f_i(x) - v_i g_i(x)] < \sum_{i=1}^{p} \tau_i[f_i(u) - v_i g_i(u)].$$

By the the V-ρ-invexity of f and g, we have

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x,u) + \sum_{i=1}^{p} \tau_i \rho_i \|\theta(x,u)\|^2 < \sum_{i=1}^{p} \tau_i v_i \zeta_i \eta(x,u) - \sum_{i=1}^{p} \tau_i v_i \rho_i \|\theta(x,u)\|^2 \tag{12}$$

for each $\xi_i \in \partial f_i(u)$ and each $\zeta_i \in \partial g_i(u)$. Hence, from the first condition in constraints of (FD2) and the assumption

$$\sum_{i=1}^{p} \tau_i \rho_i [1 + v_i] + \sum_{j=1}^{m} \lambda_j \sigma_j \geqq 0,$$

we obtain

$$\sum_{j=1}^{m} \lambda_j \mu_j \eta(x,u) + \sum_{i=1}^{p} \lambda_j \sigma_j \|\theta(x,u)\|^2 > 0,$$

which contradicts (11).

(b) Since $\tau \geq 0$, (12) holds for the inequality $\leqq$. Hence, from the assumption

$$\sum_{i=1}^{p} \tau_i \rho_i [1 + v_i] + \sum_{j=1}^{m} \lambda_j \sigma_j > 0,$$

we obtain

$$\sum_{j=1}^{m} \lambda_j \mu_j \eta(x,u) + \sum_{i=1}^{p} \lambda_j \sigma_j \|\theta(x,u)\|^2 > 0,$$

which contradicts (11).

Corollary 3.2. (Egudo [4]). Assume that the weak duality (Theorwm 3.3) holds between (FP) and (FD2). If $(\bar{u}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is a feasible solution of (FD2) such that $\bar{u}$ is a feasible solution of (FP), then $\bar{u}$ is an efficient solution of (FP) and $(\bar{u}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is an efficient solution of (FD2).

Theorem 3.5. (Strong duality). Let $\bar{x}$ be an efficient solution of (FP) and assume that $\bar{x}$ satisfies a constraint qualification for (FP_k) for at least one $k = 1, \cdots, p$. Then there exist $\bar{\tau} \in R^p$, $\bar{\lambda} \in R^m$ and $\bar{v} \in R^p$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is feasible in (FD2). If also weak duality (Theorem 3.4) holds between (FP) and (FD2), then $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is efficient for (FD2).

Proof. Since $\bar{x}$ is efficient for (FP), from Lemma 3.1, $\bar{x}$ solves (FP_k) for each $k = 1, \cdots, p$. By hypothesis there exists a k such that $\bar{x}$ satisfies a constraint qualification for (FP_k). From Theorem 3.3, there exist $\bar{\tau} \in R^p$, $\bar{\lambda} \in R^m$ and $\bar{v} \in R^p$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is feasible solution of (FD2) and $\bar{v} = \frac{f_i(\bar{x})}{g_i(x)}$, $i = 1, \cdots, p$. By the weak duality theorem (Theorem 3.4), $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is an efficient solution of (FD2).

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