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On a sign-pattern matrix and it's related algorithms for L-matrix

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Abstract

A real $m \times n$ matrix A is called an $L-matrix$ if every matrix in its qualitative class has linearly independent rows. Since the number of the sign pattern matrices of the given size is finite, we can list all patterns lexicographically. In [2], a necessary and sufficient condition for a matrix to be an L-matrix was given. We presented an algorithm which decides whether the given matrix is an L-matrix or not.

In this paper, we develope an algorithm and C-program which will determine whether a given matrix is an L-matrix or not, or an SNS-matrix or not. In addition, we have extended our algorithm to be able to classify sign-pattern matrices, and to find barely L-matrices from a given matrix and to list all $n \times n$ L-matrices.

1. Introduction

We define the *sign* of a real number a by

$$sign\, a = \begin{cases} +1 & \text{if} \quad a > 0 \\ 0 & \text{if} \quad a = 0 \\ -1 & \text{if} \quad a < 0 \end{cases}$$

The *sign pattern* of a real matrix A is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign. A real matrix A determines a *qualitative class* $Q(A)$ consisting of all matrices with the same sign pattern as A.

Consider a system of m equations in n unknowns given by

$$Ax = b \tag{1}$$

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where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

are real matrices.

The following questions are raised naturally.

"Can we solve for the signs of x_j knowing only the signs of a_{ij} and the b_i?"

This is the origin of the study of sign-solvable linear systems. In fact, this type of problem was first posed in *"Qualitative Economics"* [5] by P. A. Samuelson who discussed the possibility of determining unambiguously the qualitative behavior of solution values of a system of equations in 1947. Qualitative economists realized that a considerable body of sensible economic propositions could be expressed in a qualitative way, that is, in a form which the algebraic sign of some effect is predicted from a knowledge of the signs, only, of the relevant structural parameters of the system.

One good example of a market for a product was given in [3]. Simple economic principles tell us that as the price increases farmers will supply more number of bananas, x. When people's taste for bananas, α, is fixed, the demand for bananas decreases as the price, p, increases. When the price, p, is fixed, as people's taste α for bananas increases so does their demand for bananas. This gives a linear system of equations.

$$\begin{pmatrix} + & - \\ - & - \end{pmatrix} \begin{pmatrix} \frac{\partial p}{\partial \alpha} \\ \frac{\partial x}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ - \end{pmatrix} \tag{2}$$

Every matrix with the same sign pattern as the 2×2 matrix in (2) has a negative determinant and hence is invertible. If follows by inspection (or use of Cramer's rule) that

$$\frac{\partial p}{\partial \alpha} \quad \text{and} \quad \frac{\partial x}{\partial \alpha} \quad > \quad 0$$

independent of the magnitudes of other economic factors. Now we can conclude that the price and the number of produced bananas are increasing functions of people's taste α.

The linear system (1) is *sign-solvable* provided we can solve for the signs of the entries of x knowing only the signs of the entries of A and of b. More precisely, (1) is *sign-solvable* provided that for each matrix $\tilde{A}$ in the qualitative class $Q(A)$ and for each matrix $\tilde{b}$ in the qualitative class $Q(b)$,

$$\tilde{A}x = \tilde{b}$$

is solvable and

$$\{\tilde{x} : \text{there exists } \tilde{A} \in Q(A) \text{ and } \tilde{b} \in Q(b) \text{ with } \tilde{A}\tilde{x} = \tilde{b}\} \tag{3}$$

is entirely contained in one qualitative class. If $Ax = b$ is sign-solvable, then (3) is called the *qualitative solution class* of $Ax = b$ and is denoted by $Q(Ax = b)$. For example, the following linear system

$$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

is a sign-solvable linear system because we know the signs of x_1 and x_2 must be positive.

One class of matrices that has been heavily studied is the set of all $n \times n$ matrices with the property that every matrix with the same $(0, 1, -1)$ sign pattern is nonsingular. These are called *sign-nonsingular*, abbreviated SNS-matrix.

An $m \times n$ matrix is an *L-matrix* provided every matrix in its qualitative class has linearly independent rows. The number of rows in an L-matrix does not exceed the number of its columns ($m \leq n$). If $m = n$, an L-matrix is a sign-nonsingular matrix.

Theorem 1 [3, Theorem 1.2.1] *The homogeneous linear system $Ax = 0$ is sign-solvable if and only if every matrix in the qualitative class $Q(A)$ has linearly independent columns.*

Corollary 1 [3, Corollary 1.2.2] *If the linear system $Ax = b$ is sign-solvable, then A^T is an L-matrix.*

Recall, a square L-matrix is a sign-nonsingular matrix (SNS-matrix). Several characterizations of SNS-matrix were given in the early papers [1] and [4]. By these characterizations, we can determine whether a given matrix is sign nonsingular or not. This will help when we examine conjectures on SNS-matrices. Since many of matrix multiplications are tedious, we may use the computer for the same purpose, which is very efficient. In this paper, we prove some useful facts and develope an efficient algorithm and C-programs which will determine whether the given matrix is, an L-matrix or not, and SNS-matrix or not. In addition, we have extended our algorithm to be able to classify sign-pattern matrices, and to find *barely L-matrices*, and to list all $n \times n$ L-matrix patterns etc.

2. Algorithm for L-matrix and SNS-Matrices

Let A be an $m \times n$ real matrix. Recall that A is an L-matrix if and only if every matrix in its qualitative class $Q(A)$ has linearly independent rows. If A is an L-matrix, then every matrix obtained from A by appending column vectors is also an L-matrix. If A is an L-matrix and each of the $m \times (n-1)$ submatrices obtained from A by deleting a column is not an L-matrix, then A is called a *barely L-matrix*. Thus a barely L-matrix is an L-matrix in which every column is essential. If A is an L-matrix, then we can obtain a barely L-matrix by deleting certain columns of A. But there are barely L-matrices which are not square. For example,

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

A *signing* of order k is a nonzero $(0,1,-1)$-diagonal matrix of order k. A *strict signing* is a signing that is invertible. Let $D = \mathrm{diag}(d_1, d_2, \cdots, d_k)$ be a signing of order k with diagonal entries $d_1, d_2, \cdots, d_k$. If $k = m$, then the matrix DA is a *row signing of the matrix A*, and if D is a strict signing, then DA is a *strict row signing* of A. Similarly if k is equal to the number of columns n, then the matrix AD is a *column signing of the matrix A*, and if D is a strict signing, then AD is a *strict column signing* of A. A vector v is *balanced* provided either it is a zero vector or it has both a positive entry and a negative entry. A vector is *unisigned* provided it is not balanced. That is, a unisigned vector v is a nonzero vector all of whose nonzero entries have the same sign. A *balanced row signing of the matrix A* is a row signing of A in which all columns are balanced. A *balanced column signing* of A is a column signing of A which all rows are balanced. Using the above combinatorial definition, the L-matrix can be characterized by the following Theorem.

Theorem 2 [3, Theorem 2.1.1] *Let A be an $m \times n$ real matrix. Then A is an L-matrix if and only if every row signing of A contains a unisigned column.*

In determining whether a matrix A is L-matrix or not, it does not suffice in general to consider only strict row signing in Theorem 2. For example, let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Then every strict row singing of A unisigns its last column, but we take a signing $D = \mathrm{diag}(1, -1, 1)$, then DA has no unisigned column. Hence A is not an L-matrix. However, there are some zero patterns that we only need to consider strict row signings.

Theorem 3 [3, Theorem 2.1.2] *Let A be an $m \times n$ $(0, 1, -1)$-matrix which does not have a $p \times q$ zero submatrix for any positive integers p and q with $p + q \geq m$. Then A is an L-matrix if and only if every strict row signing of A has a unisigned column.*

For any $m \times n$ matrix A, there exist 2^m strict row signings of A. We can list all row signings of A lexicographically. Therefore we now develope an algorithm to list all row signings of A. Then we construct an algorithm determining whether the given matrix is L-matrix or not.

Now we state a remark for more efficient algorithm.

Remark 1 *For a nonzero signing D and $m \times n$ matrix A. If all column vectors of DA are balanced, then all column vectors of $-DA$ are also. And, if DA has an unsigned column vector, then $-DA$ has an unisigned column vector.*

From the above remark, we can reduce our examining process by half, $(3^m - 1)/2$. Furthermore we can use the remark for strict row signing with Theorem 3. Next, the following questions raised naturally.

Question 1 *Can we find the number of all L-matrices for a given row size m?*

Question 2 *Can we classify the set of all $n \times n$ L-matrices into some patterns ?*

In this paper, we give answers for the above questions.

In 1984, V. Klee, R. E. Lander and R. Manber [3] proved that the enumeration of all $n \times n$ L-matrices is a NP-complete problem. Even though, it is an NP-complete problem, we still need to know how to determine whether a given matrix is sign nonsingular or not. But they did not give an algorithm for finding it.

Now we develope an algorithm that answers our previous two questions, and we present a C-program for doing it. We call the program "SIGNMAT8.EXE" (Download from http://math.usu.edu/˜sglee/linear/forms.html).

Proposition 1 *If $m \times n$ matrix A is an L-matrix, then $-A$ is an $m \times n$ L-matrix.*

Proof For any $m \times n$ matrix $\tilde{A}$ in $Q(A)$, all row vectors $\{v_1, v_2, \cdots, v_m\}$ of $\tilde{A}$ in $Q(A)$ is linearly independent. Therefore the set of vectors $\{-v_1, -v_2, \cdots, -v_m\}$ is also linearly independent. This tells us $-A$ is also an L-matrix. □

Proposition 2 *If $m \times n$ matrix A is an L-matrix and P is any $m \times m$ permutation matrix, then PA is an L-matrix.*

Proof The permutation of rows does not effect the linearly independency of row vectors. This completes the proof. □

Similarly we have the following.

Proposition 3 *If $m \times n$ matrix A is an L-matrix and Q is any $n \times n$ permutation matrix, then AQ is an L-matrix.*

By using Proposition 1, 2 and 3, we can develope a much simpler algorithm for our questions. Any $n \times n$ L-matrix can be classified by some patterns. For example, an $n \times n$ L-matrix has at least $n!$ and at most $2(n!)^2$ permutation equivalent L-matrices. We now introduce our *Algorithm* for determining whether a given $m \times n$ real matrix A is an L-matrix or not.

Step 1	Input $m \times n$ sign pattern matrix A
Step 2	$i = 1$
	Repeat Step 3, 4, 5 until $i = (3^m - 1)/2$
Step 3	Generate signing D_i
Step 4	Compute row signing, $B = D_i A$
Step 5	If B has no unisigned column, then goto Step 7
Step 6	matrix A is an L-matrix ; Stop
Step 7	matrix A is not an L-matrix ; Stop

If a sign pattern matrix A satisfies the conditions of Theorem 3, then we can improve the above algorithm as follows.

	...
Step $2'$	$i = 1$
	Repeat Step $3'$, 4, 5 until $i = 2^m/2$
Step $3'$	Generate strict signing D_i
	...

Now we use the above result for determining whether a given square matrix is sign nonsingular or not. In case $m = n$, if A is an L-matrix, then it is automatically a SNS-matrix.

3. List of SNS-matrices

In this section, we list some examples of sign-nonsingular matrices, and total number of all SNS-matrices for each n.

Example 1 In case $n = 2$: SNS-matrices (total : 48)

$$\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{etc}$$

Example 2 In case $n = 3$: SNS-matrices (total : 8736)

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Example 3 In case $n = 4$: SNS-matrices (total : 9072384)

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

Example 4 In case $n = 5$: SNS-matrices

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \end{pmatrix}$$

4. C-program of the above algorithm

We made all necessary subroutines of the algorithm. All C-programs introduced in this section are compiled with Borland C ver 3.11 on MS-DOS mode. We now give the structure of the program "signmat8.exe" in this section and will show all interfaces in the next section.

SIGNMAT.C Main program

- InputSignMatrix () : Input sign pattern matrix function
- CheckLMatrix () : Check whether L-matrix or not

- SimilarMatrix () : Find similar pattern L-matrix
- BarelyLMatrix () : Find barely L-matrix
- ListLMatrix () : List all L-matrix

INMAT.C Input matrix and convert to sign pattern matrix

- ConvertSignMat () : Convert to sign pattern matrix
- InputMatrix () : Input $m \times n$ matrix

CHKLMAT.C Check whether given matrix is L-matrix or not

- IsLMatrix () : Check whether L-matrix or not
- IsBalanced () : Check whether a vector is balanced or not

INSTR.C Useful library about input-function

- iInputTextString () : Input string from console
- vXYPutChar () : Display a character
- vXYPutStr () : Display a string
- XYPutCharAttr () : Display a character with attribute at (x, y)
- HideCursor () : Toggle show/hide cursor

5. Interface of program

In this section, we show how our program "SIGNMAT8.EXE" works. The following is the main menu.

Figure 1: Main page

*

Figure 2: Input mode

After we create a new matrix by hitting the function key F1 (Input matrix), we can check by hitting the function key F2 whether it is an L-matrix or not.(Figure 1, 2, 3)

Figure 3: Check L-matrix

If the input matrix was an L-matrix, then we can list the class of similar pattern matrices with the input matrix by pressing the F3 key.(Figure 4)

Figure 4: Find similar L-matrices

Next, we can also find a barely L-matrix of the input matrix by pressing the F4 key.(Figure 5)

Figure 5: Determine barely L-matrix

We hope this program can help researches on the SNS-matrices.

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