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PACKING MEASURE AND DIMENSION OF LOOSELY SELF-SIMILAR SETS

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ABSTRACT. Let K be a loosely self-similar set. Then α-dimensional packing measure of K is the same as that of a Borel subset $K(r_1^\alpha, \cdots, r_m^\alpha)$ of K. And packing dimension of K is equal to that of $K \setminus K(r_1^\alpha, \cdots, r_m^\alpha)$ and $K(r_1^\alpha, \cdots, r_m^\alpha)$.

1. Introduction

To explain fractal sets, we sometimes use Hausdorff or packing measure and dimension. In general cases, packing dimension of a given fractal set is greater than Hausdorff dimension, however in many cases, two dimensions are equal.

In [1], S. Ikeda introduced a loosely self-similar set K and proved that Hausdorff measure and a Borel probability measure are absolutely continuous to each other on K. Moreover, S. Ikeda investigated Hausdorff measure and dimension for K and a subset of K.

In this paper, we show that packing measure and a Borel probability measure on K are absolutely continuous to each other on K (see Theorem 3.6) and then show that α-dimensional packing measures of K and $K(r_1^\alpha, \cdots, r_m^\alpha)$ are equal (see Theorem 3.7). We also show that packing dimension of K is equal to that of $K \setminus K(r_1^\alpha, \cdots, r_m^\alpha)$ and $K(r_1^\alpha, \cdots, r_m^\alpha)$ (see Theorem 3.9).

2. Preliminaries and definitions

Throughout this paper, we denote $|I|$ for the diameter of I. We define

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a δ-packing of $A \subset \mathbb{R}^d$ to be a finite or countable collection of disjoint balls $\{B_i(x_i)\}$ of radii at most δ and with centres x_i in A. For $\alpha > 0$, define

$$P^\alpha(A) = \limsup_{\delta \to 0} \left\{ \sum |B_i(x_i)|^\alpha : \{B_i(x_i)\} \text{ is a } \delta\text{- packing of } A \right\}.$$

However P^α is not countably subadditive, so in order to get an outer measure out of P^α, we define the α-dimensional packing measure of A,

$$p^\alpha(A) = \inf\left\{ \sum P^\alpha(A_n) : A_n \text{ is bounded and } A \subset \cup A_n \right\}.$$

We recall that a (outer) measure μ on $\mathbb{R}^d$ is a Borel (outer) measure if the Borel subsets of $\mathbb{R}^d$ are μ-measurable. It is well-known that p^α is a Borel measure on $\mathbb{R}^d$. Also we can define the packing dimension of A,

$$\begin{aligned} \mathrm{Dim}(A) &\equiv \sup\{\alpha > 0 : p^\alpha(A) = \infty\} \\ &= \inf\{\alpha > 0 : p^\alpha(A) = 0\}. \end{aligned}$$

Now we recall the notion of a loosely self-similar set [1].
Suppose that $\{\varphi_{i_1,i_2,\cdots,i_k} : (i_1, i_2, \cdots, i_k) \in \{1, 2, \cdots, m\}^k,\ k = 1, 2, \cdots\}$ $(m \geq 2)$ is a sequence of contraction mappings on a compact subset X of $\mathbb{R}^d$ with d-dimensional Lebesgue measure of X, $\lambda_d(X) > 0$ such that $\varphi_{i_1,i_2,\cdots,i_k} : X \longrightarrow X$ satisfies

$$|\varphi_{i_1,\cdots,i_k}(x) - \varphi_{i_1,\cdots,i_k}(y)| = r_{i_k}|x - y| \text{ for all } x, y \in X, 0 < r_{i_k} < 1$$

and

$$\varphi_{i_1,\cdots,i_{k-1},i_k}(X) \cap \varphi_{i_1,\cdots,i_{k-1},i_k'}(X) = \emptyset \quad (i_k \neq i_k').$$

Put

$$[i_1, \cdots, i_n] = \varphi_{i_1} \circ \varphi_{i_1 i_2} \circ \cdots \circ \varphi_{i_1 \cdots i_n}(X)$$

and

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\cdots,i_n) \in \{1,2,\cdots,m\}^n} [i_1, \cdots, i_n].$$

We call above K a loosely self-similar set generated by $\{\varphi_{i_1,i_2,\cdots,i_k}\}$. Since $\bigcap_{n=1}^{\infty}[i_1,i_2,\cdots,i_n]$ consists of a single point for any $w\equiv(i_1,i_2,\cdots)\in\{1,2,\cdots,m\}^{\mathbb{N}}$, we can define a bijection map φ from $\{1,2,\cdots,m\}^{\mathbb{N}}$ to K by

$$\varphi: w=(i_1,i_2,\cdots)\in\{1,2,\cdots,m\}^{\mathbb{N}}\to\varphi(w)\equiv\cap_{n=1}^{\infty}[i_1,\cdots,i_n].$$

Through the whole paper, we assume that $\{q_i\}_{i=1}^m$ satisfies $\sum_{i=1}^m q_i=1$ and $0<q_i<1(i=1,2,\cdots,m)$ and set

$$K(q_1,\cdots,q_m)=\left\{\varphi(w):\frac{N_i(w,n)}{n}\to q_i \text{ as } n\to\infty\right\},$$

where $N_i(w,n)$ denotes the cardinal number of the set $\{k:1\le k\le n, i_k=i\}$ for $w=(i_1,i_2,\cdots)\in\{1,2,\cdots,m\}^{\mathbb{N}}$. It is well-known that $K(q_1,\cdots,q_m)$ is a Borel set but not compact set.

Let $\nu_{(q_1,q_2,\cdots,q_m)}$ be the Borel probability measure on $\mathbb{R}^d$ such that $\nu_{(q_1,\cdots,q_m)}([i_1,\cdots,i_n])=\prod_{j=1}^n q_{i_j}$ for any n and any $(i_1,\cdots,i_n)\in\{1,2,\cdots,m\}$. Since $\nu_{(q_1,\cdots,q_m)}(K)=\nu_{(q_1,\cdots,q_m)}(K(q_1,\cdots,q_m))=1$, the probability measure $\nu_{(q_1,q_2,\cdots,q_m)}$ is called the $(q_1,q_2,\cdots,q_m)$ -Bernoulli measure on K [1].

3. Main results

Now we introduce some Lemmas, which have an important role in the proof of our main results.

LEMMA 3.1 [2]. *For any $F\subset\mathbb{R}^d$ we have $H^\alpha(F)\le p^\alpha(F)$, where $H^\alpha(F)$ is the α-dimensional Hausdorff measure of F.*

LEMMA 3.2 [1]. *Assume that μ is a probability Borel measure on $\mathbb{R}^d$ such that $\mu([i_1,\cdots,i_n])>0$ for any n and any $(i_1,\cdots,i_n)\in\{1,2,\cdots,m\}^n$. If*

$$a\le\liminf_{n\to\infty}\frac{\mu([i_1,\cdots,i_n])}{|[i_1,\cdots,i_n]|^\delta}\le\limsup_{n\to\infty}\frac{\mu([i_1,\cdots,i_n])}{|[i_1,\cdots,i_n]|^\delta}\le b$$

hold for any $\cap_{n=1}^{\infty}[i_1,\cdots,i_n]\in E\subset K$, then there exists a positive constant L depending only on $d, X, \lambda=\frac{1}{\min_i r_i}(>1)$ such that

$$b^{-1}\lambda^{-\delta}L^{-1}\mu^*(E)\le H^\delta(E)\le a^{-1}\mu^*(E),$$

where $\mu^*(E) = \inf\{\sum \mu(E_i) : E \subset \cup E_i, \quad \text{for Borel set } E_i\}$.

From now on, we write
$\mathcal{R}_n$ for $\{[i_1, i_2, \cdots, i_n] : (i_1, i_2, \cdots, i_n) \in \{1, 2, \cdots, m\}^n\}$, $R_n(x)$ for $x \in R_n \in \mathcal{R}_n$ and $\mathcal{R}$ for $\cup_{n=1}^{\infty} \mathcal{R}_n$.

LEMMA 3.3. *With the same hypothesis in* Lemma 3.2, *we have*

$$b^{-1}\lambda^{-\delta}L^{-1}\mu^*(E) \leq p^\delta(E) \leq \lambda^\delta a^{-1}\mu^*(E).$$

PROOF. The first inequality is obtained by Lemma 3.1 and 3.2.
We only need to show the right side inequality.
For $\rho > 0, \epsilon > 0$, set
$E_{\rho,\epsilon} = \{x \in E : (a-\epsilon)|R_n|^\delta \leq \mu(R_n) \leq (b+\epsilon)|R_n|^\delta$ or $|R_n| \geq \lambda\rho$ for any $R_n \in \mathcal{R}$ such that $x \in R_n \}$.

For $\gamma > 0, (0 <)\rho' < \rho$, let $\{B_i(x_i)\}$ be a ρ'-packing of $E_{\rho,\epsilon}$ such that for each i and for some n_i, $\lambda^{-n_i} < |B_i(x_i)| \leq \lambda^{-n_i+1}$. Then we can find $\{R_{n_i'}(x_i)\} \subset \mathcal{R}$ such that $\lambda^{-n_i} < |R_{n_i'}(x_i)| \leq \lambda^{-n_i+1}$, and so $\lambda^{-1}|B_i(x_i)| < |R_{n_i'}(x_i)| \leq \lambda|B_i(x_i)|$ for any i.
Therefore

$$\begin{aligned} P_{\rho'}^\delta(E_{\rho,\epsilon}) &\leq \sum_i |B_i(x_i)|^\delta + \gamma \\ &\leq \lambda^\delta \sum |R_{n_i'}(x_i)|^\delta + \gamma \\ &\leq \lambda^\delta \sum (a-\epsilon)^{-1}\mu(R_{n_i'}(x_i)) + \gamma \\ &\leq \lambda^\delta (a-\epsilon)^{-1}\mu^*(E_{\rho,\epsilon}(\rho')) + \gamma \end{aligned}$$

where $E(\rho')$ is a closed ρ'-neighborhood of E.
By letting $\rho' \downarrow 0$, we have $P^\delta(E_{\rho,\epsilon}) \leq \lambda^\delta(a-\epsilon)^{-1}\mu^*(\overline{E_{\rho,\epsilon}}) + \gamma$. Since $E_{\rho,\epsilon} \uparrow E$ as $\rho \downarrow 0$, we have $P^\delta(E) \leq \lambda^\delta(a-\epsilon)^{-1}\mu^*(E) + \gamma$. Therefore we have $p^\delta(E) \leq \lambda^\delta a^{-1}\mu^*(E)$. □

LEMMA 3.4 [1]. *Assume that μ is a positive finite Borel measure on $\mathbb{R}^d$ satisfying $\mu([i_1, \cdots, i_n]) > 0$ for any n and any $(i_1, \cdots, i_n) \in \{1, 2, \cdots, m\}^n$. If $E \subset K$ with $\mu^*(E) > 0$ satisfies*

$$a \leq \liminf_{n\to\infty} \frac{\log \mu([i_1, \cdots, i_n])}{\log(|[i_1, \cdots, i_n]|)} \leq \limsup_{n\to\infty} \frac{\log \mu([i_1, \cdots, i_n])}{\log(|[i_1, \cdots, i_n]|)} \leq b$$

for any $\cap_{n=1}^{\infty}[i_1, \cdots, i_n] \in E$, *then*

$$a \leq \dim_H(E) \leq b$$

where $\dim_H(E)$ *is the Hausdorff dimension of* E.

LEMMA 3.5. *With the same hypothesis in* Lemma 3.4, *we have*

$$a \leq \mathrm{Dim}(E) \leq b.$$

PROOF. Using Lemma 3.1 and Lemma 3.4, we have $a \leq \mathrm{Dim}(E)$.

To show $\mathrm{Dim}(E) \leq b$, it is sufficient to show that $p^{b+\epsilon}(E) < \infty$ for any $\epsilon > 0$. Set

$$E_{\rho,\epsilon} = \{x \in E : |R_n|^{b+\epsilon} \leq \mu(R_n) \text{ or } |R_n| \geq \rho \text{ for any } R_n \in \mathcal{R} \text{ such that } x \in R_n\}.$$

Now we can proceed with an argument similar to the proof of Lemma 3.3. That is, for δ-packing $\{B_i(x_i)\}$ of $E_{\rho,\epsilon}$ with $\lambda^{-n_i} < |B_i(x_i)| \leq \lambda^{-n_i+1}$ for some $n, (\delta < \rho)$, we can get $\{R_{n'_i}(x_i)\} \subset \mathcal{R}$ such that $\lambda^{-1}|B_i(x_i)| < |R_{n'_i}(x_i)| < \lambda|B_i(x_i)|$ for any i.

So,

$$\begin{aligned} P^{b+\epsilon}(E_{\rho,\epsilon}) &\leq \sum |B_i(x_i)|^{b+\epsilon}, \quad |B_i| \leq \delta \\ &\leq \lambda^{b+\epsilon} \sup \sum |R_{n'_i}(x_i)|^{b+\epsilon} \\ &\leq \lambda^{b+\epsilon} \sup \sum \mu(R_{n'_i}(x_i)) \\ &\leq \lambda^{b+\epsilon} \mu(\mathbb{R}^d). \end{aligned}$$

Since $E_{\rho,\epsilon} \uparrow E$ as $\rho \downarrow 0$, $P^{b+\epsilon}(E_{\rho,\epsilon}) < \infty$, therefore $p^{b+\epsilon}(E) < \infty$ for any $\epsilon > 0$. □

For simplicity of notation, we write $r_k = r_{i_k}$ for contraction rate of $\varphi_{i_1,i_2,\cdots,i_k}$.

THEOREM 3.6. *Assume that* $(q_1, q_2, \cdots, q_m)$ *satisfies* $\sum_{i=1}^{m} q_i = 1$ *and* $0 < q_i < 1$ $(i = 1, 2, \cdots, m)$. *Let* α *be the unique solution of* $\sum_{k=1}^{m} {r_k}^{\alpha} = 1$. *Then a probability measure* $\nu_{({r_1}^{\alpha}, {r_2}^{\alpha}, \cdots, {r_m}^{\alpha})}$ *and* α-*dimensional packing measure* p^{α} *are absolutely continuous to each other on* K.

PROOF. Assume $q_i = r_i^\alpha$ for each i, $i = 1, 2, \cdots, m$. Then for all $w = (i_1, i_2, \cdots, i_n, \cdots) \in \{1, 2, \cdots, m\}^{\mathbb{N}}$ and $n \in \mathbb{N}$,

$$\frac{\nu_{(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)}([i_1, \cdots, i_n])}{|[i_1, \cdots, i_n]|^\alpha} = \frac{\prod_i^m r_i^{\alpha N_i(w,n)}}{\prod_{i=1}^m r_i^{\alpha N_i(w,n)} \cdot |X|^\alpha} = |X|^{-\alpha}.$$

Since $\nu_{(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)}([i_1, \cdots, i_n]) > 0$ for any $i_1, \cdots, i_n, n$, by Lemma 3.3, we have

$$|X|^\alpha \lambda^{-\alpha} L^{-1} \nu_{(r_1^\alpha, \cdots, r_m^\alpha)}(E) \le p^\alpha(E) \le |X|^\alpha \lambda^\alpha \nu_{(r_1^\alpha, \cdots, r_m^\alpha)}(E)$$

for any Borel set $E \subset K$. This achieves the proof. □

The following Theorem shows that α-dimensional packing measure on $K \setminus K(r_1^\alpha, \cdots, r_m^\alpha)$ is 0. That is, α-dimensional packing measure on K concentrates on $K(r_1^\alpha, \cdots, r_m^\alpha)$.

THEOREM 3.7. *Let α be as in* Theorem 3.6. *Then*

(i) $p^\alpha(K \setminus K(r_1^\alpha, \cdots, r_m^\alpha)) = 0$

(ii) $p^\alpha(K) = p^\alpha(K(r_1^\alpha, \cdots, r_m^\alpha))$

(iii) $0 < p^\alpha(K) < \infty$.

PROOF. (i) $K(q_1, \cdots, q_m)$ is a Borel set for any $(q_1, \cdots, q_m)$ and $\nu_{(r_1^\alpha, \cdots, r_m^\alpha)}(K \setminus K(r_1^\alpha, \cdots, r_m^\alpha)) = 0$, so by using Lemma 3.3, $p^\alpha(K \setminus K(r_1^\alpha, \cdots, r_m^\alpha)) = 0$.

(ii) Since p^α is an outer measure,

$$\begin{aligned} p^\alpha(K(r_1^\alpha, \cdots, r_m^\alpha)) &\le p^\alpha(K) \\ &\le p^\alpha(K(r_1^\alpha, \cdots, r_m^\alpha)) + p^\alpha(K \setminus K(r_1^\alpha, \cdots, r_m^\alpha)) \\ &= p^\alpha(K(r_1^\alpha, \cdots, r_m^\alpha)). \end{aligned}$$

(iii) Noting that $\nu_{(r_1^\alpha, \cdots, r_m^\alpha)}(K(r_1^\alpha, \cdots, r_m^\alpha)) = 1$ we have, by Lemma 3.3,

$$\begin{aligned} |X|^\alpha L^{-1} \lambda^{-\alpha} \nu_{(r_1^\alpha, \cdots, r_m^\alpha))}(K(r_1^\alpha, \cdots, r_m^\alpha)) &\le p^\alpha(K(r_1^\alpha, \cdots, r_m^\alpha)) \\ &\le |X|^\alpha \lambda^\alpha \nu_{(r_1^\alpha, \cdots, r_m^\alpha)}(K(r_1^\alpha, \cdots, r_m^\alpha)). \end{aligned}$$ □

REMARK 3.8. Noting Theorem 3.7 (ii) and the fact $\nu_{(q_1,q_2,\cdots,q_m)}(K) = \nu_{(q_1,q_2,\cdots,q_m)}(K(q_1,q_2,\cdots,q_m))$, we see by Theorem 3.6 that a Borel probability measure $\nu_{(q_1,q_2,\cdots,q_m)}$ and $\beta(q_1,q_2,\cdots,q_m)$-dimensional packing measure are absolutely continuous to each other on $K(q_1,q_2,\cdots,q_m)$ if and only if $(q_1,q_2,\cdots,q_m) = (r_1^\alpha, r_2^\alpha,\cdots,r_m^\alpha)$.

The next Theorem shows that $K\backslash K(r_1^\alpha,\cdots,r_m^\alpha)$ and K are equivalent in the view of packing dimension.

THEOREM 3.9. *Let* α *be as in* Theorem 3.6. *Then*

(i) $\mathrm{Dim}(K) = \mathrm{Dim}(K({r_1}^\alpha,\cdots,{r_m}^\alpha)) = \alpha$
$= \mathrm{Dim}(K \setminus K({r_1}^\alpha,\cdots,{r_m}^\alpha))$

(ii) *for any* $(q_1,q_2,\cdots,q_m)$ *satisfying* $\sum_{i=1}^m q_i = 1$, *for* $0 < q_i < 1$, $\mathrm{Dim}(K(q_1,\cdots,q_m)) = \sum_{i=1}^m q_i\log q_i/\sum_{i=1}^m q_i\log r_i \le \alpha$ *and the equality is attained only in the case of* $(q_1,\cdots,q_m) = ({r_1}^\alpha,\cdots,{r_m}^\alpha)$.

PROOF. (ii) ; By the definition of $K(q_1,\cdots,q_m)$ and all $(i_1,i_2,\cdots) \in \{1,2,\cdots,m\}^{\mathbb{N}}$ such that $\cap_{i=1}^\infty[i_1,\cdots,i_n] \in K(q_1,\cdots,q_m)$,

$$\lim_{n\to\infty}\frac{\log\nu_{(q_1,\cdots,q_m)}([i_1,\cdots,i_n])}{\log|[i_1,\cdots,i_n]|} = \lim_{n\to\infty}\frac{\sum_{i=1}^m N_i(w,n)\log q_i}{\sum_{i=1}^m N_i(w,n)\log r_i} = \frac{\sum_{i=1}^m q_i\log q_i}{\sum_{i=1}^m q_i\log r_i}.$$

Since $\nu_{(q_1,\cdots,q_m)}(K(q_1,\cdots,q_m)) = 1$, by Lemma 3.2,

$$\mathrm{Dim}K(q_1,\cdots,q_m) = \frac{\sum_{i=1}^m q_i\log q_i}{\sum_{i=1}^m q_i\log r_i}$$

for any $(q_1,\cdots,q_m)$.
Hence $\sum_{i=1}^m q_i\log q_i/\sum_{i=1}^m q_i\log r_i \le \alpha$ and the equality holds if and only if $q_i = r_i^\alpha$, $i = 1,2,\cdots,m$.

For (i), let's show that $\alpha = \mathrm{Dim}(K \setminus K({r_1}^\alpha,\cdots,{r_m}^\alpha))$.
Suppose that $\{q_{i,k}\}_{i=1}^m$, $k = 1,2,\cdots$, is a sequence of probability vectors such that $0 < q_{i,k} < 1$, $\sum_{i=1}^m q_{i,k} = 1$, $\lim_{k\to\infty} q_{i,k} = r_i^\alpha$ and $(q_{1,k},q_{2,k},\cdots, q_{m,k}) \ne (r_1^\alpha,r_2^\alpha,\cdots,r_m^\alpha)$.

Then by Theorem 3.9-(ii), we see

$$\begin{aligned}\alpha &\geq \mathrm{Dim}(K \setminus K(r_1^\alpha, \cdots, r_m^\alpha)) \\ &\geq \mathrm{Dim}(K(q_{1,k}, q_{2,k}, \cdots, q_{m,k})) \\ &= \frac{\sum_{i=1}^m q_{i,k} \log q_{i,k}}{\sum_{i=1}^m q_{i,k} \log r_i} \qquad \text{for any } k.\end{aligned}$$

Letting $k \to \infty$, we have $\alpha = \mathrm{Dim}(K \setminus K(r_1^\alpha, \cdots, r_m^\alpha))$. □

4. Example

EXAMPLE 4.1. Define two sequences of contraction maps $\{\varphi_{i_1,i_2,\cdots,i_n}\}$ and $\{\psi_{i_1,i_2,\cdots,i_n}\}$ for $(i_1, i_2, \cdots, i_n) \in \{1,2\}^n$, $n = 1, 2, \cdots$.

Put $X = [0,1]^2$. Suppose that

$$\begin{aligned}&\varphi_i, \psi_i : X \to X, \quad i = 1, 2 \\ &\varphi_1 = \psi_1 : (x, y) \to \left(\frac{4}{9}x, \quad \frac{4}{9}y \right) \\ &\varphi_2 : (x, y) \to \left(\frac{1}{9}x + \frac{8}{9}, \quad \frac{1}{9}y \right) \\ &\psi_2 : (x, y) \to \left(\frac{1}{9}x + \frac{8}{9}, \quad \frac{1}{9}y + \frac{8}{9} \right).\end{aligned}$$

Then we define

$$\begin{cases} \varphi_{i_1,i_2,\cdots,i_n} = \varphi_{i_n} \\ \psi_{i_1,i_2,\cdots,i_n} = \begin{cases} \psi_{i_n}, & n = 1 \\ \varphi_{i_n}, & i_1 = 1,\ n \geq 2 \\ \psi_{i_n}, & i_1 = 2,\ n \geq 2 . \end{cases} \end{cases}$$

Put

$$K_\varphi = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,i_2,\cdots,i_n)\in\{1,2\}^n} \varphi_{i_1} \circ \varphi_{i_1,i_2} \circ \cdots \circ \varphi_{i_1,i_2,\cdots,i_n}(X)$$

and

$$K_\psi = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,i_2,\cdots,i_n)\in\{1,2\}^n} \psi_{i_1} \circ \psi_{i_1,i_2} \circ \cdots \circ \psi_{i_1,i_2,\cdots,i_n}(X).$$

Then $\mathrm{Dim}K_\varphi = \mathrm{Dim}K_\psi = \frac{1}{2} = \alpha$ (by Theorem 3.6). In fact, for any Borel set B,

$$p^\alpha(B \cap K_\varphi) = \int_B d\nu^\varphi_{(\frac{2}{3},\frac{1}{3})}(w)$$

$$p^\alpha(B \cap K_\psi) = \int_B \left(I_{[1]_\psi} + I_{[2]_\psi} \cdot \sqrt{2}^\alpha\right) d\nu^\psi_{(\frac{2}{3},\frac{1}{3})}(w),$$

where I_A is the indicator function of A, $[1]_\psi \equiv \psi_1(X)$, $[2]_\psi \equiv \psi_2(X)$ and $\nu^\varphi_{(\frac{2}{3},\frac{1}{3})}$, $\nu^\psi_{(\frac{2}{3},\frac{1}{3})}$ denote $(\frac{2}{3},\frac{1}{3})$-Bernoulli measures on K_φ, K_ψ respectively.

Hence, α-dimensional packing measure and the probability measure $\nu^\varphi_{(\frac{2}{3},\frac{1}{3})}$ are coincident on K_φ, but α-dimensional packing measure and $\nu^\psi_{(\frac{2}{3},\frac{1}{3})}$ are absolutely continuous to each other on K_ψ.

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