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A NOTE ON THE STRONG LAW OF LARGE NUMBERS FOR WEIGHTED SUMS OF NEGATIVELY DEPENDENT RANDOM VARIABLES

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ABSTRACT. Some conditions on the strong law of large numbers for weighted sums of negative quadrant dependent random variables are studied. The almost sure convergence of weighted sums of negatively associated random variables is also established, and then it is utilized to obtain strong laws of large numbers for weighted averages of negatively associated random variables.

1. Introduction

Many recent papers have been concerned with concepts of dependence for families of random variables (see for example Block, Savits and Shaked (1982), Ebrahimi and Ghosh (1981), Karlin and Rinott (1980) and the references therein). A finite family $\{X_1, \cdots, X_m\}$ is said to be associated if $Cov\ (f(X_1, \cdots, X_m),\ g(X_1, \cdots, X_m)) \geq 0$ for any real coordinatewise nondecreasing (nonincreasing) functions f and g on R^m, such that this covariance exists. It is said to be negatively associated (NA) if for any disjoint subsets $A, B \subset \{1, \cdots, m\}$ and any real coordinatewise nondecreasing (nonincreasing) functions f on R^A, g on R^B, $Cov(f(X_k,\ k \in A),\ g(X_k,\ k \in B)) \leq 0$. An infinite family of random variables is associated (negatively associated) if every finite subfamily is associated (negatively associated). Two random variables X and Y are negative quadrant dependent (NQD) if $P[X > x,\ Y > y] \leq P[X > x]P[Y > y]$, for all, $x, y \in R$, and positive

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quadrant dependent (PQD) if $P[X > x, Y > y] \geq P[X > x]\ P[Y > y]$ for all, $x, y \in R$. These concepts of dependence were introduced by Lehmann (1966), Esary, Proschan and Walkup (1967), and Joag-Dev and Proschan (1983), respectively.

For a sequence of random variables $\{X_j : j \geq 1\}$ and a sequence of positive numbers $\{w_j : j \geq 1\}$ the almost sure convergence of the weighted partial sums $\sum_{j=1}^{n} w_j X_j$ enjoys a large literature of investigation (see for example, Yu (1990), Rosalsky (1993), Cuzick (1995) and the references therein). For a sequence of nonnegative random variables $\{X_j : j \geq 1\}$ and a sequence of positive numbers $\{w_j : j \geq 1\}$ Etemadi (1983) obtained a strong law of large numbers of the weighted partial sums $W_n^{-1} \sum_{j=1}^{n} w_j(X_j - EX_j)$ where $W_n = \sum_{j=1}^{n} w_j$,

$$(1) \qquad w_n/W_n \longrightarrow 0 \ and \ W_n \longrightarrow \infty \ as \ n \to \infty$$

and applied it to obtain the strong law of large numbers for the weighted averages.

For a sequence of associated random variables $\{X_j : j \geq 1\}$ Birkel (1989) proved the strong law of large numbers and for a sequence of negatively associated random variables $\{X_j : j \geq 1\}$ Matula (1992) established the almost sure convergence of $\sum_{j=1}^{n}(X_j - EX_j)/n$ to zero as $n \to \infty$ only using the maximal inequality.

Petrov (1996) examined the connection between general moment conditions and the strong law of large numbers (SLLN) for iid random variables. In this note, we extend Theorem 1 of Petrov (1996) on the strong law of large numbers for identically distributed pairwise independent random variables to the pairwise NQD case. We put $T_n = \sum_{j=1}^{n} w_j X_j$ and $W_n = \sum_{j=1}^{n} w_j$, and then study the almost sure convergence of $(T_n - ET_n)/W_n$ to zero as $n \to \infty$, for the case where the random variables are either NQD or negatively associated and a sequence $\{w_j : j \geq 1\}$ of positive numbers satisfies (1) and we also apply this result to obtain almost sure convergence for the weighted averages of negatively associated random variables.

2. An extension of Petrov's strong law of large numbers

First we extend Theorem 1 in Pretrov (1996) for pairwise identically independent random variables to the NQD case.

Let $f(x)$ be an even continuous function that is positive and strictly increasing in the region $x > 0$ and satisfying the condition $f(x) \to \infty$ as $x \to \infty$. We put

$$a_n = f^{-1}(n) \tag{2}$$

where f^{-1} is the inverse of f and $a_{n-1} / a_n = O(1)$. By the properties of f we have $a_n \uparrow \infty$ (see Petrov (1996)).

THEOREM 2.1. *Let $\{X_j : j \geq 1\}$ be a sequence of pairwise NQD random variables with the same distribution. If*

$$S_n / a_n \to 0 \ a.\ s \tag{3}$$

then

$$Ef(X_1) < \infty. \tag{4}$$

PROOF. By Lemma 3 in Matula (1992) it follows from (2) and (3) that $\sum_{n=1}^{\infty} P(|X_1| \geq f^{-1}(n)) < \infty$. Therefore,

$$\sum_{n=1}^{\infty} P(f(X_1) \geq n) < \infty. \tag{5}$$

For an arbitrary random variable Y the conditions $\sum_{n=1}^{\infty} P(|Y| \geq n) < \infty$ and $E|Y| < \infty$ are equivalent. Therefore, it follows from (5) that (4) holds. Theorem 2.1 is proved. □

Next, we consider the SLLN for weighted partial sums of pairwise NQD random variables.

LEMMA 2.1. (Matula, 1992). *If $\{X_j : j \geq 1\}$ is a sequence of pairwise NQD random variables, and $\{f_j : j \geq 1\}$ is a sequence of nondecreasing functions $f_j : R \to R$, then $\{f_j(X_j) : j \geq 1\}$ is also pairwise NQD.*

LEMMA 2.2. *Let $\{X_j : j \geq 1\}$ be a sequence of pairwise NQD random variables and $\{w_j : j \geq 1\}$ a sequence of positive numbers satisfying* (1). *Let $T_n = \sum_{j=1}^{n} w_j X_j$ and $W_n = \sum_{j=1}^{n} w_j$. If*

$$T_n / W_n \to 0 \ a.\ s. \tag{6}$$

then

$$\sum_{n=1}^{\infty} P(|w_n X_n| \geq W_n) < \infty. \tag{7}$$

PROOF. Put $Y_n = w_n X_n$. Then Y_n's are pairwise NQD by Lemma 2.1. Note that $sup_{n\geq 1} W_{n-1} \ / \ W_n \leq 1$ according to condition (1). Thus the result follows by Lemma 3 of Matula (1992). □

Let $g(x)$ be an even continuous function that is positive and strictly increasing in the region $x > 0$ and satisfying the condition $g(x) \to \infty$ as $x \to \infty$. We put

$$b_n = g^{-1}(n) = W_n \ / \ w_n \tag{8}$$

where g^{-1} is the inverse of g. By properties of g we have $W_n \ / \ w_n \ \uparrow \ \infty$.

THEOREM 2.2. *Let $\{X_j : j \geq 1\}$ be a sequence of pairwise NQD random variables with same distribution. If* (6) *holds then*

$$E(g(X_1)) < \infty. \tag{9}$$

PROOF. If (6) holds, then

$$\begin{aligned} \sum_{n=1}^{\infty} P(|w_n X_n| \geq W_n) &= \sum_{n=1}^{\infty} P(|X_1| \geq W_n \ / \ w_n) \\ &= \sum_{n=1}^{\infty} P(|X_1| \geq g^{-1}(n)) < \infty \end{aligned}$$

by Lemma 2.2 and equality (8). Therefore, we obtain

$$\sum_{n=1}^{\infty} P(g(|X_1|) \geq n) < \infty. \tag{10}$$

For an arbitrary random variable Y the conditions $\sum_{n=1}^{\infty} P(|Y| \geq n) < \infty$ and $E|Y| < \infty$ are equivalent. Therefore it follows from (10) that (9) holds. Thus the proof is complete. □

THEOREM 2.3. *Let $\{X_j : j \geq 1\}$ be a sequence of random variables with common distributions. If*

$$\sum_{k=n}^{\infty} \left(\frac{w_k}{W_k} \right) = O\left(n \frac{w_n}{W_n} \right) \tag{11}$$

then (9) *implies* (6).

SKETCH OF PROOF. It follows from (8), (9) and (10) that

$$\sum_{n=1}^{\infty} P(|w_n X_n| \geq W_n) = \sum_{n=1}^{\infty} P(|X_1| \geq b_n) < \infty. \tag{12}$$

By setting $b_0 = 0$ and $F(x) = P(X_1 < x)$ from (11) and (12) we have

$$\begin{aligned}(13)\qquad \sum_{n=1}^{\infty} \frac{1}{b_n} \int_{-b_n}^{b_n} |x| dF(x) &\leq \sum_{n=1}^{\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k P(b_{k-1} \leq |X_1| < b_k) \\ &= \sum_{k=1}^{\infty} b_k P(b_{k-1} \leq |X_1| < b_k) \sum_{n=k}^{\infty} \frac{1}{b_n} \\ &\leq c \sum_{k=1}^{\infty} k P(b_{k-1} \leq |X_1| < b_k) \\ &= c \sum_{k=1}^{\infty} P(|X_1| \geq b_k) < \infty.\end{aligned}$$

By the similar arguments of the proof of Theorem 2 in Matikainen and Petrov (1980) from (12) and (13) the desired result is obtained. □

3. The SLLN for weighted averages of NA random variables

According to Property P_6 of Joag-Dev and Proschan (1983) we obtain the following lemma:

LEMMA 3.1. *Let $\{X_j : j \geq 1\}$ be a sequence of NA random variables and let $\{f_j : j \geq 1\}$ be a sequence of nondecreasing functions $f_j : R \to R$. Then $\{f_j(X_j) : j \geq 1\}$ is also a sequence of NA random variables.*

LEMMA 3.2. *Let $\{X_j : j \geq 1\}$ be a sequence of NA random variables with $EX_j = 0$, $EX_j^2 < \infty$ and $\{w_j : j \geq 1\}$ a sequence of positive numbers. Let $T_n = \sum_{j=1}^{n} w_j X_j$ then for every $\epsilon > 0$,*

$$(14)\qquad P[max(|T_1|, \cdots, |T_n|) > \epsilon] \leq 8\epsilon^{-2} \sum_{j=1}^{n} w_j^2 Var(X_j)$$

PROOF. Let $Y_j = w_j X_j$. Then $T_n = \sum_{j=1}^{n} Y_j$, and by Lemma 3.1 $\{Y_j : j \geq 1\}$ is clearly a sequence of NA random variables since $w_j > 0$. Thus from Lemma 4 of Matula (1992) we get (14).

From Lemmas 3.1 and 3.2 we derive the following theorem for weighted partial sum of NA random variables by the similar arguments in the proof of strong law of large numbers for NA random variables in Matula (1992) (see Theorem 3 and Lemma 4 of [9]).

THEOREM 3.1. *Let $\{X_j : j \geq 1\}$ be a sequence of NA random variables with $EX_j^2 < \infty$ and $\{w_j : j \geq 1\}$ a sequence of positive numbers. If $\sum_{j=1}^{\infty} w_j^2 VarX_j < \infty$, then $\sum_{j=1}^{\infty} w_j(X_j - EX_j)$ converges.*

Proof. Let $Y_j = w_j X_j$. Then $\{Y_j : j \geq 1\}$ is a sequence of NA random variables according to Lemma 3.1. Thus by Theorem 3 of Matula (1992) from (14) the result follows.

The above theorem and the Kronecker lemma imply the following corollary:

COROLLARY 3.1. *Let $\{X_j : j \geq 1\}$ be a sequence of NA random variables with $EX_j^2 < \infty$ and $\{w_j : j \geq 1\}$ a sequence of positive number satisfying (1). Let $T_n = \sum_{j=1}^{n} w_j X_j$ and $W_n = \sum_{j=1}^{n} w_j$. If*

$$\sum_{j=1}^{\infty} w_j^2 VarX_j / W_n^2 < \infty,$$

then

$$(T_n - ET_n)/W_n \to a.\ s.\ as\ n \to \infty.$$

REMARK. The strong law of large number for negatively associated random variables established in Matula (1992) is then the case where the w_j's are identically one. (see Corollary in Matula (1992))

Now we consider the strong law of large numbers for weighted averages under negative association assumption.

THEOREM 3.2. *Let $\{X_j : j \geq 1\}$ be a sequence of negatively associated random variables with $EX_j^2 < \infty$. Let $S_n = \sum_{j=1}^{n} X_j$. Assume*

$$\begin{aligned} &(a) \quad EX_j > 0 \ for\ all\ j, \\ &(b) \quad \{EX_j : j \geq 1\}\ satisfies\ (1), \\ &(c) \quad \sum_{j=1}^{\infty} Var(X_j)/(ES_j)^2 < \infty. \end{aligned}$$

Then, as $n \to \infty$ $S_n/ES_n \to 1$ a. s.

Proof. Let $Y_j = X_j/EX_j$ and $w_j = EX_j$. Then according to Lemma 3.1 $\{Y_j : j \geq 1\}$ is a sequence of negatively associated random variables with $EY_j = 1$ and $EY_j^2 < \infty$ and $\{w_j : j \geq 1\}$ is a sequence of positive numbers satisfying (1) by (b). We also have

$$X_j = (EX_j)Y_j = w_j Y_j,\ S_n = \sum_{j=1}^{n} X_j = \sum_{j=1}^{n} w_j Y_j \ and\ W_n = ES_n.$$

Thus it follows from (c) that

$$\sum_{j=1}^{\infty} w_j^2 Var(Y_j)/W_j^2 < \infty.$$

Thus by Corollary 3.1,

$$\left\{ \sum_{j=1}^{\infty} w_j Y_j - E(\sum_{j=1}^{n} w_j Y_j) \right\} \ / \ W_n \to 0 \ a.\ s.$$

which yields $(S_n - ES_n)/ES_n \to 0\ a.\ s.$, that is, $S_n/ES_n \to 1\ a.\ s.$

Finally, if one is interested in what is called logarithmic averages we have:

COROLLARY 3.2. *Let $\{X_j : j \geq 1\}$ be a sequence of negatively associated random variables with $EX_j^2 < \infty$. Let $S_n = \sum_{j=1}^{n} X_j$.* Assume

$$(a)\quad EX_j > 0 \ for\ all\ j,$$
$$(b)\quad \{EX_j : j \geq 1\}\ satisfies\ (1),$$
$$(c)\quad \sum_{j=1}^{\infty} \frac{Var(X_j)}{(ES_j)^2 (logS_j)^2} < \infty.$$

Then, as $n \to \infty$ $(\log ES_n)^{-1} \sum_{j=1}^{n} X_j/ES_j \to 1\ a.\ s.$

PROOF. We will use the arguments in the proof of Corollary 3 in Etemadi (1983). Let $Y_j = X_j/EX_j$ and $w_j = EX_j/ES_j$. Then $EY_j = 1$ and $w_j Y_j = X_j/ES_j$. From (b) we have $W_n \sim logES_n$. Therefore from (c) it follows that

$$\sum_{j=1}^{\infty} w_j^2 Var(Y_j)/W_j^2 < \infty.$$

By Corollary 3.1

$$(\sum_{j=1}^{n} w_j Y_j - E(\sum_{j=1}^{n} w_j Y_j))/W_n = [(\sum_{j=1}^{n} X_j/ES_j)/logES_n] - 1 \to 0 \ a.\ s.$$

since $W_n \sim logES_n$, $EY_j = 1$ and $W_n = \sum_{j=1}^{n} w_j$. Thus the proof is complete.

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