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ON AN ERROR OF TRAPEZOIDAL RULE

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Abstract. We show that if $r \leq 2$, the average error of the Trapezoidal rule is proportional to $n^{-\min\{r+1,\ 3\}}$ where n is the number of mesh points on the interval $[0,\ 1]$. As a result, we show that the Trapezoidal rule with equally spaced points is optimal in the average case setting when $r \leq 2$.

1. Introduction

Because the available informations are limited, many numerical computations in science and engineering can only be solved approximately. If information about f is typically provided by few function values, such as $N(f) = [f(x_1), f(x_2), \ldots, f(x_n)]$, the solution is approximated by a numerical method. Therefore we have the error between the true and the approximate solutions.

The error between the true solution and the approximation depends on a problem setting. In the *worst case setting*, the error of a numerical scheme is defined by its worst performance with respect to the given class of functions. In this paper, we concentrate on another setting, the *average case setting*. In this setting, we assume that the class F of input functions is equipped with a probability measure. Then the average case error of an algorithm is defined by its expectation, rather than by its worst case performance. The average case analysis is important and significant number of results have already been obtained (see, e.g., [5] and the references cited therein).

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It is well known that the average case setting requires the space of functions to be equipped with a probability measure. In this paper, we choose a probability measure μ_r which is a variant of an *r-fold Wiener measure* ω_r. The probability measure ω_r is a *Gaussian measure* with zero mean and correlation function given by $M_{\omega_r}(f(x)\,f(y)) = \int_F f(x)f(y)\,\omega_r(df) = \int_0^1 \frac{(x-t)_+^r}{r!}\frac{(y-t)_+^r}{r!}\,dt$, where $(z-t)_+^r = [\max\{0, (z-t)\}]^r$. Equivalently, f distributed according to ω_r can be viewed as a Gaussian stochastic process with zero mean and autocorrelation given above. However, since ω_r is concentrated on functions with boundary conditions $f(0) = f'(0) = \cdots = f^{(r)}(0) = 0$, we choose to study a slightly modified measure μ_r that preserves basic properties of ω_r, yet does not require any boundary conditions. More precisely, we assume that a function f, as a stochastic process, is given by

$$f(x) = f_1(x) + f_2(1-x), \quad x \in [0,1],$$

where f_1 and f_2 are independent and distributed according to ω_r. Then the corresponding probability measure μ_r is a zero mean Gaussian with the correlation function given by

$$M_{\mu_r}(f(x)\,f(y)) = \int_0^1 \frac{(x-t)_+^r(y-t)_+^r + (t-x)_+^r(t-y)_+^r}{r!\;r!}\,dt.$$

We study the problem of approximating an integral $I(f) = \int_0^1 f(x)\,dx$ for $f \in F = C^r[0,1]$, assuming that the class of integrands is equipped with the probability measure μ_r.

2. Basic Definitions

In the integration problem, we compute an approximation to the integral $I(f) = \int_0^1 f(x)dx$, where $I : F \to \mathbb{R}$, with $f \in F = C^r[0,1]$. This approximation to $I(f)$ is computed based on n function values. That is, the available *information* $N(f)$ about the integrand f is given by $N(f) = [f(x_1), f(x_2), \cdots, f(x_n)]$, $x_i \in [0,1]$. The number n of function values is called the *cardinality* of N, and is denoted by *card*(N). Given $y = [y_1, \cdots, y_n] = N(f)$, the approximation to $I(f)$ is provided by $\phi(y) = \phi(N(f))$, where $\phi : \mathbb{R}^n \to \mathbb{R}$, called an *algorithm*, is an arbitrary mapping. Numerical quadratures $\phi(y) = \sum_{i=1}^{n} a_i f(x_i)$ with appropriately chosen weights $a_i \in \mathbb{R}$ are specific examples of algorithms. They include *composite Newton-Cotes quadratures.* Since we analyze the *composite Trapezoidal rule*, we now recall the definition and basic properties of *Trapezoidal rule*, see also e.g., [1]. In composite *Trapezoidal rule*, we let $x_0 = 0$, $x_n = 1$, and $x_i - x_{i-1} = h_i$, $i = 1, 2, \dots, n$. On each subinterval $[x_{i-1}, x_i]$, the integral $I_i(f) \equiv \int_{x_{i-1}}^{x_i} f(x)\,dx$ is approximated by

$$T_i(f) = \frac{h_i}{2}\{f(x_{i-1}) + f(x_i)\}.$$

Then, $I(f)$ is approximated by $I(f) = \sum_{i=1}^{n} I_i(f) \approx T(N(f)) = \sum_{i=1}^{n} T_i(f)$.

For the average case setting, we assume that the space $F = C^r[0,1]$ is equipped with a probability measure μ_r which is a variant of the *r-fold Wiener measure.* In order to define it, we first recall basic properties of the classical *r-fold Wiener measure* ω_r, see [2], [4] and [6]. It is a Gaussian measure with zero mean and correlation function given by

$$M_{\omega_r}(f(x)\,f(y)) = \int_F f(x)f(y)\,\omega_r(df) = \int_0^1 \frac{(x-t)_+^r}{r!}\,\frac{(y-t)_+^r}{r!}\,dt.$$

More precisely, we assume that a function f, as a stochastic process, is given by $f(x) = f_1(x) + f_2(1-x)$, where f_1 and f_2 are independent and distributed according to ω_r. Equivalently, this leads to the probability measure μ_r defined on σ-field of the space $C^r[0,1]$ that is zero mean Gaussian with the correlation function given by

$$\begin{aligned} M_{\mu_r}(f(x)\,f(y)) &= \int_0^1 \frac{(x-t)_+^r(y-t)_+^r + (1-x-t)_+^r(1-y-t)_+^r}{r!\,r!}\,dt \\ &= \int_0^1 \frac{(x-t)_+^r(y-t)_+^r + (t-x)_+^r(t-y)_+^r}{r!\,r!}\,dt. \end{aligned}$$

The *average error* of an algorithm ϕ that uses N is defined by

$$\begin{aligned} e^{avg}(\phi, N; \mu_r) &= \left(M_{\mu_r}\left([I(f) - \phi(N(f))]^2\right)\right)^{1/2} \\ &= \left(\int_F |I(f) - \phi(N(f))|^2 \mu_r(df)\right)^{1/2} \end{aligned}$$

It is known, see [3], that for the *r-fold Wiener measure* ω_r, the average error of any algorithm that uses information of cardinality n is bounded from below by

$$e^{avg}(\phi, N; \omega_r) = \Omega\left(n^{-(r+1)}\right),^{1} \quad \forall\phi,\ \forall N,\ card(N) = n.$$

3. Average case error of Trapezoidal rule

Recall that the space $F = C^r[0,1]$ is equipped with the probability measure μ_r defined in chapter 2. The error $I(f)-T(N(f))$ of *Trapezoidal rule* equals

$$I(f) - T(N(f)) = \sum_{i=1}^{n} Z_i, \quad \text{where } Z_i = Z_i(f) = I_i(f) - T_i(f).$$

Since f is a zero-mean Gaussian process, Z_i's are zero-mean Gaussian random variables with covariances given in the following lemma.

LEMMA 3.1. *For* $r \le 2$,

$$M_{\mu_r}(Z_i Z_j) = \delta_{ij} \cdot c_r \cdot h_i^{2r+3} \quad \text{for} \quad i \le j,$$

where δ_{ij} *is the Kronecker delta and the constant* c_r *is independent of* h_i*'s and equals respectively:* $c_0 = \frac{1}{12}$, $c_1 = \frac{1}{60}$, *and* $c_2 = \frac{13}{2520}$.

PROOF. Let $Z_{i1} = Z_i(f_1)$ and $Z_{i2} = Z_i(f_2)$. Then $Z_i(f) = Z_{i1} + Z_{i2}$, and due to the independence of f_1 and f_2, we have $M_{\mu_r}(Z_i Z_j) = M_{\omega_r}(Z_{i1} Z_{j1}) + M_{\omega_r}(Z_{i2} Z_{j2})$. For $i \le j$,

$$\begin{aligned} &M_{\omega_r}(Z_{i1} Z_{j1}) \\ &= \int_0^1 \left[\int_{x_{i-1}}^{x_i} \frac{(x-t)_+^r}{r!}\, dx - A_{i1}(t)\right] \left[\int_{x_{j-1}}^{x_j} \frac{(y-t)_+^r}{r!}\, dy - A_{j1}(t)\right] dt \\ &= \int_0^1 L_{i1}(t) \cdot L_{j1}(t)\, dt, \end{aligned}$$

[1] $f(n) = \Omega(g(n))$ means that there is a positive constant C such that $f(n) \ge C g(n)$, $\forall n$.

where L_{i1} is the first term and L_{j1} is the second term in the above integral, and $A_{i1}(t) = T_i\left(\frac{(\cdot - t)_+^r}{r!}\right)$. Since $L_{i1}(t) = 0$ when $t \in [x_i, 1]$, we have

$$M_{\omega_r}(Z_{i1}Z_{j1}) = \int_0^{x_i} L_{i1}(t) \cdot L_{j1}(t)\, dt.$$

Similarly,

$$\begin{aligned} &M_{\omega_r}(Z_{i2}Z_{j2}) \\ &= \int_{x_{j-1}}^1 \left[\int_{x_{i-1}}^{x_i} \frac{(t-x)_+^r}{r!}\, dx - A_{i2}(t)\right]\left[\int_{x_{j-1}}^{x_j} \frac{(t-y)_+^r}{r!}\, dy - A_{j2}(t)\right] dt \\ &= \int_{x_{j-1}}^1 L_{i2}(t) \cdot L_{j2}(t)\, dt, \end{aligned}$$

where L_{i2} is the first term and L_{j2} is the second term in the above integral, and $A_{i2}(t) = T_i\left(\frac{(t-\cdot)_+^r}{r!}\right)$. Since $L_{j2}(t) = 0$ when $t \in [0, x_{j-1}]$, we therefore have

$$M_{\mu_r}(Z_iZ_j) = \int_0^{x_i} L_{i1}(t) \cdot L_{j1}(t)\, dt + \int_{x_{j-1}}^1 L_{i2}(t) \cdot L_{j2}(t)\, dt.$$

Since *Trapezoidal rule* is exact for polynomials of degree ≤ 2, $L_{j1}(t) = 0$ for $t \le x_i$ and $L_{i2}(t) = 0$ for $t \ge x_{j-1}$. Thus, $M_{\mu_r}(Z_iZ_j) = 0$, and hence, Z_i and Z_j are independent when $i < j$. For $i = j$, let $z = \frac{x - x_{i-1}}{h_i}$ and $u = \frac{t - x_{i-1}}{h_i}$. Then

$$\begin{aligned} &M_{\omega_r}(Z_{i1}^2) \\ &= \int_{x_{i-1}}^{x_i} \left[\int_{x_{i-1}}^{x_i} \frac{(x-t)_+^r}{r!}\, dx - T_i\left(\frac{(\cdot - t)_+^r}{r!}\right)\right]^2 dt \\ &= \int_0^1 \left[\int_0^1 \frac{{h_i}^r(z-u)_+^r}{r!}\, h_i dz - \frac{h_i}{2}\left\{\frac{{h_i}^r(0-u)_+^r}{r!} + \frac{{h_i}^r(1-u)_+^r}{r!}\right\}\right]^2 h_i du \\ &= c_{r1}h_i^{2r+3}, \end{aligned}$$

where

$$c_{r1} = \int_0^1 \left[\int_0^1 \frac{(z-u)_+^r}{r!}\, dz - \frac{1}{2}\left\{ \frac{(0-u)_+^r}{r!} + \frac{(1-u)_+^r}{r!} \right\} \right]^2 du.$$

Similarly, $M_{\omega_r}(Z_{i2}^2) = c_{r2} h_i^{2r+3}$, where

$$c_{r2} = \int_0^1 \left[\int_0^1 \frac{(u-z)_+^r}{r!}\, dz - \frac{1}{2}\left\{ \frac{(u-0)_+^r}{r!} + \frac{(u-1)_+^r}{r!} \right\} \right]^2 du.$$

We now calculate $c_r = c_{r1} + c_{r2}$. For $r = 0$,

$$c_{01} = \int_0^1 \left[\int_u^1 dz - \frac{1}{2} \right]^2 du = \int_0^1 \left[\frac{1}{2} - u \right]^2 du = \frac{1}{24}$$

and

$$c_{01} = \int_0^1 \left[\int_0^u dz - \frac{1}{2} \right]^2 du = \int_0^1 \left[u - \frac{1}{2} \right]^2 du = \frac{1}{24}.$$

For $r = 1$,

$$\begin{aligned} &c_{11} \\ &= \int_0^1 \left[\int_0^u (z-u)_+\, dz + \int_u^1 (z-u)_+\, dz - \frac{1}{2}\left\{ (0-u)_+ + (1-u)_+ \right\} \right]^2 du \\ &= \int_0^1 \left[\int_u^1 (z-u)\, dz - \frac{1}{2}(1-u) \right]^2 du = \int_0^1 \left[\frac{1}{2}u^2 - \frac{1}{2}u \right]^2 du \\ &= \frac{1}{120} \end{aligned}$$

and

$$\begin{aligned} &c_{12} \\ &= \int_0^1 \left[\int_0^u (u-z)_+\, dz + \int_u^1 (u-z)_+\, dz - \frac{1}{2}\left\{ (u-0)_+ + (u-1)_+ \right\} \right]^2 du \\ &= \int_0^1 \left[\int_0^u (u-z)\, dz - \frac{1}{2}(u-0) \right]^2 du = \int_0^1 \left[\frac{1}{2}u^2 - \frac{1}{2}u \right]^2 du \\ &= \frac{1}{120}. \end{aligned}$$

For $r = 2$,

$$c_{21} = \int_0^1 \left[\int_0^u \frac{(z-u)_+^2}{2}\, dz + \int_u^1 \frac{(z-u)_+^2}{2}\, dz - \frac{1}{2}\left\{ \frac{(0-u)_+^2}{2} + \frac{(1-u)_+^2}{2} \right\} \right]^2 du$$
$$= \int_0^1 \left[\int_u^1 \frac{(z-u)^2}{2}\, dz - \frac{1}{2}\frac{(1-u)^2}{2} \right]^2 du$$
$$= \int_0^1 \left[\frac{1}{6}(1-u)^3 - \frac{1}{4}(1-u)^2 \right]^2 du$$
$$= \frac{13}{5040}$$

and

$$c_{22} = \int_0^1 \left[\int_0^u \frac{(u-z)_+^2}{2}\, dz + \int_u^1 \frac{(u-z)_+^2}{2}\, dz - \frac{1}{2}\left\{ \frac{(u-0)_+^2}{2} + \frac{(u-1)_+^2}{2} \right\} \right]^2 du$$
$$= \int_0^1 \left[\int_0^u \frac{(u-z)^2}{2}\, dz - \frac{1}{2}\frac{(u-0)^2}{2} \right]^2 du$$
$$= \int_0^1 \left[\frac{1}{6}u^3 - \frac{1}{4}u^2 \right]^2 du$$
$$= \frac{13}{5040}.$$

Therefore $c_0 = \frac{1}{12}$, $c_1 = \frac{1}{60}$ and $c_2 = \frac{13}{2520}$. This completes the proof. □

In the next theorem that is the main theorem of this paper, we show that the *Trapezoidal rule* with equally spaced points is optimal in the average case setting when $r \leq 2$.

THEOREM 3.2. *For any information N_n of cardinality n,*

$$e^{avg}(S, N_n; \mu_r) \;=\; \Omega\left(n^{-\min\{r+1,\,3\}}\right).$$

Furthermore, for $r \leq 2$, Trapezoidal rule at equally spaced points is almost optimal among all algorithms that use n functions values at arbitrary points.

PROOF. Assume $r \leq 2$. Since Z_i's are independent,

$$e^{avg}(S, N_n; \mu_r)^2 = \sum_{i=1}^{n} M_{\mu_r}(Z_i^2) = c_r \cdot \sum_{i=1}^{n} h_i^{2r+3}$$

with c_r given in Lemma 3.1. To minimize the above expression, we need to solve

$$\frac{\partial}{\partial h_j} \sum_{i=1}^{n} h_i^{2r+3} = 0, \quad \text{for } j = 1, 2, \dots, n,$$

subject to $\sum_{i=1}^{n} h_i = 1$. Then, since $h_n = 1 - \sum_{i=1}^{n-1} h_i$,

$$\begin{aligned} &\frac{\partial}{\partial h_j} \left(\sum_{i=1}^{n-1} h_i^{2r+3} + \left(1 - \sum_{i=1}^{n-1} h_i\right)^{2r+3} \right) \\ &= (2r+3) h_j^{2r+2} - (2r+3) \left(1 - \sum_{i=1}^{n-1} h_i\right)^{2r+2} \\ &= 0, \text{ for } j = 1, \dots, n-1. \end{aligned}$$

Thus, we have

$$h_j = 1 - \sum_{i=1}^{n-1} h_i = h_n \text{ for } j = 1, \dots, n-1.$$

Hence, $\sum h_i^{2r+3}$ is minimized when all h_i's are equal. Let $h = h_i$ for all i. Then, we have

$$\begin{aligned} e^{avg}(S, N_n; \mu_r)^2 &= c_r \sum_{i=1}^{n} h_i^{2r+3} \geq c_r \sum_{i=1}^{n} h^{2r+3} \\ &= \frac{c_r}{2} h^{2r+2}. \end{aligned}$$

This completes the case of $r \leq 2$. □

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