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Mathematical Simulation of Seawater Intrusion

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Abstract

The subject of this research is to determine the optimal pumping rate so that seawater can not intrude so much to the freshwater region.

There are several ingradients affecting the fluctuation of the interface: some geological parameters, fluid parameters, the precipitation, artificial recharge and discharge(due to pumping) are such ones. The parameter of particular interest is the pumpage of freshwater. In this article all the parameters are assumed to be known except the freshwater pumping rate. By considering a suitable inverse or parameter estimation problem we want to determine the pumpage which will not make the interface rise over the permissible bound.

1 Introduction

Groundwater is one of the most important resources in the world. In many areas, water supplies for industrial, domestic, and agricultural uses are dependent on groundwater. Both the quantity and quality of groundwater may vary with environmental changes and human activities. If the groundwater managemnet problem is not seriously considered, overextraction may lead to groundwater mining, saltwater intrusion, and land subsidence. Groundwater pollution is a serious environmental problem that may damage human health, destroy the ecosystem, and cause water shortage. Since simulation models can provide forecasts of future states of groundwater systems, the optimal protection or rehabilitation strategy may be found by incorporating a simulation model into a management model.

This paper is concerned with saltwater intrusion in groundwater quality modeling. The emphasis is on numerical techniques. Saltwater intrusion into the freshwater region can be a problem in coastal area where rates of groundwater pumping are excessive. The problem can be avoided by appropriate well field design and by drilling relief wells to keep the saltwater away from the fresh groundwater source.

In this paper, we want to determine the optimal pumping rate that seawater can not encroach so much to the freshwater region. There are several ingradients affecting the fluctuation of the interface: some geological parameters, fluid parameters, the precipitation, artificial recharge and discharge(due to pumping) are such ones. The parameter of particular interest is the pumpage of freshwater. All the parameters are assumed to be known except the freshwater pumping rate. In fact, some of them can be found through suitable experiments and parameter estimation technique, c.f. see [?]. Thus we regard the fluctuation of the interface as the consequence of the pumpage. By considering a suitable inverse or parameter estimation problem we want to determine the pumpage which will not make the interface rise over the permissible bound.

This article is organized as follows. In section 2, the mathematical theories concerning our subject is briefly reviewed. In section 3, the numerical schemes both for forward problem and for inverse problem are presented. Also, a simple algorithm is described. In section 4, we test our numerical scheme with a simple test model. Concluding remarks and some comments will be made in section 5.

2 Fundamental theories

Here, we briefly review the mathematical model describing the behaviour of seawater-freshwater interface and address the parameter estimation problem.

The interface between the seawater region and the freshwater region in an unconfined aquifer can be determined by solving a IBVP for a system of nonlinear partial differential equations.

The governing equations for the flow in each seawater and freshwater region are obtained from the Mass Conservation and Darcy's law :

$$\mathrm{S_s}\frac{\partial \mathrm{h_s}}{\partial \mathrm{t}} + \mathrm{div}(\mathrm{q_s}) = 0, \qquad \mathrm{q_s} = -\mathrm{K_s}\nabla \mathrm{h_s}, \tag{2.1}$$

$$\mathrm{S_f}\frac{\partial \mathrm{h_f}}{\partial \mathrm{t}} + \mathrm{div}(\mathrm{q_f}) = 0, \qquad \mathrm{q_f} = -\mathrm{K_f}\nabla \mathrm{h_f}. \tag{2.2}$$

(??) and (??) holds in the saltwater region and in the freshwater region, respectively. Here, $\mathrm{S_s}$ and $\mathrm{S_f}$ are the specific yields, $\mathrm{K_s}$ and $\mathrm{K_f}$ are the hydraulic conductivities, and $\mathrm{h_s}$ and $\mathrm{h_f}$ denote the hydraulic heads of the seawater and freshwater regions, respectively.

The main difficulty with this basic model (??)–(??) is that we do not know the saltwater and freshwater regions until we know the hydraulic heads (the solution) themselves. So, we are confronted with the free boundary problem. It can be reduced to a simpler problem (without free boundary) if we admit the following

Hypotheses 1.1 (Dupuit assumption). 1. Flow lines are assumed to be horizontal and equipotential vertical.

2. The hydraulic gradient is assumed to be equal to the slope of the free surface and to be invariant with depth.

Under the Hypotheses 1.1, by averaging (??)–(??) in the vertical direction, we may assume that the hydraulic heads $\mathrm{h} = (\mathrm{h_s}, \mathrm{h_f})$ both for seawater and for freshwater are independent of the depth, so that they are the functions defined on a two-dimensional domain $\Omega \subset \mathbb{R}^2$, and satisfies the following system of partial differential equations :

$$(S_s b_s + n d_s^*)\frac{\partial h_s}{\partial t} - (n d_f^*)\frac{\partial h_f}{\partial t} = \mathrm{div}\,[b_s(K_s\cdot\nabla)h_s] + Q_s, \tag{2.3}$$

$$-(n d_s^*)\frac{\partial h_s}{\partial t} + (S_f b_f + n d_f^* + n)\frac{\partial h_f}{\partial t} = \mathrm{div}\,[b_f(K_f\cdot\nabla)h_f] + Q_f, \tag{2.4}$$

where n is the porosity of the aquifer, b_s and b_f are the vertical widths of the seawater and freshwater regions. Also, Q_s and Q_f denote the source-sink terms and $d_s^* = (d_s - d_f)^{-1}d_s$, $d_f^* = (d_s - d_f)^{-1}d_f$, where d_s and d_f denote the fluid densities. We will denote by Γ the boundary of Ω. For the detailed dicussions, see [?, ?, ?].

To solve the system (??)-(??) it is needed to impose initial and boundary conditions. In this article, we assumed the following ones:

$$\begin{aligned}[(K_s\cdot\nabla)h_s]\cdot\vec{n} = g_s;\\ [(K_f\cdot\nabla)h_f]\cdot\vec{n} = g_f;\end{aligned} \quad \text{on } \Gamma, \text{ for all } t\in[0,T]; \tag{2.5}$$

$$\begin{aligned}h_s(x,y,0) = h_{s0}(x,y);\\ h_f(x,y,0) = h_{f0}(x,y);\end{aligned} \quad \text{for } (x,y)\in\Omega, \tag{2.6}$$

where $\vec{n}$ denotes the unit outer normal vector on the boundary Γ of the reference domain Ω.

Notice that the model (??)-(??) is nonlinear since the functions b_s and b_f are related to h_s and h_f by

$$b_s = [d_s^* h_s - d_f^* h_f] - z_0; \quad b_f = h_f - [d_s^* h_s - d_f^* h_f], \tag{2.7}$$

where z_0 denotes the elevation of the bottom of the aquifer. Conversely, we can obtain the hydraulic heads (h_s, h_f) from the vertical widths (b_s, b_f) by solving the following linear system :

$$d_s^* h_s - d_f^* h_f = b_s + z_0; \quad -d_s^* h_s + (1 + d_f^*)h_f = b_f. \tag{2.8}$$

Once we get a solution $h = (h_s, h_f)$ for (??)-(??), the seawater-freshwater interface z_1 and the water table z_2 are given by functions $z_1, z_2 : \Omega\times[0,T]\to\mathbb{R}$,

$$z_1 = d_s^* h_s - d_f^* h_f; \quad z_2 = h_f. \tag{2.9}$$

Notice that the precipitation take part in Q_f as the source. Thus, we will decompose Q_f as $Q_f(x,y,t) = p(x,y,t) + q(x,y,t)$, where p denotes the precipitation and q the artificial recharge and pumpage. Notice also that the boundary conditions should be affected by the circumstances, especially by the precipitation history p. Throughout this article it is assumed that the flow is stable at initial time $t = 0$. If we denote by g_{f0} the stable flow rate of the freshwater at the boundary Γ, then $g_f(x,y,t) = g_{f0}(x,y) + G_f(p(t))$, where G_f is a function of precipitation p.

Now we consider the parameter estimation problem. The general theory for parameter estimation in an abstract setting can be found in [?].

As we have indicated, all the parameters including initial and boundary conditions are assumed to be known except the source-sink term Q_f. Recall that we assume that Q_f is of the form $Q_f = p + q$, where p is the term due to the precipitation and q the artificial recharge-pumpage. For simplicity, we further assume that the artificial recharge-discharge q is of the form

$$q(x, y, t) = q^1(x, y)q^2(t), \tag{2.10}$$

where q^1 is a known function. So, let us take $\mathbf{Q} = \{q\} = C[0, T]$ as the parameter space. For a fixed $q \in \mathbf{Q}$, let us denote by

$$h_s(x, y, t; q) \quad \text{and} \quad h_f(x, y, t; q)$$

the solution of (**??**)–(**??**) with the corresponding parameter. Then, we also denote by

$$z_1(x, y, t; q) \quad \text{and} \quad z_2(x, y, t; q)$$

the resulting seawater-freshwater interface and the water table surface, respectively. We want to pump out the fresh water as much as possible as long as the interface is kept unchanged. Thus, we consider the following optimization problem.

Problem (ID). Let $z^w = (z_1^w, z_2^w)$ be the wanted pair of seawater-freshwater interface and the water table. Find $q^* \in \mathbf{Q}$ that minimizes the cost functional

$$J(q) := \|\Phi(q) - z^w\|_{l^2}^2 = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}[z_1(x_i, y_i, t_j; q) - z_1^w(x_i, y_i)]^2 + \\ + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}[z_2(x_i, y_i, t_j; q) - z_2^w(x_i, y_i)]^2.$$

Here, the observation space is $Z = \mathbb{R}^{2m_1m_2}$ and the parameter-to-output mapping $\Phi : \mathbf{Q} \to Z$ is defined by $q \mapsto \{(z_1(x_i, y_i, t_j; q), z_2(x_i, y_i, t_j; q))\}_{1\le i\le m_1,\, 1\le j\le m_2}$.

The cost functional might be chosen differently along the type of possible observations. But, in most practical cases, the observations could be possible only for finite number of points and only at finite number of observation times.

Notice that this problem is infinite-dimensional. So, for actual computation, we need to approximate the problems by a sequence of finite dimensional problems. This will be addressed in the next section.

3 Numerical schemes

Following the standard procedure of the Galerkin FEM, we get an approximated solution of (**??**)–(**??**). The detailed process can be found in [**?**], which is summarized as follows.

Let $\{V^N\}_{N\in\mathbb{N}}$ be a sequence approximating the state space, where

$$V^N = \mathrm{span}\,\{\zeta_1, \ldots, \zeta_N\}.$$

Let us fix $N \in \mathbb{N}$. To obtain an approximate solution $\hat{h} = \hat{h}^N = (\hat{h}_s, \hat{h}_f)$ we set

$$\hat{h}_s = \textstyle\sum_{j=1}^N H_{sj}(t)\zeta_j(x,y); \quad \hat{h}_f = \textstyle\sum_{j=1}^N H_{fj}(t)\zeta_j(x,y)$$

and put them into the weak formulation. If we define the column vector H as

$$H = [H_{s1}(t), \cdots, H_{sN}(t), H_{f1}(t), \cdots, H_{fN}(t)]^{tr},$$

then H satisfies the following system of ODE :

$$[A]H + [B]\frac{dH}{dt} + [F] = 0, \tag{3.1}$$

where the matrices [A], [B] and the column vector [F] are defined by

$$[A] = \begin{bmatrix} \langle [b_s(K_s \cdot \nabla)\zeta_i], \nabla\zeta_j\rangle_\Omega & 0 \\ 0 & \langle [b_f(K_f \cdot \nabla)\zeta_i], \nabla\zeta_j\rangle_\Omega \end{bmatrix}_{2N\times 2N},$$

$$[B] = \begin{bmatrix} \langle [b_s S_s + nd_s^*]\,\zeta_i, \zeta_j\rangle_\Omega & -\langle nd_f^*\zeta_i, \zeta_j\rangle_\Omega \\ -\langle nd_s^*\zeta_i, \zeta_j\rangle_\Omega & \langle [b_f S_f + nd_f^* + n]\,\zeta_i, \zeta_j\rangle_\Omega \end{bmatrix}_{2N\times 2N},$$

$$[F] = \begin{bmatrix} -\langle Q_s, \zeta_i\rangle_\Omega - \langle b_s g_s, \zeta_i\rangle_\Gamma \\ -\langle Q_f, \zeta_i\rangle_\Omega - \langle b_f g_f, \zeta_i\rangle_\Gamma \end{bmatrix}_{2N\times 1},$$

where $\langle , \rangle$ denotes the usual L^2 inner product. On the other hand, the initial data is given by

$$H(0) = \begin{bmatrix} H_{sj}(0) \\ H_{fj}(0) \end{bmatrix} = \begin{bmatrix} \langle \zeta_i, \zeta_j\rangle_\Omega & 0 \\ 0 & \langle \zeta_i, \zeta_j\rangle_\Omega \end{bmatrix}^{-1} \begin{bmatrix} \langle h_{s0}, \zeta_i\rangle_\Omega \\ \langle h_{f0}, \zeta_i\rangle_\Omega \end{bmatrix}. \tag{3.2}$$

Notice that (**??**) is still nonlinear since the matrices [A], [B] and [F] depend on the solution. So, we solve the IVP (**??**)–(**??**) by an iterative method. For notational convenience, let us define maps $[A] = [\mathbf{A}](b_s, b_f)$, $[B] = [\mathbf{B}](b_s, b_f)$ and $[F] = [\mathbf{F}](b_s, b_f)$, whose meanings are clear from above. We also define maps $\mathbf{b}_s$ and $\mathbf{b}_f$ by (**??**).

If we discretize the time derivative by the backward difference, we get a system of algebraic equations :

$$[A]_n^m H_n^{m+1} + \frac{1}{\Delta t}[B]_n^m \left[H_n^{m+1} - H_{n-1}\right] + [F]_n^m = 0, \tag{3.3}$$

where subscript n denotes the n-th time step, superscript m denotes the m-th iteration, H_n the hydraulic heads to be solved at time $n\Delta t$, and H_{n-1} the known vector of hydarulic heads at time $(n-1)\Delta t$.

Algorithm `g_bFDM` (Galerkin-backward FDM scheme).

```
INPUT : N(dimension of spatial finite elements),
        M(mesh size in time),
        MI(max number of iterations), ε(tolerance);
```

0. Set $\Delta t = T/M$;

1. Determine H_0 from initial data;

   Set

   $h_{s,0} = \sum_{j=1}^{N} H_{0,j}\zeta_j$; $h_{f,0} = \sum_{j=1}^{N} H_{0,N+j}\zeta_j$;

   $b_{s,0} = \mathbf{b}_s(h_{s,0}, h_{f,0})$; $b_{f,0} = \mathbf{b}_f(h_{s,0}, h_{f,0})$;

2. for $n = 1, 2, \cdots, M$;

   Take a initial guess H_n^0 for H_n;

   Set

   $h_{s,n}^0 = \sum_{j=1}^{N} H_{n,j}^0\zeta_j$; $h_{f,n}^0 = \sum_{j=1}^{N} H_{n,N+j}^0\zeta_j$;

   $b_{s,n}^0 = \mathbf{b}_s(h_{s,n}^0, h_{f,n}^0)$; $b_{f,n}^0 = \mathbf{b}_f(h_{s,n}^0, h_{f,n}^0)$;

   FLAG=0;

   for $m = 0, 1, 2, 3, \ldots, MI$;

   Set

   $[A]_n^m = [\mathbf{A}](b_{s,n}^m, b_{f,n}^m)$;

   $[B]_n^m = [\mathbf{B}](b_{s,n}^m, b_{f,n}^m)$;

   $[F]_n^m = [\mathbf{F}](b_{s,n}^m, b_{f,n}^m)$;

   Calculate H_n^{m+1} by solving

   $$[A]_n^m H_n^{m+1} + \frac{1}{\Delta t}[B]_n^m \left[H_n^{m+1} - H_{n-1}\right] + [F]_n^m = 0;$$

   Set

   $h_{s,n}^{m+1} = \sum_{j=1}^{N} H_{n,j}^{m+1}\zeta_j$; $h_{f,n}^{m+1} = \sum_{j=1}^{N} H_{n,N+j}^{m+1}\zeta_j$;

   $b_{s,n}^{m+1} = \mathbf{b}_s(h_{s,n}^{m+1}, h_{f,n}^{m+1})$; $b_{f,n}^{m+1} = \mathbf{b}_f(h_{s,n}^{m+1}, h_{f,n}^{m+1})$;

   If $\max\{|b_{s,n}^{m+1} - b_{s,n}^m|, |b_{f,n}^{m+1} - b_{f,n}^m|\} < \varepsilon$;

   Set

   $H_n = H_n^{m+1}$;

   $h_{s,n} = \sum_{j=1}^{N} H_{n,j}\zeta_j$; $h_{f,n} = \sum_{j=1}^{N} H_{n,N+j}\zeta_j$;

   $b_{s,n} = \mathbf{b}_s(h_{s,n}, h_{f,n})$; $b_{f,n} = \mathbf{b}_f(h_{s,n}, h_{f,n})$;

   FLAG=1;

   break;

   end;

   end;

   If FLAG==0 try again with another initial guess;

   end;

```
3. for n = 0,1,2,···,M;  z_{1,n} = z_1(h_{s,n}, h_{f,n});  z_{2,n} = z_1(h_{s,n}, h_{f,n});
   end;

OUTPUT : {z_{1,n}}_{n=0}^M (seawater-freshwater interface),
         {z_{2,n}}_{n=0}^M (watertable).
```

Now we consider the approximation scheme for the parameter estimation process.

First, we construct a sequence $\{\mathbf{Q}_K\}_{K\in\mathbb{N}}$ of finite dimensional subspaces of the parameter space $\mathbf{Q} \subset C[0,T]$. Given a natural number $K \in \mathbb{N}$, let PL^K be the set of all continuous functions on $[0,T]$ which is piecewise linear with respect to the nodes $\{(i/K)T\}_{0\le i\le K}$. Let I^K be the linear spline interpolating operator on $[0,T]$, that is, for $f \in C[0,T]$, $I^K f$ is the unique continuous function on $[0,T]$ which is linear on each subinterval $[(i/K)T, ((i+1)/K)T]$, $0 \le i \le K-1$. Let us set $\mathbf{Q}_K = I^K\mathbf{Q}$.

Next, we construct a sequence $\{H^N\}$ of finite dimensional subspaces of the state space $C(\Omega \times [0,T])$. ¿From now on, we consider only rectagular domains, that is, Ω is of the form $\Omega = [0,M_1] \times [0,M_2]$. Define H^N to be the set of all continuous functions on $[0,M_1] \times [0,M_2] \times [0,T]$ which is linear on each standard finite element.

Finally we define a sequence of finite dimensional problems approximating Problem (ID).

Problem $(ID)_K^N$. Let $\{(x_i, y_i)\}_{1\le i\le m_1}$ be a sequence of points in Ω and $\{t_j\}_{1\le j\le m_2}$ a sequence of times in $[0,T]$. Find $q^* \in \mathbf{Q}_K$ that minimizes

$$J^N(q) := \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left[z_1^N(x_i,y_i,t_j;q) - z_1^w(x_i,y_i)\right]^2 + \\ + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left[z_2^N(x_i,y_i,t_j;q) - z_2^w(x_i,y_i)\right]^2,$$

where $z^N(q)$ is the approximated solution described in section 2.

We hope that each $(ID)_K^N$ has a solution q_K^N and that the sequence $\{q_K^N\}$ converges to a solution of the original problem (ID). More precisely, it is desired to prove the following statements :

(C1) For each N, K, there exists a solution q_K^N of Problem $(ID)_K^N$ and there exists a subsequence of $\{q_K^N\}$ converging to an element in $\mathbf{Q}$.

(C2) For every convergent subsequence $\{q_{K_k}^{N_k}\}$ of $\{q_K^N\}$ the following properties hold :

(a) its limit is a solution $q^* \in \mathbf{Q}$ of (ID).

(b) $\|h^{N_k}(q_{K_k}^{N_k}) - h(q^*)\| \to 0$ as $k \to \infty$.

(c) $|J^{N_k}(q_{K_k}^{N_k}) - J(q^*)| \to 0$ as $k \to \infty$.

A parameter estimation scheme satisfying the above conditions (C1) and (C2) is called function space parameter estimation convergent(FSPEC) [**?**, p.61]. We shall leave the rigorous proof for FSPEC of our approximation scheme to a future manuscript.

4 Numerical results

To illustrate the parameter estimation convergence we present examples. All the numerical computations were executed on a SUN-ULTRASPARC workstation in POSTECH under the MATLAB environment.

As the first attempt, we further simplify the situation by assuming the flow is horizontal and one-dimensional. So we assume that the spatial dimension is one.

The domain Ω, the final time T and the geological parameters are assumed as follows :

$\Omega =]0,1[\,(\mathrm{km})$; $T = 3\,(\mathrm{month})$;
$n = 0.4$;
$d_s = 1.025$; $d_f = 1.0$; $d_s^* = 41$; $d_f^* = 40$;
$K_s(x) = 3.84[\frac{m}{day}]$; $K_f(x) = 4.80[\frac{m}{day}]$;
$S_s(x) = 0.50$; $S_f(x) = 0.40$.

The initial and boundary conditions and the source-sink terms are assumed as of the following ones :

$h_{f0}(x) = -0.0324(x - 1.2727)^2 + 6.0024$;
$h_{s0}(x) = -0.0189(x + 0.6000)^2 + 5.9318$;
$g_s(0,t) = +0.0079$; $g_s(1,t) = -0.0209$;
$g_f(0,t) = -0.0356 - 0.5p(t)$; $g_f(1,t) = +0.0076 + 0.5p(t)$;
$Q_s(x,t) = 0$; $Q_f(x,t) = p(t) + q^1(x)q^2(t)$,

where p denotes the precipitation and $q = q^1(x)q^2(t)$ the artificial recharge-discharge. We choose $q^1(x)$ as a step function

$$q^1(x) = \begin{cases} 1 & \text{if} \quad x \in]0.5 - \frac{1}{8}, 0.5 + \frac{1}{8}[, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.1}$$

This is designed to model the pumping-recharging well at the position 0.5.

The above model implies the following :

$z_0 = 0$;
$z_1(x,0) = +41\left[-0.0189(x + 0.6000)^2 + 5.9318\right]$
$-40\left[-0.0324(x - 1.2727)^2 + 6.0024\right]$;
$z_2(x,0) = -0.0324(x - 1.2727)^2 + 6.0024$.

Figure 1 illustrates the initial free surfaces and the position of well. The parameters and initial head distributions (h_{s0}, h_{f0}) are chosen so that they satisfy the compatibility condition (**??**).

To solve the forward problem we choose finite element spaces as follows : For $N \in \mathbb{N}$ fixed, let $\{L_j^N\}_{0 \le j \le N}$ be the piecewise linear spline basis elements, i.e.,

$$L_j^N(x) = L(Nx - j),$$

where

$$L(s) = \begin{cases} s + 1 & \text{for } -1 \le s \le 0, \\ 1 - s & \text{for } 0 \le s \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, we set $V^N = \text{span}\,\{L_j^N\}_{0 \le j \le N}$.

To calculate the spatial integral in [A], [B] and [F] the Trapezoidal rule is adopted with the additional 5 divisons of each subinterval.

For the parameter estimation process, we choose the initial interfaces as the wanted interfaces, that is $z_1^w = z_{10}$ and $z_2^w = z_{20}$. Five observation points and twenty observation times

$$x_i = 0.25\,i, \quad i = 0, 1, 2, 3, 4; \quad t_j = 0.15\,j, \quad j = 1, \ldots, 20 \tag{4.2}$$

are chosen.

Example 4.1. First, we assume that the precipitation (in and out of the region) is given by

$$p(t) = \min\{0, (t - 0.5)(t - 1.5)\} \cos(\pi t). \tag{4.3}$$

We have assumed that there is no pumping at initial time, that is, $q(0) = 0$. Table 1 and Figure 2 show the convergence of parameter estimation. The OLS-error means the output-least-squared error

$$\sum_{i=0}^{4}\sum_{j=1}^{20}[z_1(x_i, t_j; q) - z_{10}(x_i)]^2 + \sum_{i=0}^{4}\sum_{j=1}^{20}[z_2(x_i, t_j; q) - z_{20}(x_i)]^2, \tag{4.4}$$

where x_i and t_j are as in (**??**). Figure 3 illustrate the behaviour of interfaces with the estimated pumpage.

Example 4.2. Here, we assume that the precipitation out of the region is known as (**??**). So, the boundary flux conditions for the freshwater becomes

$g_f(0, t) = -0.0356;$

$g_f(1, t) = +0.0076 + 0.5 \min\{0, (t - 0.5)(t - 1.5)\} \cos(\pi t).$

We have assumed that there is no pumping at initial time, that is, $q(0) = 0$. Table 2 and Figure 4 describe the convergence of estimation process. The OLS-error is the same as that in Example 4.1. Figure 5 shows the behaviour of interfaces with the estimated pumping rate.

In the above examples, for the minimizing process, we adopt the Finite-Difference-Levenberg-Marquadt method[**?**], which has been popularly used for the least squared error functionals.

5 Conclusions

In determining the optimal pumping rate in the coastal areas, where excessive pumpage cause the intrusion of seawater into the freshwater region, we proposed a parameter estimation approximation scheme. Numerical experiments showed the convergence of our approximation scheme. The rigorous proof will be our future research.

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Fig. 1. Initial interfaces

Fig. 2. [estimation] precipitation in and out of the region:
estimated $q^2(t)$

Fig. 3. [estimation] precipitation in and out of the region: resulted solution

Fig. 4. [estimation] precipitation out of the region: estimated $q^2(t)$

Fig. 5. [estimation] precipitation out of the region: resulted solution

Table 1. [Estimation] Precipitation in and out of the region

K	M	N	tol	OLS-error
9	8	8	1.0e-04	2.5798
	16	16	1.0e-05	2.7726
	32	32	1.0e-06	2.9083
11	8	8	1.0e-04	2.5798
	16	16	1.0e-05	2.7701
	32	32	1.0e-06	2.9065
15	8	8	1.0e-04	2.5799
	16	16	1.0e-05	2.7687
	32	32	1.0e-06	2.9039

Table 2. [Estimation] Precipitation out of the region

K	M	N	tol	OLS-error
9	8	8	1.0e-04	2.3946
	16	16	1.0e-05	2.5604
	32	32	1.0e-06	2.6777
11	8	8	1.0e-04	2.3946
	16	16	1.0e-05	2.5571
	32	32	1.0e-06	2.6752
15	8	8	1.0e-04	2.3946
	16	16	1.0e-05	2.5549
	32	32	1.0e-06	2.6719