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ON THE PETTIS-DIVISOR PROPERTY FOR DUNFORD-PETTIS OPERATORS

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ABSTRACT. In this paper it is shown that Dunford-Pettis operators obey the "Pettis-divisor property": if T is a Dunford-Pettis operator from $L_1(\mu)$ to a Banach space X, then there is a non-Pettis representable operator $S : L_1(\mu) \longrightarrow L_1(\mu)$ such that $T \circ S$ is Pettis representable.

1. Introduction

Throughout this paper suppose X and Y are Banach spaces and X^* is the dual space of X. Write $B(X,Y)$ for the set of all bounded linear operators from X to Y, and $L_1(\mu)$ for the Banach space of all Lebesgue integrable functions on $[0,1]$, where μ is the Lebesgue measure on $[0,1]$.

In 1987 Petrakis [3,6] introduced the class of nearly representable operators from X to Y. These are the operators that map X-valued uniformly bounded martingales [2] that are Cauchy in the Pettis norm into Y-valued martingales that converge almost everywhere.

In 1988 Park [5] introduced the class of nearly Pettis representable operators from X to Y. These are the operators that map X-valued uniformly bounded martingales that are Cauchy in the Pettis norm into Y-valued martingales that converge in Pettis norm. Clearly nearly representable operators are also nearly Pettis representable operators. It is well known that the uniformly bounded X-valued martingales (ξ_n)

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correspond to the operators T which maps $L_1(\mu)$ into X. This correspondence is obtained by taking

$$T(\varphi) = \lim_{n \longrightarrow \infty} \int \xi_n(t)\varphi(t)dt$$

and

$$\xi_n = \sum_{E \in \Pi_n} \frac{T(\chi_E)}{\mu(E)} \chi_E$$

where Π_n is the nth dyadic partition of $[0,1]$, i.e.,

$$\Pi = \{I_{n,k} | I_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n}), k = 1, 2, \cdots, 2^n - 1\} \bigcup I_{n,2^n},$$

$n = 0, 1, 2, \cdots$ and $I_{n,2^n} = [\frac{2^n-1}{2^n}, 1]$.

In 1980 Bourgain [1] showed that

(1.1) The martingale (ξ_n) is Pettis-Cauchy if and only if $\lim_{n \longrightarrow \infty} \| \int \xi_n \varphi_n \| = 0$, whenever (φ_n) is an $L_\infty(\mu)$-bounded weakly null sequence in $L_1(\mu)$. And he showed that

(1.2) An operator $T : L_1(\mu) \longrightarrow X$ is Dunford-Pettis if and only if the corresponding martingale (ξ_n) is Pettis-Cauchy.

In 1987, M. Petrakis [6] also proved a "representability-divisor property" for Dunford-Pettis operators: if $T : L_1(\mu) \longrightarrow X$ is any Dunford-Pettis operator, then there is a non-representable operator $S : L_1(\mu) \longrightarrow L_1(\mu)$ such that $T \circ S$ is representable. In this paper we show that Petrakis' result remains true when "representable operator" is replaced by "Pettis representable operator" [7].

2. Pettis-divisor property for Dunford-Pettis operators

We begin with:

DEFINITION 2.1 [5]. An operator $T : X \longrightarrow Y$ is called a nearly Pettis representable operator if it maps every X-valued $L_\infty(\mu, X)$-bounded dyadic martingale that is Pettis Cauchy into a Y-valued martingale that converges in Pettis norm.

Also in [5], nearly Pettis representability was characterized in terms of Pettis representable operators.

THEOREM 2.2 [5]. *An operator $T : X \longrightarrow Y$ is nearly Pettis representable if and only if whenever $D : L_1(\mu) \longrightarrow X$ is a Dunford-Pettis operator $T \circ D : L_1(\mu) \longrightarrow Y$ is Pettis representable.*

REMARK. If Y is separable, then two notions of nearly Pettis representability and nearly representability coincide.

The following example shows a gap between Pettis representability and nearly Pettis representability.

EXAMPLE 2.3. The Volterra operator $V : L_1(\mu) \longrightarrow C[0,1]$ defined by

$$Vf(t) = \int_0^t f d\mu \quad (0 \leq t \leq 1, f \in L_1(\mu))$$

is a Dunford-Pettis operator which is nearly Pettis representable but not Pettis representable.

PROOF. By an argument of Lewis [4, Theorem], V is a Dunford-Pettis operator which is not Bochner representable. But since $C[0,1]$ is separable it follows that V is not Pettis representable. Also by an argument of Bourgain [1, Corollary 8], V is nearly representable and hence nearly Pettis representable. □

In [6], Petrakis gave a "representability-divisor property" for Dunford-Pettis operators.

LEMMA 2.4 [6]. *Let $T : L_1(\mu) \longrightarrow X$ be any Dunford-Pettis operator. Then there is a non representable operator $S : L_1(\mu) \longrightarrow L_1(\mu)$ such that $T \circ S$ is representable.*

Since every representable operator is Pettis representable, one might ask if Lemma 2.4 is true for "Pettis representable" in place of "representable". We will show that it is the case in Theorem 2.6 below. To do this we need to recall:

LEMMA 2.5 [5]. *An operator $T \in B(L_1(\mu), X)$ is a Pettis representable operator if and only if the martingale associated with T converges in Pettis norm.*

We now have:

THEOREM 2.6. *Let $T : L_1(\mu) \longrightarrow X$ be any Dunford-Pettis operator. Then there is a non Pettis representable operator $S : L_1(\mu) \longrightarrow L_1(\mu)$ such that $T \circ S$ is Pettis representable.*

PROOF. Since T is a Dunford-Pettis operator, we may assume that $\frac{3}{2}T(B_{L_1(\mu)}) \subset W$, where W is an open ball of X. Use the proof of [6, Theorem 28] to construct a tree $(\psi_{n,k})_{k=1,2,\cdots,2^n,n=0,1,2,\cdots}$ of positive functions in $L_\infty(\mu)$ and a system $(B_{n,k})_{k=1,2,\cdots,2^n,n=0,1,2,\cdots}$ of open balls of X such that

(1) $1 < \| \psi_{n,k} \|_1 < 2$ for all $n = 0,1,2,\cdots$, $k = 1,2,\cdots,2^n$,

(2) $\| \psi_{n+1,2k-1} - \psi_{n+1,2n} \|_1 > \frac{1}{2}$ for all $n = 0,1,2,\cdots$, $k = 1,2,\cdots,2^n$,

(3) B_{nk} has center at $T\psi_{n,k}$ and radius $r_{n,k}$ at most $\frac{1}{2^n}$,

(4) $B_{n,k} \subset W$ for all n,k and $\bar{B}_{n+1,2k-1} \bigcup \bar{B}_{n+1,2k} \subset B_{n,k}$.

Let $\xi_n = \frac{1}{2^n}\sum_{k=1}^{2^n} h_{n,k}\psi_{n,k}$, where $h_{n,k} = 2^n\chi_{I_{n,k}}$, $I_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n}]$, $1 \le k \le 2^n$, $n = 0,1,2,\cdots$. Then $\| \sum_{k=1}^{2^n}(-1)^k\psi_{n,k} \| \ge 2^{n-1}$ and the martingale $(T\xi_n)$, which is associated with $T \circ S$, converges [6]. But (ξ_n) is not convergent in Pettis norm. Consider the Rademacher functions (φ_n) where $\varphi_n(t) = sign(\sin 2^n\pi t)$, $0 \le t \le 1$. Then (φ_n) is an L_∞-bounded weakly null sequence in $L_1(\mu)$. Now

$$\begin{aligned}
\left\| \int \xi_n \varphi_n l \right\| &= \int \left| \int 2^{-n} \sum_{k=1}^{2^n} h_{n,k}(t)\psi_{n,k}\varphi_n(t)dt \right| d\mu \\
&= \int \left| \sum_{k=1}^{2^n} \psi_{n,k} \int_{In,k} \varphi_n(t)dt \right| d\mu \\
&= 2^{-n} \int \left| \sum_{k=1}^{2^n} (-1)^k \psi_{n,k} \right| d\mu \\
&\ge \frac{1}{2}.
\end{aligned}$$

Thus (ξ_n) is not Pettis Cauchy by (1.2) in section 1 and hence (ξ_n) is not convergent in Pettis norm. Hence $S : L_1(\mu) \longrightarrow L_1(\mu)$ which is associated to (ξ_n) is not Pettis representable by Lemma 2.5. $\square$

COROLLARY 2.7. *Let $T : L_1(\mu) \longrightarrow X$ be any nearly Pettis representable operator. Then there is a non Dunford-Pettis operator $S : L_1(\mu) \longrightarrow L_1(\mu)$ such that $T \circ S$ is Pettis representable.*

PROOF. Suppose $T : L_1(\mu) \longrightarrow X$ is nearly Pettis representable. By an argument of Bourgain [1. Theorem 5], T is Dunford-Pettis. By Theorem 2.6 there is a non Pettis representable operator $S : L_1(\mu) \longrightarrow L_1(\mu)$ such that $T \circ S$ is Pettis representable. In this case by (1.2) in the introduction, S is also a non Dunford-Pettis operator. This completes the proof. □

Corollary 2.7 enables us to think of a new set of operators in $B(L_1(\mu), L_1(\mu))$. We will call such operators near Dunford-Pettis operators, and will denote them as $NDP(N, X)$ operators, where $N : L_1(\mu) \longrightarrow X$ is a nearly Pettis representable operator. And thus

$$NDP(N, X) = \{T \in B(L_1(\mu), L_1(\mu)) | N \circ T : L_1(\mu) \longrightarrow X \text{ is Pettis representable}\}.$$

REMARK. By Corollary 2.7 the set $NDP(N, X)$ is strictly larger than $DP(L_1(\mu), L_1(\mu))$. And by Example 2.3 the identity operator $I \in B(L_1(\mu), L_1(\mu))$ is not an $NDP(V, C[0,1])$-operator.

LEMMA 2.8. *Let $T : X \longrightarrow Y$ is nearly Pettis representable and $S : Z \longrightarrow X$ is any operator. Then $T \circ S : Z \longrightarrow Y$ is nearly Pettis representable.*

PROOF. Let $T : X \longrightarrow Y$ be nearly Pettis representable and $S : Z \longrightarrow X$ be an operator. Let $D : L_1(\mu) \longrightarrow Z$ be a Dunford-Pettis operator. Then $S \circ D : L_1(\mu) \longrightarrow X$ is a Dunford-Pettis operator. Since T is nearly Pettis representable, $T \circ (S \circ D) = (T \circ S) \circ D : L_1(\mu) \longrightarrow Y$ is Pettis representable. Hence $T \circ S$ is nearly Pettis representable. □

DEFINITION 2.9. A Banach space X has the near Weak Radon-Nikodym property (NWRNP) if every nearly Pettis representable operator from $L_1(\mu)$ to X is Pettis representable.

THEOREM 2.10. *Let X^* be a dual Banach space of X. Then X^* has the $NWRNP$ if and only if $B(L_1(\mu), L_1(\mu)) = NDP(N, X^*)$ for every nearly Pettis representable operator $N : L_1(\mu) \longrightarrow X^*$.*

PROOF. Let $T \in B(L_1(\mu), L_1(\mu))$ and $N : L_1(\mu) \longrightarrow X^*$ be any nearly Pettis representable operator. Then $N \circ T : L_1(\mu) \longrightarrow X^*$ is nearly Pettis representable by Lemma 2.8. Hence $N \circ T$ is Pettis representable, i.e., $T \in NDP(N, X^*)$. Conversely let $N : L_1(\mu) \longrightarrow X^*$ be a nearly Pettis representable operator. Then since $I : L_1(\mu) \longrightarrow L_1(\mu)$ is an $NDP(N, X^*)$-operator, $N = N \circ I$ is Pettis representable. Thus X^* has the $NWRNP$. □

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