Comm. Korean Math. Soc. **13** (1998), No. 4, pp. 913–931

INVARIANT CUBATURE FORMULAS OVER A UNIT CUBE

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ABSTRACT. Using invariant theory, new invariant cubature formulas over a unit cube are given by imposing a group structure on the formulas. Cools and Haegemans [Computing 40, 139-146 (1988)] constructed invariant cubature formulas over a unit square. Since there exists a problem in directly extending their ideas over the unit square which were obtained by using a concept of good integrity basis to some constructions of invariant cubature formulas over the unit cube, a Reynold operator will be used to obtain new invariant cubature formulas over the unit cube. In order to practically find integration nodes and weights for the cubature formulas, it is required to solve a system of nonlinear equations. With an IMSL subroutine DUNLSF which is used for solutions of the system of nonlinear equations, we shall give integration nodes for the new invariant cubature formulas over the unit cube depending on each degree of polynomial precision.

1. Introduction

The construction of cubature formulas for integration over simplexes and cones was first given by Hammer, Marlowe and Stroud[8]. Hammer, Wymore and Stroud[10, 9] obtained cubature formulas for low degrees by taking the way to prescribe all kinds of integration nodes. Many cubature formulas[22] for two particular weight functions were gained by solving a system of non-linear equations for unknown parameters in the manner described in Hammer and Stroud[9]. Since integration formulas with high degree of polynomial precision were needed, especially for shell analysis about sophisticated curved finite element, Laursen and Gellert[12] studied such formulas for triangles. For low degrees Reddy

Received November 22, 1997. Revised June 6, 1998.
1991 Mathematics Subject Classification: 65D30, 41A55.
Key words and phrases: cubature formulas, integration formulas.

and Shippy[15, 16] reduced the number of integration nodes over a triangle required for achieving a desired integration accuracy by giving changes in prescribing forms of integration nodes. In fact, Reddy and Shippy's assertions can be justified by some properties of invariant cubature formulas which are explained in Section 2. Since a concept of invariant polynomials was needed in studying error-correcting codes, Sloane[19] was interested in the theory of invariant polynomials which was one of the main branches of nineteenth century mathematics. Using the invariant theory, Cools and Haegemans[1] constructed invariant cubature formulas over a unit square by imposing a group structure on the formulas.

On the other hand, after Stroud and Secrest[22] discussed the relationship between integration formulas and orthogonal polynomials much efforts were done to extend their results to more general cases [20, 3 and 4]. Haegemans and Piessens[6, 7] constructed cubature formulas of degree 7, 9 and 11 for symmetric planar regions by using orthogonal polynomials. There have been proceeded efforts to reduce the number of integration nodes required for achieving a desired accuracy. Recently Gaussian cubature formulas having integration nodes which were based on the common zeros of either quasi-orthogonal polynomials or orthogonal polynomials were treated in 2-dimensional case[18, 17].

In this story of cubature formulas, it is natural and important to construct cubature formulas for 3-dimensional case. But there exist some problems in such a way to find cubature formulas by using either orthogonal or quasi-orthogonal polynomials. Firstly, $\dim\Pi^d_{k-1}$ distinct real common zeros of kth degree (quasi)orthogonal polynomials should exist in order to construct a Gaussian cubature formula of degree of precision $2k-1$ where Π^d_{k-1} is the set of polynomials of total degree $k-1$ in d variables. Secondly it is difficult to actually calculate $\dim\Pi^d_{k-1}$ distinct real common zeros of kth degree polynomials even though they exist[24]. Therefore the main purpose of this paper is to create new cubature formulas over a unit cube through utilizing both Molien's formula which Cools and Haegemans[1] treated in order to construct cubature formulas over a unit square and some properties of Reynold operator which are explained in Section 4. Since there exist problems in directly applying some ideas of Cools and Haegemans to some constructions of invariant cubature formulas over the unit cube, the Reynold operator will be used to obtain new invariant cubature formulas over the unit cube. One among key points in extending 2-dimensional case to 3-dimensional case is to generate a group, denoted by G^{rot} in this paper, satisfying some conditions which are mentioned in Section 3.

In Section 2, we shall treat general properties known about invariant cubature formulas and mention Molien's formula.

In Section 3, after defining a group G^{rot} we shall derive each dimension of homogeneous G^{rot}-invariant polynomials of each degree.

In Section 4, new invariant cubature formulas over the unit cube will be obtained by solving a system of non-linear equations which is derived through some properties of the Reynold operator. Also the IMSL library[11], DUNLSF, is used for solutions of the system of nonlinear equations.

In Section 5, our results are examined and compared with the known results for low degrees.

In Appendix we shall give integration nodes for invariant cubature formulas over the unit cube depending on each degree of polynomial precision.

2. Invariant cubature formula

We shall treat general properties known about invariant cubature formulas and mention Molien's formula needed to proceed our assertion.

Let $C(R^d)$ be all continuous functions defined over d-dimensional vector space R^d and let Π_k^d be the set of polynomials of degree k in d variables. Let $I : C(R^d) \to R$ be a linear functional of the form

$$I(f) = \int_\Omega w(x)f(x)dx \tag{2.1}$$

where $f \in C(R^d)$, $\Omega \subset R^d$ is a integration region and $w(x)$ is a weight function. We shall assume a notation N as the natural number, that is a set $\{1, 2, 3, \cdots\}$. An approximation of I,

$$I_n(f) = \sum_{i=1}^{n} w_i f(x_i) \quad (n \in N) \tag{2.2}$$

is called a cubature formula with weight $w_i \in R\setminus\{0\}$ and nodes $x_i \in R^d$. I_n is said to be of degree k if both $I(p) = I_n(p)$ whenever $p \in \Pi_k^d$ and $I(q) \neq I_n(q)$ for at least one $q \in \Pi_{k+1}^d$. Let G be a finite group of linear transformations $g : R^d \to R^d$. A function $f(x)$ is said to be invariant with respect to the group G if $f(g(x)) = f(x)$ for any $g \in G$ and $x \in \Omega$. Assume that the integration region Ω remains unchanged under all transformations $g \in G$ and the weight function $w(x)$ is invariant with respect to G. A cubature formula I_n is said to be G-invariant if $I_n(f) = I_n(f \circ g)$ holds for every $g \in G$ and $f \in C(R^d)$. When a G-orbit was given as a set $\{g(y) : g \in G\}$ for a given $y \in R^d$, Münzel and Renner[14] showed that I_n is a G-invariant cubature formula if and only if the set of nodes is the union of several G-orbits and weights corresponding to nodes of the same G-orbit are equal. For $p \in \Pi_k^d$ the Reynold operator[23] " $*$ " is defined as

$$p_* = \sum_{g\in G} \frac{p \circ g}{|G|} \tag{2.3}$$

where $|G|$ denotes the order of G. Let all invariant polynomials of degree k in d-variables with respect to G be given by

$$I_{k,G}^d = \{p \in \Pi_k^d : p \circ g = p \text{ for every } g \in G\} \tag{2.4}$$

and let $C_{k,G}^d$ denote the direct complement of $I_{k,G}^d$ in Π_k^d. When the cubature formula (2.2) is G-invariant the following results[5] can be obtained:

$$I(p) = I_n(p) = 0 \tag{2.5}$$

for every $p \in C_{k,G}^d$ and every $n \in N$.

Moreover, Cools and Haegemans[2] mentioned that in order for the formula (2.2) to be exact for all polynomials in Π_k^d it is necessary and sufficient that (2.2) is exact for all those polynomials which are invariant with respect to G if the formula (2.2) is G-invariant. We arrive at the position to write down Molien's formula that played an important role in developing Cools and Haegemans's assertion[1]. It is assumed that for each $i \geq 0$ homogeneous G-invariant polynomials of degree i form a finite dimensional vector space over R of dimension c_i.

THEOREM 2.1 (Molien's formula).
Let $w_1(g), \cdots, w_d(g)$ be the eigenvalues of $g \in G$. Then

$$\sum_{i=0}^{\infty} c_i t^i = \frac{1}{|G|} \sum_{g \in G} \frac{1}{(1 - w_1(g)t) \cdots (1 - w_d(g)t)}. \tag{2.6}$$

PROOF. See [19]. □

3. Molien's formula for G^{rot}

For a proper G-invariant cubature formula, the group G must satisfy two properties: the integration region Ω remains unchanged under all transformations $g \in G$ and the weight function $w(x)$ is invariant with respect to G. For example, Cools and Haegemans[1] selected that $w(x) = 1$ and

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \tag{3.1}$$

and constructed G-invariant cubature formulas over a unit square. What group G can be adopted in order to make G-invariant cubature formulas over a unit cube ? The key points in extending 2-dimensional case to 3-dimensional case lie in constructing invariant cubature formulas with properties of both efficiency and appropriateness. That is, one stresses the necessity of a group having as less order as possible in order to reduce total number of integration nodes required for achieving a desired accuracy and the other does the need of the group inducing invariant cubature formulas with evenly distributed integration nodes over integration region. From now, we shall assume both the integration region Ω as a unit cube, that is a set $\{(x, y, z) \in R^3 : |x| \le 1, |y| \le 1 \text{ and } |z| \le 1\}$, and the weight function $w(x)$ as an identity function.

We introduce the following matrices:

$$R_{xy} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

$$R_{yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{3.3}$$

$$R_{xz} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

Note that R_{xy} plays a role of 90^0 rotation in xy-plane, R_{yz} also does in yz-plane and R_{xz} also does in xz-plane.

DEFINITION 3.1. Let H be a set of linear transformations $h : R^d \rightarrow R^d$. A group $\langle h_1, h_2, \cdots, h_k \rangle$ is said to be the group generated by the generators h_i's $\in H$ where $k \in N$ and $i = 1, 2, \cdots, k$.

DEFINITION 3.2.

$$G^{rot} = \langle R_{xy}, R_{yz}, R_{xz} \rangle. \tag{3.5}$$

Then we obtain the following Lemma 3.3.

LEMMA 3.3.

$$|G^{rot}| = 24. \tag{3.6}$$

PROOF. $G^{rot} = \{I, R_{xy}, R_{xy}^2, R_{xy}^3, R_{yz}, R_{yz}^2, R_{yz}^3, R_{xz}, R_{xz}^2, R_{xz}^3, R_{xy}R_{xz}, R_{xy}^2R_{xz}, R_{xy}^3R_{xz}, R_{xy}R_{xz}^2, R_{xy}^3R_{xz}^2, R_{xy}R_{xz}^3, R_{xy}^2R_{xz}^3, R_{xy}^3R_{xz}^3, R_{yz}R_{xy}^2, R_{yz}R_{xz}, R_{yz}R_{xy}^2R_{xz}, R_{yz}R_{xz}^2, R_{yz}R_{xy}^2R_{xz}^3, R_{yz}^3R_{xy}^2R_{xz}\}$. □

Now we apply the group G^{rot} to Theorem 2.1. Fortunately we obtain a very simple form in the following Lemma 3.4.

LEMMA 3.4. *The right hand side of* (2.6) *can be written as*

$$\frac{1 - t^3 + t^6}{(1-t^2)(1-t^3)(1-t^4)} \text{ or } \frac{1 + t^9}{(1-t^2)(1-t^4)(1-t^6)}. \tag{3.7}$$

LEMMA 3.4. *The right hand side of* (2.6) *can be written as*

$$\frac{1-t^3+t^6}{(1-t^2)(1-t^3)(1-t^4)} \quad \text{or} \quad \frac{1+t^9}{(1-t^2)(1-t^4)(1-t^6)}. \tag{3.7}$$

PROOF. After determining all eigenvalues of each element of G^{rot}, it is calculated straightforwardly. □

Thus we obtain a main result in this section.

THEOREM 3.5. *Assume that* $G = G^{rot}$ *in* Theorem 2.1. *Then for nonnegative integer* i, *the dimensions* c_i*'s of homogeneous* G^{rot}*-invariant polynomials of degree* i *are given in the followings:*

$$c_1 = c_3 = c_5 = c_7 = 0, \quad c_0 = c_2 = c_9 = c_{11} = 1. \tag{3.8}$$

$$c_4 = 2, \ c_6 = 3, \ c_8 = 4, \ c_{10} = 5. \tag{3.9}$$

$$c_i = c_{i-2} + c_{i-4} - c_{i-8} - c_{i-10} + c_{i-12} \quad (i \geq 12). \tag{3.10}$$

PROOF. By (2.6) and the second term of (3.7) we know that

$$1 + t^9 = \left(\sum_{i=0}^{\infty} c_i t^i \right) (1 - t^2 - t^4 + t^8 + t^{10} - t^{12}). \tag{3.11}$$

(3.8), (3.9) and (3.10) are obtained by comparing both sides of (3.11).□

4. Classifications of G^{rot}-orbits

We obtain a useful result (4.1) through relations between some properties of the Reynold operator and all invariant polynomials of degree k in d-variables with respect to G.

THEOREM 4.1. *Assume the " $*$ " and $I^d_{k,G}$ are defined as (2.3) and (2.4). Then*

$$I^d_{k,G} = \{p_* : p \in \Pi^d_k\}. \tag{4.1}$$

PROOF. We will denote the right hand side of (4.1) as RHS. Let $p \in I^d_{k,G}$. Then for each $g \in G$, $p \circ g = p$. We obtain that

$$\sum_{g\in G} p \circ g = \sum_{g \in G} p = |G|p. \tag{4.2}$$

Therefore

$$p = \frac{1}{|G|} \sum_{g \in G} p \circ g. \tag{4.3}$$

(i.e.) $p \in$ RHS. Let $p_* \in$ RHS. For each $h \in G$,

$$p_* \circ h = \frac{1}{|G|} \sum_{g\in G} p \circ g \circ h = \frac{1}{|G|} \sum_{g \in G} p \circ g \tag{4.4}$$

because G is a group. Thus (4.1) is proved. □

By using a concept of good integrity basis Cools and Haegemans[1] found linear independent homogeneous invariant polynomials over the unit square depending on each degree. But it is not reasonable to apply the concept of good integrity basis to our case, that is the construction of G^{rot}-invariant cubature formulas over the unit cube, because the result of (3.7) forms a shape which is not easy in noticing a good integrity basis for $I^3_{G^{rot}}$. In order to overcome the above problem we shall use the property given as (4.1). By acting the Reynold operator on each monomial given in Table 4.1 we obtain linear independent homogeneous G^{rot}-invariant polynomials depending on each degree. We shall call such monomials given in Table 4.1 as generators to generate the linear independent homogeneous G^{rot}-invariant polynomials. The given generators in Table 4.1 are obtained through observing results which are derived by acting the Reynold operator on all monomials depending upon each degree. In fact most values of the results are zero. That is, calculations are not complicated thanks to symmetric position of integration nodes induced by G^{rot}. As the degrees are increased, the generators for higher degree can be obtained through the above explained procedure. But we restrict our attention until degree 12. Note that the dimension of homogeneous invariant polynomials of each degree is given in Theorem 3.5.

Table 4.1. Generators depending on each degree

degree	generators
0	1
2	x^2
4	x^4, x^2y^2
6	$x^6, x^4y^2, x^2y^2z^2$
8	$x^8, x^6y^2, x^4y^4, x^4y^2z^2$
9	x^5y^3z
10	$x^{10}, x^8y^2, x^6y^4, x^6y^2z^2, x^4y^4z^2$
11	x^7y^3z
12	$x^{12}, x^{10}y^2, x^8y^4, x^8y^2z^2, x^6y^6, x^6y^4z^2, x^4y^4z^4$

We stand at the position to discuss about all kinds of G^{rot}-orbits introduced in Section 2. All kinds of G^{rot}-orbits are induced by the following three cases of nodes:

case 1 (x , x , x).
case 2 (x , x , y), where $x \neq y$.
case 3 (x , y , z), where $x \neq y \neq z$.

The case 1 is divided into type I and III in Table 4.2, the case 2 into type II, IV and V and the case 3 into type VI and VII. Thus all

G^{rot}-orbits are classified in the following Table 4.2. By identifying (2.1) with (2.2) given as G^{rot}-invariant cubature formula for each generator which is given in Table 4.1 it is obtained a system of nonlinear equations. The IMSL library[11], DUNLSF, is used for solutions of the system of nonlinear equations. The DUNLSF is a very complicated subroutine with 12 parameters, requires initial guess estimates and different initial estimates may give rise to very different results. When we execute a program containing DUNLSF as a subroutine, a good choice of initial guess estimates can save the runtime of the program. Thus we obtain all results about the found $(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$ depending on each degree in the following Table 4.3 and such all results are found as optimal in the sense that it will cause a system of conflicting nonlinear equations if we choose another combination of $(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$ to induce less total nodes depending on each degree of polynomial precision. For example, for degree 7 if we choose another combination to induce less nodes than 27 nodes it will cause conflicting nonlinear equations.

Table 4.2. All types of G^{rot}-orbits

types	number per type	weights	representative for each orbit	#
I	t_1	ω_1	$(0,0,0)$	1
II	t_2	$\omega_{2,i}$	$(x_i,0,0)$	6
III	t_3	$\omega_{3,i}$	(x_i,x_i,x_i)	8
IV	t_4	$\omega_{4,i}$	$(x_i,x_i,0)$	12
V	t_5	$\omega_{5,i}$	(x_i,x_i,y_i)	24
VI	t_6	$\omega_{6,i}$	$(x_i,y_i,0)$	24
VII	t_7	$\omega_{7,i}$	(x_i,y_i,z_i)	24

The "#" column denotes the order of each orbit obtained by acting the group G^{rot} on representative for the orbit.

Table 4.3. Optimal choice for $(t_1, t_2, t_3, t_4, t_5, t_6, t_7)$

degree	dim	t_1	t_2	t_3	t_4	t_5	t_6	t_7	# of nodes
1	1	1	0	0	0	0	0	0	1
3	2	0	1	0	0	0	0	0	6
5	4	0	1	1	0	0	0	0	14
7	7	1	1	1	1	0	0	0	27
7	7	1	1	0	0	0	0	1	31
7	7	0	1	1	0	1	0	0	38
8	11	1	1	2	0	0	0	1	47
9	12	1	2	2	0	0	1	0	53
9	12	1	1	1	0	0	1	1	63
10	17	1	2	2	0	0	0	2	77
11	18	1	2	2	1	1	0	1	89
11	18	1	1	3	1	1	0	1	91
12	25	1	3	3	1	1	1	1	127
12	25	1	2	2	1	1	1	2	137

5. Discussion

Cools and Haegemans's results[1] over the unit square are known as best in the sense that the number of integration nodes which are used in cubature formulas depending on each degree of polynomials precision is minimal under the assumption that the location of the integration nodes should form a symmetric shape. Since we can obtain the Cools and Haegemans's results if our construction of cubature formulas over the unit cube is restricted to the unit square, we can expect good results over the unit cube. For low degrees Hammer and Stroud[9] made some cubature formulas over the unit cube not by our method to use invariant properties but by a classical method to prescribe all kinds of integration nodes, their results except for degree 5 are known as best in the above-mentioned sense about Cools and Haegemans's results and their results are given in Table 5.1. Exceptionally, for degree 5 Stroud[21] constructed 13-point cubature formula over the unit cube by a particular choice of integration nodes. For low degrees results equal to the Hammer and Stroud's products are given in Appendix. For degree 7 new two formulas are additionally obtained, one is 31-node formula having both all interior nodes to the unit cube and only one negative weight and the other is 38-node formula having both all positive weights and all interior nodes to the unit cube. For degree 8 new 47-node formula having both all positive weights and all interior nodes to the unit cube is obtained. For degree 9 new two formulas are given, one is 53-node formula having both some negative weights and some exterior nodes to the unit cube and the other is 63-node formula having both all positive weights and all interior nodes to the unit cube. For degree 10 new 77-node formula having both negative weights and some exterior nodes to the unit cube is obtained. For degree 11 new two formulas are given, one is 89-node formula having negative weights and the other is 91-node formula having both all positive weights and all interior nodes. For degree 12 new two formulas are obtained, one is 127-node formula having both negative weights and exterior nodes to the unit cube and the other is 137-node formula having both some negative weights and nearly all interior nodes. Thus all results are given in Table 5.1. Note that some cubature formulas over the unit cube can be obtained by taking the way to consecutively product one dimensional Gaussian cubature formula and such formulas are defined as Gaussian product cubature formulas. The "product" column in Table 5.1 means the total number of nodes for Gaussian product cubature formulas depending on degrees.

For the purpose of exactness of results, the following integral over the unit cube Ω is considered: For nonnegative integer α, β and γ

$$T_k = \iiint_\Omega \sum_{0\le\alpha+\beta+\gamma\le k} x^\alpha y^\beta z^\gamma dxdydz. \tag{5.1}$$

In particular, the integral results for cubature formulas of high degrees are compared in Table 5.2. The exact values in Table 5.2 are directly calculated. In fact the difference between the exact value of T_k and the approximate value by the nodes given in Appendix corresponding to each degree k of polynomial precision is at least exact up to 15 decimal digits.

For the purpose of comparison of results, the following integrals over the unit cube Ω are considered at Table 5.3 and Table 5.4.

$$V_1 = \iiint_\Omega cos(x+y+z)dxdydz = 6sin(1) - 2sin(3). \tag{5.2}$$

$$(5.3) \qquad V_2 = \iiint_\Omega exp(x+y+z)dxdydz = e^3 - 3e + 3e^{-1} - e^{-3}.$$

$$(5.4) \quad V_3 = \iiint_\Omega \sqrt{x+y+z+3}\,dxdydz = \frac{8}{105}(\sqrt{6^7} - 3*2^7 + 3*\sqrt{2^7}).$$

For even n the Composite Simpson's rule with n subintervals can be written as

$$\int_a^b f(x)dx \simeq \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right]$$

where $f(x)$ is a function defined on a closed interval [a,b], $h = (b-a)/n$ and $x_j = a + jh$ for each $j = 0, 1, \cdots, n$. The "Product Composite Simpson's rule" columns in Table 5.3 are obtained by taking the way to consecutively product one dimensional Composite Simpson's rule.

Table 5.1. Comparison of cubature formulas over a unit cube

degree	[21]	[9]	product	Appendix
1		1	1	1
3		6	8	6
5	13	14,19	27	14
7		27,34	64	27,31,38
8				47
9			125	53,63
10				77
11			216	89,91
12				127,137

The "product" column is explained in Section 5.

Calculations are executed in double precision on PC equipped with Pentium processor through utilizing Microsoft Fortran Powerstation Compiler[13]. In Appendix, the nodes for G^{rot}-invariant cubature formula over the unit cube depending on each degree of polynomial precision are given. Note that invariant cubature formulas in n-dimensional space can be considered by using the techniques presented in this paper if it is possible to find a group G in n-dimensional space corresponding to the group G^{rot} in our case.

Table 5.2. Comparison of numerical results

T_k	cubature formulas	exact values
T_7 (38 nodes)	.151280423280423D+02	.151280423280423D+02
T_8 (47 nodes)	.168956613756614D+02	.168956613756614D+02
T_9 (63 nodes)	.168956613756614D+02	.168956613756614D+02
T_{10} (77 nodes)	.183814526214526D+02	.183814526214526D+02
T_{11} (91 nodes)	.183814526214526D+02	.183814526214526D+02
T_{12} (137 nodes)	.198192604714509D+02	.198192604714509D+02

Table 5.3. Product Composite Simpson's rule

integrals	exact values	Product	Composite Simpson's	rule
		n=2(27 nodes)	n=8(4,913 nodes)	n=32(35,937 nodes)
V_1	4.7665859	4.8571602	4.7668985	4.7665871
V_2	12.984543	13.178602	12.985382	12.984546
V_3	13.640450	13.623247	13.640284	13.640449

Table 5.4. Cubature formulas by integration nodes given in Appendix

integrals	exact values	deg=7(27 nodes)	deg=7(38 nodes)	deg=8(47 nodes)
V_1	4.7665859	4.7657660	4.7660666	4.7665805
V_2	12.984543	12.983436	12.983816	12.984549
V_3	13.640450	13.641093	13.640964	13.640441

APPENDIX. Nodes for invariant cubature formulas

degree = 1 nodes= 1

weight	x	y	z
8.000000000000000	.000000000000000	.000000000000000	.000000000000000

degree = 3 nodes = 6

weight	x	y	z
1.333333333333333	1.000000000000000	.000000000000000	.000000000000000

degree = 5 nodes = 14

weight	x	y	z
.886426592797784	.795822425754222	.000000000000000	.000000000000000
.335180055401662	.758786910639328	.758786910639328	.758786910639328

degree = 7 nodes = 27

weight	x	y	z
.788073482744211	.000000000000000	.000000000000000	.000000000000000
.499369002307720	.848418011472252	.000000000000000	.000000000000000
.478508449425127	.652816472101691	.652816472101691	.652816472101691
.032303742334037	1.106412898626717	1.106412898626717	.000000000000000

degree = 7 nodes = 31

weight	x	y	z
-1.275362318840587	.000000000000000	.000000000000000	.000000000000000
.871111111111112	.585540043769119	.000000000000000	.000000000000000
.168695652173913	.694470135991705	.937161638568208	.415659267604065

degree = 7 nodes = 38

weight	x	y	z
.295189738262623	.901687807821291	.000000000000000	.000000000000000
.404055417266202	.408372221499475	.408372221499475	.408372221499475
.124850759678944	.859523090201055	.859523090201055	.414735913727988

degree = 8 nodes = 47

weight	x	y	z
.451903714875209	.000000000000000	.000000000000000	.000000000000000
.299379177352344	.782460796435947	.000000000000000	.000000000000000
.300876159371237	.488094669706371	.488094669706371	.488094669706371
.049484325587704	.862218927661482	.862218927661482	.862218927661482
.122872389222467	.281113909408340	.944196578292009	.697574833707236

degree = 9 nodes = 53

weight	x	y	z
.551726616220070	.000000000000000	.000000000000000	.000000000000000
-.039815382483699	1.168919328946080	.000000000000000	.000000000000000
.291276920125688	.813846360001057	.000000000000000	.000000000000000
.097942099058285	.821976265563318	.821976265563318	.821976265563318
.365400985686203	.515144823456534	.515144823456534	.515144823456534
.093031644998837	.650007853956632	1.017168937265364	.000000000000000

degree = 9 nodes = 63

weight	x	y	z
.475494777063152	.000000000000000	.000000000000000	.000000000000000
.204420910853443	-.715325892816490	.000000000000000	.000000000000000
.285138399553968	-.495344936668094	-.495344936668094	-.495344936668094
.114000479465275	.574088934357702	.919080086799122	.000000000000000
.053369210592410	.556745749451503	.901819045276726	.901819045276726

degree = 10 nodes = 77

weight	x	y	z
-.420319339835990	.000000000000000	.000000000000000	.000000000000000
.091997951678989	.928948149609317	.000000000000000	.000000000000000
.360442603230160	.453893457051262	.000000000000000	.000000000000000
.125224364468651	.510353784105911	.510353784105911	.510353784105911
.053615880583340	.842746471299711	.842746471299711	.842746471299711
.151750984165498	.251657335074872	.593030194462684	.856662793563785
.026372101249717	.373600837415278	1.019819719217013	.843706182617256

degree = 11 nodes = 89

weight	x	y	z
.313745381025670	.000000000000000	.000000000000000	.000000000000000
.261504969818370	.713380567938806	.000000000000000	.000000000000000
-.031924390408240	.912650951986862	.000000000000000	.000000000000000
.091458658607678	.749631791047098	.749631791047098	.749631791047098
.235430080669983	.447730316999687	.447730316999687	.447730316999687
.123866698630940	.815452385224494	.815452385224494	.000000000000000
.015007743761030	.975064418273972	.975064418273972	.599272378395484
.076961458102344	.952194555140010	.378469352111259	.378469352111259

degree = 11 nodes = 91

weight	x	y	z
.310761779537515	.000000000000000	.000000000000000	.000000000000000
.202477073612799	.812614334099629	.000000000000000	.000000000000000
.080444795437134	.745551245202776	.745551245202776	.745551245202776
.014740557641599	.911057234173168	.911057234173168	.911057234173168
.244143729769331	.402153741690512	.402153741690512	.402153741690512
.145199345860119	.734668286997006	.734668286997006	.000000000000000
.022614296138822	.941244857210604	.941244857210604	.353902814596628
.061441994097835	.965099665512710	.450799935114511	.450799935114511

degree = 12 nodes = 127

weight	x	y	z
-.073928031695602	.000000000000000	.000000000000000	.000000000000000
.010477901534402	1.511521614586340	.000000000000000	.000000000000000
-.009844095092341	1.518036150896987	.000000000000000	.000000000000000
.256679894701887	.478945364485381	.000000000000000	.000000000000000
.008524241405641	.931296821014687	.931296821014687	.931296821014687
.027746619705925	.800861835516696	.800861835516696	.800861835516696
.189643044383209	.501261758656610	.501261758656610	.501261758656610
.005698078501484	1.010788881812456	1.010788881812456	.000000000000000
.059563828742259	.850441900836910	.850441900836910	.385834359819586
.121048699150332	.875969864984466	.468783976480477	.000000000000000
.013319040393071	1.029692626670935	.642769304172527	.642769304172527

degree = 12 nodes = 137

weight	x	y	z
.196405496216727	.000000000000000	.000000000000000	.000000000000000
.193518411882601	.702454015165759	.000000000000000	.000000000000000
-.060876307453592	.906623578327549	.000000000000000	.000000000000000
.164609369967711	.358398560260120	.358398560260120	.358398560260120
.011560384831739	-.916622619558752	-.916622619558752	-.916622619558752
.015913522550276	.960988458433379	.960988458433379	.000000000000000
.043785409496396	.853137369797976	.853137369797976	.459595305781123
.084757991764383	.433691226656289	.930448523814206	.000000000000000
.015715740215575	1.002531510733000	.680321831958566	.680321831958566
.081050090532408	.685919914795626	.685919914795629	.304234417435723

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