Comm. Korean Math. Soc. **13** (1998), No. 4, pp. 875–888

GENERALIZED INVERSES IN NUMERICAL SOLUTIONS OF CAUCHY SINGULAR INTEGRAL EQUATIONS

S. KIM

ABSTRACT. The use of the zeros of Chebyshev polynomial of the first kind $T_{4n+4}(x)$ and second kind $U_{2n+1}(x)$ for Gauss-Chebyshev quadrature and collocation of singular integral equations of Cauchy type yields computationally accurate solutions over other combinations of $T_n(x)$ and $U_m(x)$ as in [8]. We show that the coefficient matrix of the overdetermined system has the generalized inverse. We estimate the residual error using the norm of the generalized inverse.

1. Introduction

An integral equation of the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{g(t)dt}{t-s}+\lambda\int_{-1}^{1}g(t)\kappa(s,t)dt=f(s),\quad -1<s<1, \tag{1}$$

where $g(s)$ is an unknown function to be determined is called a Cauchy Singular Integral Equation (CSIE). The singular integral is interpreted as a Cauchy principle value. In many problems of physical interest, κ are matrices of functions, and g and f are vector functions. The integral equation derived from (1) by putting $\kappa(t,s)=0$, viz,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{g(t)dt}{t-s}=f(x),\quad -1<s<1, \tag{2}$$

Received April 30, 1997. Revised June 5, 1998.

1991 Mathematics Subject Classification: 65.

Key words and phrases: generalized inverses, numerical solutions of integral equations.

Research supported by Korean Ministry of Education, BSRI-97-1430 and KOSEF 97-0701-02-01-5.

is called the dominant equation. The classical theory based on the properties of sectionally holomorphic function includes theorems of the Fredholm type derived through a process of regularization. Few problems, however, admit closed form analytical solutions. While in principle numerical techniques may be applied to the regularized equation, direct methods based on quadrature and collocation or Galerkin approximation-minimum residual error in an appropriate norm have been popular since their inception in the paper by Erdogen and Gupta [4]. The integral equations considered by Erdogen and Gupta are special cases of (1).

In a quadrature-collocation scheme the integrals are replaced by suitable formulae for numerical integration [1, 3, 7]. These formulae use the values, of the function to be determined, at specified nodes called abscissae of integration. A system of algebraic equations is obtained ignoring the quadrature error and satisfying the resulting equation at a finite set of collocation points. The error in the approximation will depend on the quadrature error, which is expected to decrease for the larger number of nodes, and the characteristics of the linear algebraic system. In particular, the error will be significantly effected by the proximity of the eigenvalues of the coefficient matrix to the parameter λ. Thus, for reliably accurate solutions, it is desirable to have a variety of techniques available, which are amendable to estimation of error.

Three principal sources of error in the numerical solution of singular integral equations are as follows:

1. Quadrature error, especially if the abscissae of integration and weights have to be computed.
2. Sampling error, which will be incurred if the derivatives of the function f and the kernal κ are large.
3. Poor conditioning of the matrix due to the proximity of the eigenvalues.

In this paper, we consider the equation (2). We investigate the behavior of the proposed techniques for the complete equation (1) because the Fredholm term

$$\int_{-1}^{1} g(t)\kappa(t,s)dt$$

usually plays a role of moderate perturbation. Equation (2) has a solution of index -1 if

$$\int_{-1}^{1} \frac{f(x)dx}{\sqrt{1-x^2}} = 0. \tag{3}$$

If the equation (1) has a bounded solution, then

$$\int_{-1}^{1} (1-x^2)^{-1/2} \left\{ f(x) - \int_{-1}^{1} \kappa(x,t) g(t) dt \right\} dx = 0. \tag{4}$$

The condition (4) cannot be checked except the case

$$\int_{-1}^{1} (1-x^2)^{-1/2} \kappa(x,t) dx = 0,$$

where $g(t)$ is not involved in (4). One of the cases that does not involve $g(t)$ in (4) is when $\kappa(x,t)$ is an odd function.

Numerical method for solving (2) by [5] sets

$$g(t) = \phi(t)(1-t^2)^{1/2}.$$

Then, the zeros of $T_{2n+2}(x)$ and $U_{2n+1}(x)$ have been used for Gauss-Chebyshev quadrature when $\phi(t)$ is odd and $f(x)$ is even, and a consistent $(n+1) \times n$ overdetermined system of equations is obtained. The odd-and-even splitting of a problem will reduce the arithmetic of solving the linear system to approximately one quarter of the system.

It is shown in [8] that the use of the zeros of $T_{2d(n+1)}(x)$ and $U_{2n+1}(x)$ where $d = 1, 2, 4, \ldots$, for the quadrature yields $d(n+1) \times n$ overdetermined system of equations and $d = 2$ provides the most accurate solutions among other choices of $d's$ in computational tests. We used even integers for $d \geq 2$ since odd d's did not determine the element of the coefficient matrix. Increasing d has the effect of taking more collocation points. Solving the resulting overdetermined linear system, however, may introduce large errors to the solution. In this paper, we find the generalized inverse matrix of the coefficient matrix for $d = 2$. Norms of the generalized inverse of the coefficient matrix is computed to study the error 3 mentioned above and explain the accuracy of the computed solutions.

In section 2, we will describe overdetermined systems. Then, we find the generalized inverse matrix for the coefficient matrix in section 3.

2. Overdetermined systems

We first derive the linear system of algebraic equations. The Chebyshev polynomial of the first kind $T_k(x)$ is defined by

$$T_k(x) = \cos(k \cos^{-1} x), \quad k = 0, 1, 2, \cdots .$$

Let $\{t_i^{(k)}\}$ denote the zeros of $T_k(x)$ and $t_1^{(k)}$ be the zero nearest to 1 and $t_k^{(k)}$ nearest to -1. That is,

$$t_i^{(k)} = \cos\left(\frac{(i-1/2)\pi}{k}\right), \quad i = 1, 2, \cdots, k.$$

The Chebyshev polynomial of the second kind $U_k(x)$ is given by

$$U_k(x) = \frac{\sin\{(k+1)\cos^{-1} x\}}{\sin\{\cos^{-1}(x)\}}.$$

Let $\{s_j^{(k)}\}$ denote the zeros of $U_k(x)$ with $s_1^{(k)}$ being nearest to 1 and $s_k^{(k)}$ nearest to -1, i.e.,

$$s_j^{(k)} = \cos\left(\frac{j\pi}{k+1}\right), \quad i = 1, 2, \cdots, k.$$

Note that the zeros are distributed symmetrically about the origin.

We have [5]

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{(1-t^2)}\phi(t)dt}{t-x} = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{(1-t^2)}(\phi(t)-\phi(x))dt}{t-x} - x\phi(x).$$

If $\{s_j^{(2n)}\}$ are used as abscissae for quadrature and the integral is evaluated for $x = t_k^{(2n+1)}$, then, in a discretized form,

(5)

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{(1-t^2)}\phi(t)dt}{t-x} \approx \frac{1}{2n+1}\sum_{j=1}^{2n} \frac{(1-s_j^2)\phi(s_j)}{s_j - t_i}, \quad i = 1, 2, \cdots, 2n+1.$$

If $\{s_j^{(2n)}\}$ are used as abscissae and $\{t_i^{(d(2n+1))}\}$, $d = 2, 4, \cdots$, as collocation, then, as shown in [8], where $x = t_i^{d(2n+1)}$,

$$\begin{aligned}
&\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{(1-t^2)}(\phi(t)-\phi(x))dt}{t-x} - x\phi(x)\\
&= \frac{1}{2n+1}\sum_{j=1}^{2n}\frac{(1-s_j^2)(\phi(s_j)-\phi(x))}{s_j-x} - x\phi(x)\\
&= \frac{1}{2n+1}\sum_{j=1}^{2n}\frac{(1-s_j^2)\phi(s_j)}{s_j-x} + \phi(x)\left(\frac{1}{2n+1}\sum_{j=1}^{2n}\frac{1-s_j^2}{x-s_j} - x\right)\\
&= \frac{1}{2n+1}\sum_{j=1}^{2n}\frac{(1-s_j^2)\phi(s_j)}{s_j-x} + \phi(x)\left(\frac{xU_{2n}(x)-T_{2n+1}(x)}{U_{2n}(x)} - x\right)\\
&= \frac{1}{2n+1}\sum_{j=1}^{2n}\frac{(1-s_j^2)\phi(s_j)}{s_j-x} - \phi(x)\frac{T_{2n+1}(x)}{U_{2n}(x)}.
\end{aligned}$$

Note that if x is a zero of $T_{2n+1}(x)$, then $-\phi(x)\frac{T_{2n+1}(x)}{U_{2n}(x)}$ is zero, which is the same discretized form as (5). We use Lagrange interpolation formula for $\phi(x)$ as follows. Let

$$\phi(x) = \sum_{j=1}^{2n}\phi(s_j)\frac{U_{2n}(x)}{x-s_j}\cdot\frac{1}{U'_{2n}(s_j)}.$$

Then,

$$-\phi(x)\frac{T_{2n+1}(x)}{U_{2n}(x)} = -\sum_{j=1}^{2n}\phi(s_j)\frac{T_{2n+1}(x)}{(x-s_j)U'_{2n}(s_j)}.$$

Therefore, the discretized form is

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{(1-t^2)}\phi(t)dt}{t-x} = \frac{1}{2n+1}\sum_{j=1}^{2n}\frac{(1-s_j^2)\phi(s_j)}{s_j-x} - \sum_{j=1}^{2n}\phi(s_j)\frac{T_{2n+1}(x)}{(x-s_j)U'_{2n}(s_j)}.$$

If we use $s_j^{(2n+1)}$ for abscissae and $t^{(d(2n+2))}$ as collocation points, the discrete form is

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{(1-t^2)}\phi(t)dt}{t-x}$$

$$(6) \qquad = \frac{1}{2n+2}\sum_{j=1}^{2n+1}\frac{(1-s_j^2)\phi(s_j)}{s_j-t_i} - \sum_{j=1}^{2n+1}\phi(s_j)\frac{T_{2n+2}(t_i)}{(t_i-s_j)U'_{2n+1}(s_j)},$$

where $i = 1, 2, \cdots, d(2n+2)$, $d = 2, 4, \cdots$.

It remains to check whether the condition (3) is satisfied in the equation

$$(7) \qquad \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{(1-t^2)}\phi(t)dt}{t-x} = f(x), \quad -1 < x < 1.$$

Note that $\phi(t)$ is an odd function of t if $f(x)$ is even and $\phi(t)$ is an even function of t if $f(x)$ is odd in (7). The condition (3) is satisfied in the latter case.

If $f(x)$ is odd and $\phi(t)$ is even, then a bounded solution always exists. Suppose that $\phi(s_{n+1}) = 0$ for $s_{n+1} = 0$. When we use $s_j^{(2n+1)}$ for abscissae and $t_i^{(d(2n+2))}$ as the collocation points, using the symmetry, we can write (6) as

$$(8) \qquad \sum_{j=1}^{n}\left[\frac{1}{(n+1)}\frac{t_i(1-s_j)\phi(s_j)}{s_j^2-t_i^2} - \frac{2t_iT_{2n+2}(t_i)\phi(s_j)}{(t_i^2-s_j^2)U'_{2n+1}(s_j)}\right] = f(t_i),$$

where $i = 1, 2, \cdots, d(n+1)$, $d = 2, 4, \cdots$.

Let us consider the case that $f(x)$ is even, $\phi(t)$ is an odd function, and (3) is satisfied. The generalized inverse for (8) can be obtained similarly. If we use $s_j^{(2n+1)}$ for abscissae and $t_i^{(d(2n+2))}$ as the collocation points, the discrete form of the condition (3) is

$$\sum_{i=1}^{d(2n+2)} f(t_i) = 0.$$

Equivalently, using that $f(x)$ is even,

$$\sum_{i=1}^{d(n+1)} f(t_i) = 0.$$

Using the formula (6), $s_{n+1} = 0$, $\phi(s_{n+1}) = 0$ for continuous ϕ, we have the equations

$$\sum_{j=1}^{n} \left[\frac{1}{(n+1)} \frac{s_j(1-s_j^2)\phi(s_j)}{s_j^2 - t_i^2} - \phi(s_j) \frac{2s_j T_{2n+2}(t_i)}{(t_i^2 - s_j^2)U'_{2n+1}(s_j)} \right] = f(t_i), \tag{9}$$

where $i = 1, 2, \cdots, d(n+1)$, $d = 2, 4, \cdots$.

The overdetermined system can be represented in matrix form as

$$A\mathbf{x} = \mathbf{b} \tag{10}$$

where A is a $d(n+1) \times n$ matrix whose (i,j)th element for (9) is

$$\frac{1}{n+1} \cdot \frac{s_j^2(1-s_j^2)}{s_j^2 - t_i^2} - \frac{2s_j^2 \cdot T_{2n+2}(t_i)}{(t_i^2 - s_j^2)U'_{2n+1}(s_j)}, \tag{11}$$

$\mathbf{x}$ is the n vector whose ith component is $\phi(s_j)/s_j$, and $\mathbf{b}$ is the $d(n+1)$ vector whose ith component is $f(t_i)$. Therefore, we have the overdetermined system whose number of rows depends on d.

Identities Involving Zeros of Chebyshev Polynomials

We will use some of the following identities involving zeros of $T_n(x)$ and $U_{n-1}(x)$ in finding the generalized inverse of the coefficient matrix obtained using the Gauss-Chebyshev quadrature.

Sums with zeros of $T_n(x)$ and $U_{n-1}(x)$: [2]

$$
\begin{aligned}
\sum_{j=1}^{n-1}(t_k - s_j)^{-1} &= t_k(1-t_k^2)^{-1}, \qquad k = 1,2,\dots,n,\\
\sum_{j=1}^{n-1}(t_k - s_j)^{-2} &= n(1-t_k^2)^{-1} - (1+t_k^2)(1-t_k^2)^{-2},\\
& \qquad k = 1,2,\dots,n,\\
\sum_{k=1}^{n}(s_j - t_k)^{-1} &= 0, \qquad j = 1,2,\dots,n-1, \qquad (12)\\
\sum_{k=1}^{n-1}(s_j - t_k)^{-2} &= n^2(1-s_j^2)^{-1}, \quad j = 1,2,\dots,n-1,\\
\sum_{k=1}^{n}(1-t_k^2)^{-1} &= n^2\\
\sum_{k=1}^{n}(1-t_k)^{-2} &= \frac{n^2(2n^2+1)}{3},\\
\sum_{j=1}^{n} t_j &= 0,\\
\sum_{j=1}^{n} t_j^2 &= \frac{n}{2}.
\end{aligned}
$$

Sums with the positive zeros of $T_{2n}(x)$ and $U_{2n-1}(x)$:

$$\sum_{k=1}^{n}(t_k^2 - s_j^2)^{-1} = 0, \tag{13}$$

$$\sum_{k=1}^{n}(s_j^2 - t_k^2)^{-2} = n^2[s_j^2(1-s_j^2)^{-1}]. \tag{14}$$

3. Generalized inverse of A

We first show that the coefficient matrix (11) has the generalized inverse when $d = 2$.

LEMMA 3.1. *Let* t_k *are the zeros of* $T_{4(n+1)}$, $k = 1, \ldots, 4n+4$ *and* s_i *are the positive zeros of* $U_{2n+1}(x)$, $i = 1, \ldots, n$. *Then,*

$$\sum_{k=1}^{2n+2} \frac{1}{t_k^2 - s_i^2} = 0. \tag{15}$$

PROOF. Since t_k are the zeros of $T_{4(n+1)}$, $k = 1, \ldots, 4n+4$ and, by (12),

$$\begin{aligned} \sum_{k=1}^{2n+2} \frac{1}{t_k - s_i} &= -\sum_{k=2n+3}^{4n+4} \frac{1}{t_k - s_i} \\ &= -\sum_{k=1}^{2n+2} \frac{1}{-t_k - s_i} \\ &= \sum_{k=1}^{2n+2} \frac{1}{t_k + s_i}. \end{aligned}$$

Hence,

$$\sum_{k=1}^{2n+2} \left(\frac{1}{t_k - s_i} - \frac{1}{t_k + s_i} \right) = 0,$$

which equals (15). □

For the positive zeros of $T_{4n+4}(x)$ and $U_{4n+3}(x)$, by (13) and (14),

$$\sum_{k=1}^{2n+2} (t_k^2 - s_i^2)^{-1} = 0, \tag{16}$$

$$\sum_{k=1}^{2n+2} (s_j^2 - t_k^2)^{-2} = (2n+2)^2 [s_j^2 (1 - s_j^2)^{-1}]. \tag{17}$$

for $j = 1, \ldots, 2n+2$, where $s_j = \cos \frac{j\pi}{4n+4}$. Since the zeros of $U_{4n+3}(x)$, $\cos \frac{2j\pi}{4n+4}$, $j = 1, \cdots, 2n+1$ are the zeros of $U_{2n+1}(x)$, (16) and (17) hold for the positive zeros of $T_{4n+4}(x)$ and $U_{2n+1}(x)$.

THEOREM 3.1. *Let t_i be the positive roots of $T_{4(n+1)}$ and s_j be the positive roots of U_{2n+1}, $i = 1, \ldots, 2n+2$, $j = 1, \ldots, n+1$. Let A denote the coefficient matrix of* (11) *whose (i,j)-th element is*

$$a_{ij} = \frac{1}{n+1}\left(\frac{s_j^2(1-s_j^2)}{s_j^2 - t_i^2}\right) - \frac{2s_j^2 \cdot T_{2n+2}(t_i)}{(t_i^2 - s_j^2)U'_{2n+1}(s_j)}.$$

Then, (i,j)-th element of the generalized inverse of A such that $A^+A = I_n$ is

$$a_{ij}^+ = \frac{1}{10(n+1)}\left(\frac{1}{s_i^2 - t_j^2}\right) - \frac{2\,T_{2n+2}(t_j)}{10(1-s_i^2)(t_j^2 - s_i^2)U'_{2n+1}(s_i)}.$$

PROOF. With simple calculations, we can show that A^+ satisfies the four Moore-Penrose conditions [6].

$$\begin{aligned}\sum_{k=1}^{2n+2} a_{ik}^+ a_{kj} &= \frac{1}{10(n+1)^2}\sum_{k=1}^{2n+2}\left(\frac{1}{(s_i^2 - t_k^2)} - \frac{2(n+1)T_{2n+2}(t_k)}{(1-s_i^2)(t_k^2 - s_i^2)U'_{2n+1}(s_i)}\right) \\ &\quad \times \left(\frac{s_j^2(1-s_j^2)}{(s_j^2 - t_k^2)} - \frac{2(n+1)s_j^2\,T_{2n+2}(t_k)}{(t_k^2 - s_j^2)U'_{2n+1}(s_j)}\right).\end{aligned}$$

Here,

$$T_{2n+2}(t_k) = \cos\frac{(2k-1)\pi}{4}, \quad k = 1, \ldots, 2n+2.$$

If k varies from $1, \ldots, 4n+4$, the values of $T_{2n+2}(t_k)$ varies $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \ldots, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$. The values of $T_{2n+2}(t_k)$ have symmetry and along with the symmetry of t_k,

$$\sum_{k=1}^{4n+4}\frac{T_{2n+2}(t_k)}{s_j - t_k} = 0,$$

hence,

$$\sum_{k=1}^{2n+2}\frac{T_{2n+2}(t_k)}{s_j^2 - t_k^2} = 0.$$

The symmetry of the values of $T_{2n+2}(t_k)$ results that if the terms with t_k^2 are summarized with $T_{2n+2}(t_k)$, then the results are half the summation of

t_k^2 taking the sign of $\frac{T_{2n-2}(t_k)}{t_k^2}$ with the biggest absolute value, e.g.,

$$\left|\sum_{k=1}^{2n+2} \frac{T_{2n+2}(t_k)}{t_k^2}\right| = (2n+2)^2 = \frac{1}{2}\sum_{k=1}^{2n+2} \frac{1}{t_k^2},$$

where $sign(\sum_{k=1}^{2n+2} \frac{T_{2n+2}(t_k)}{t_k^2}) = sign(T_{2n+2}(t_{k'}))$, k' is the index of $\left(\max_k \left|\frac{T_{2n+2}(t_k)}{t_k^2}\right|\right)$.

$$\left|\sum_{k=1}^{2n+2} \frac{T_{2n+2}(t_k)}{1-t_k^2}\right| = (2n+2)^2 = \frac{1}{2}\sum_{k=1}^{2n+2} \frac{1}{1-t_k^2},$$

where $sign(\sum_{k=1}^{2n+2} \frac{T_{2n+2}(t_k)}{1-t_k^2}) = sign(T_{2n+2}(t_{k'}))$, k' is the index of $\left(\max_k \left|\frac{T_{2n+2}(t_k)}{1-t_k^2}\right|\right)$.

$$\left|\sum_{k=1}^{2n+2} \frac{T_{2n+2}(t_k)}{(s_j^2-t_k^2)^2}\right| = \frac{(2n+2)^2}{2s_j^2(1-s_j^2)} = \frac{1}{2}\sum_{k=1}^{2n+2} \frac{1}{(s_j^2-t_k^2)^2}.$$

where $sign(\sum_{k=1}^{2n+2} \frac{T_{2n+2}(t_k)}{(s_j^2-t_k^2)^2}) = sign(T_{2n+2}(t_{k'}))$, k' is the index of $\left(\max_k \left|\frac{T_{2n+2}(t_k)}{(s_j^2-t_k^2)^2}\right|\right)$. □

For $i \neq j$, after simple computation using (15), we have

$$\sum_{k=1}^{2n+2} \frac{T_{2n+2}(t_k)}{(s_j^2-t_k^2)(t_k^2-s_i^2)} = 0, \text{ and } \sum_{k=1}^{2n+2} \frac{T_{2n+2}^2(t_k)}{(t_k^2-s_i^2)(t_k^2-s_j^2)} = 0.$$

$$\begin{aligned}\sum_{k=1}^{2n+2} a_{ik}^+ a_{kj} &= \frac{1}{10(n+1)^2}\sum_{k=1}^{2n+2} \frac{s_j^2(1-s_j^2)}{(s_j^2-t_k^2)(s_i^2-t_k^2)} \\ &= \frac{1}{10(n+1)^2} s_j^2(1-s_j^2)\sum_{k=1}^{2n+2}\left[\frac{1}{(s_i^2-t_k^2)} - \frac{1}{(s_j^2-t_k^2)}\right]\frac{1}{s_j^2-s_i^2}.\end{aligned}$$

Hence, $\sum_{k=1}^{2n+2} a_{ik}^+ a_{kj} = 0$ for $i \neq j$ by (15).

If $i = j$, we have

$$\sum_{k=1}^{2n+2} \frac{T_{2n+2}(t_k)}{(s_i^2-t_k^2)^2} = (-1)^i \frac{(2n+2)^2}{2s_i^2(1-s_i^2)}, \quad \text{and} \sum_{k=1}^{2n+2} \frac{T_{2n+2}^2(t_k)}{(t_k^2-s_i^2)^2} = \frac{(2n+2)^2}{2s_i^2(1-s_i^2)}.$$

With

$$U'_{2n+1}(s_i) = (-1)^i(2n+2)\frac{1}{(1-s_i^2)}$$

and the observation on the zeros of $T_{4n+4}(x)$ and $U_{2n+1}(x)$, we have

$$\begin{aligned}
&\sum_{k=1}^{2n+2} a_{ik}^{+} a_{kj} \\
= \;& \frac{1}{10(n+1)^2}\sum_{k=1}^{2n+2}\frac{1}{(s_i^2-t_k^2)^2}\left(s_i^2(1-s_i^2)+\frac{4(n+1)\; s_i^2\; T_{2n+2}(t_k)}{U'_{2n+1}(s_i)}\right) \\
&+\frac{1}{10(n+1)^2}\sum_{k=1}^{2n+2}\frac{T_{2n+2}^2}{(s_i^2-t_k^2)^2}\frac{4(n+1)^2 s_i^2}{(1-s_i^2)(U'_{2n+1}(s_i))^2} \\
= \;& 1.
\end{aligned}$$

We can find $(A^T)^+$, similarly. We observe that

$$A^+ = \Gamma A^T,$$

where (i,k)-th element of Γ is

$$\gamma_{ik} = \frac{1}{10 s_i^2(1-s_i^2)}\delta_{ik}.$$

Norms of inverse matrices

THEOREM 3.2.

$$\| A^+ \|_2 = \frac{2}{\sqrt{10}}\text{cosec}\frac{\pi}{n+1},$$

PROOF. From $A^+ = \Gamma A^T$, we have

$$A^+(A^T)^+ = \Gamma.$$

Hence,

$$\begin{aligned}
\| A^+ \|_2^2 &= \max_i\{10^{-1}s_i^{-2}(1-s_i^2)^{-1}\}, \\
\| A^+ \|_2 &= \frac{2}{\sqrt{10}}\text{cosec}\frac{\pi}{n+1}.
\end{aligned}$$

In the case of $d = 4, 6, \ldots,$ we cannot compute the generalized inverses as for $d = 2$ because $T_{2n+2}(t_k)$, where t_k are the positive zeros of T_{8n+8}, $T_{16n+16}, \ldots$, varies as k increases. For $d = 1$, the coefficient matrix of the linear system A is an $(n+1)\times(n+1)$ matrix whose (i,j)-th element is

$$a_{ij} = \frac{1}{(n+1)}\frac{s_j^2(1-s_j^2)}{s_j^2-t_i^2}.$$

A^{-1} is represented by

$$a_{ij}^{-1} = \frac{1}{4(n+1)}\frac{1}{(s_i^2-t_j^2)},$$

and, $\| A^{-1} \| = \mathrm{cosec}\frac{\pi}{n+1}$. Hence, we have $\| A^{-1} \| \geq \| A^{+} \|$ by Theorem 3.2. □

We can estimate the error **e** using the norms of the inverse matrices in the computed solution in terms of the residual error **r** by means of the inequality

$$\| \mathbf{e} \| \leq \| A^{+} \| \| \mathbf{r} \| .$$

We showed in computation in [8] that the residual errors $\| r \|$ for the case of $d = 1, 4, 6, \ldots$, are bigger than $d = 2$, and $d = 2$ provides the most accurate solutions. In this paper, we have found the generalized inverse of the coefficient matrix A and shown that the increased computational cost by increasing the size of A can be reduced using the generalized inverse. Solving the linear system takes at most $O(n^2)$ for $2(n+1)\times n$ matrix, whereas gauss-newton method employed in the solution of (10) in [5] is $O(n^3)$. We have also estimated the error for $d = 2$ and shown that it is smaller than that of $d = 1$.

ACKNOWLEDGEMENT. The author would like to thank a referee for his careful reading and helpful suggestions.

References

[1] C. T. H. Baker, *The Numerical Treatment of Integral Equations*, Oxford, Clarendon Press, 1977.

[2] K. M. Case, *Sum Rules for Zeros of Polynomials,* J. Math. Phys. **21** (1980), 702-708.

[3] D. Elliott, *The Classical Collocation Method For Singular Integral Equations,* SIAM J. Numer. Anal. **19** (1982), no. 4.

[4] F. Erdogan and G. D. Gupta, *On the Numerical Solution of Singular Integral Equations,* Quart. Appl. Math. **29** (1972), 525-534.

[5] Erica Jen and R. P. Srivastav, *Solving Singular Integral Equations Using Gaussian Quadrature and Overdetermined System,* Comp. & Math with Applic. **9** (1983), no. 4, 625-632.
[6] G. Golub and V. Loan, *Matrix Computation*, 2nd Ed., The Johns Hopkins Univ. Press, 1989.
[7] A. Gerasoulis, *On the Existence of Approximate Solutions for Singular Integral Equations of Cauchy-type Discretized by Gauss-Chebyshev Quadrature Formulae,* BIT **21** (1981), 277-380.
[8] S. Kim, *Numerical Solutions of Singular Integral Equations Using GSVD techniques*, to appear in Comp. & Math. with Appl. (1998).
[9] R. P. Srivastav, *Numerical Solutions of Singular Integral Equations Using Gauss-type Formulae I:Quadrature and Collocation on Chebyshey Nodes,* IMA Journal of Numerical Analysis **3** (1983), 305-318.

Department of Mathematics
Ewha Womans University
Seoul 120-750, Korea