THE ANANLYSIS OF WILSON'S NONCONFORMING MULTIGRID ALGORITHM FOR SOLVING THE ELASTICITY PROBLEMS

Kab Seok Kang and Do Young Kwak

Abstract. In this paper we consider multigrid algorithms for solving elasticity problems by using Wilson's nonconforming finite element method. We consider two types of intergrid transfer operators which is needed to define the multigrid algorithm and prove convergence of $\mathcal{W}$-cycle mutigrid algorithm and uniform condition number estimates for the variable $\mathcal{V}$-cycle multigrid preconditioner.

1. Introduction

There are various ways of solving the elasticity problems. One can use conforming or nonconforming methods, which use elements not of class C^0. In [12], it has been shown that Wilson's element which is used in practice by engineers to solve the elasticity problems in two dimensions, passed the patch-test and the errors on the stresses and displacements are asymptotically of order h and h^2 respectively.

The convergence of conforming multigrid algorithm for elliptic problems was given by many others, for example [1, 2, 3, 4]. The convergence of nonconforming multigrid algorithm for elliptic problems was given in [5, 6, 7, 8]. To show the convergence of nonconforming multigrid algorithms, it is usually necessary to show the stability of the intergrid transfer operators [5, 6]. In this paper, we introduce two types of intergrid transfer operators for Wilson's nonconforming finite element method and show that the intergrid transfer operators satisfy the stablity condition if the Lamé constant λ is small.

Another property which is necessary for the convergence of multigrid algorithms is , so called, "regularity and approximation property"([4, 5]). Under certain regularity assumptions on the solutions, we are able to verify the regularity and approximation property in section 6, thus show that the convergence factor of $\mathcal{W}$-cycle multigrid algorithm is $C/(C+m^{1/2})$ for sufficiently large smoothing number m and the variable $\mathcal{V}$-cycle multigrid preconditioner has uniform condition number.

The outline of this paper is as follows. In section 2, we recall the variational formulation of an elasticity problem. In section 3, we define the Wilson's nonconforming finite element method for elasticity problem. In section 4 and 5 we define two types of intergrid transfer operators and verify the stability of the intergrid transfer operators. Finally, in section 6, we define the multigrid algorithm and give the convergence proof and some numerical results.

1991 *Mathematics Subject Classification.* Primary 65N30.

Key words and phrases. multigrid method, Wilson's nonconforming, elasticity problems.

2. Elasticity problem

Let Ω be a bounded open subset of the x-y plane with a Lipschitz-continuous boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ where the measure of Γ_0 is strictly positive and $\Gamma_1 = \Gamma - \Gamma_0$.

For any $\mathbf{v} = (v_1, v_2) \in (H^1(\Omega))^2$, we let

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad 1 \le i, j \le 2, \tag{2.1}$$

$$\sigma_{ij}(\mathbf{v}) = \lambda(\text{div } \mathbf{v})\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{v}), \quad 1 \le i, j \le 2, \tag{2.2}$$

where the constants $\lambda \ge 0$ and $\mu > 0$ appearing in the relationship (2.2) between the stresses σ_{ij} and the strains ϵ_{ij} are the coefficients of Lamé of the continuum and $\delta_{ij} = 0 (i \ne j)$, $\delta_{ii} = 1$.

We consider the problem

$$\begin{aligned} -\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\sigma_{ij}(\mathbf{u}) &= f_i \quad \text{in } \Omega, \qquad 1 \le i \le 2, \\ u_i &= 0 \quad \text{on } \Gamma_0, \qquad 1 \le i \le 2, \\ \sum_{j=1}^{2} \sigma_{ij}(\mathbf{u}) n_j &= g_i \quad \text{on } \Gamma_1, \qquad 1 \le i \le 2, \end{aligned} \tag{2.3}$$

where $\mathbf{n} = (n_1, n_2)$ denotes the outer normal on Γ. The solution $\mathbf{u}$ of (2.3) is the displacements relative to an equilibrium state of a homogenous and isotropic elastic continuum $\bar{\Omega}$ under the action of distributed body forces $\mathbf{f} = (f_1, f_2)$ per unit volume and external loading $\mathbf{g} = (g_1, g_2)$ per unit area, the displacements being specified and equal to zero along the subset Γ_0 of Γ.

To get a weak form of (2.3), we consider the space

$$V = \{\mathbf{v} = (v_1, v_2) \in (H^1(\Omega))^2 : v_i = 0 \text{ on } \Gamma_0, 1 \le i \le 2\}. \tag{2.4}$$

We let the bilinear form $a(\cdot, \cdot)$ be defined on $V \times V$ by

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \sum_{i,j=1}^{2} \sigma_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}) dxdy \\ &= \lambda \int_\Omega \text{div } \mathbf{u} \text{div } \mathbf{v} dxdy + 2\mu \int_\Omega \sum_{i,j=1}^{2} \epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}) dxdy, \end{aligned} \tag{2.5}$$

and we let the linear form $F(\cdot)$ be defined on V by

$$F(\mathbf{v}) = \int_\Omega (f_1 v_1 + f_2 v_2) dxdy + \int_{\Gamma_1} (g_1 v_1 + g_2 v_2) ds, \tag{2.6}$$

where $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_1)$, $i = 1, 2$, and s denote a curvilinear coordinate along Γ_1.

The weak form of (2.3) can be formulated as follows:

Find the dispalcements $\mathbf{u} = (u_1, u_2) \in V$ such that

$$a(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \text{for all } \mathbf{v} = (v_1, v_2) \in V. \tag{2.7}$$

We denote a seminorm and a norm over the Sobolev space $H^m(\Omega)$ as $|\cdot|_{m,\Omega}$ and $\|\cdot\|_{m,\Omega}$ respectively. For any $\mathbf{v} = (v_1, v_2) \in V$, we denote the expressions $(|v_1|^2_{m,\Omega} + |v_2|^2_{m,\Omega})^{1/2}$ and $(\|v_1\|^2_{m,\Omega} + \|v_2\|^2_{m,\Omega})^{1/2}$ as $|\mathbf{v}|_{m,\Omega}$ and $\|\mathbf{v}\|_{m,\Omega}$, respectively. Using the Korn's inequality,

$$\|\mathbf{v}\|_{1,\Omega} \le c \left(\sum_{i,j=1}^{2} \|\epsilon_{ij}(\mathbf{v})\|^2_{0,\Omega} + \|\mathbf{v}\|^2_{0,\Omega} \right)^{1/2} \qquad \text{for all } \mathbf{v} \in (H^1(\Omega))^2$$

and the fact that the measure of Γ_0 is strictly positive,

$$\mathbf{v} \in V \to \left(\sum_{i,j=1}^{2} \|\epsilon_{ij}(\mathbf{v})\|^2_{0,\Omega} \right)^{1/2} \tag{2.8}$$

is a norm over the space V that is equivalent to the norm $\|\cdot\|_{1,\Omega}$. As a consequence, we get, for a constant $c > 0$ depending only on Ω,

$$a(\mathbf{v}, \mathbf{v}) \ge 2\mu \sum_{i,j=1}^{2} \|\epsilon_{ij}(\mathbf{v})\|^2_{0,\Omega} \ge c\|\mathbf{v}\|^2_{1,\Omega} \qquad \text{for all } \mathbf{v} \in V. \tag{2.9}$$

On the other hand, we have

$$a(\mathbf{u}, \mathbf{v}) \le c|\mathbf{u}|_{1,\Omega}|\mathbf{v}|_{1,\Omega} \qquad \text{for all } \mathbf{u}, \mathbf{v} \in V. \tag{2.10}$$

By the Lax-Milgram lemma, (2.9) and (2.10) imply that problem (2.7) has a unique solution $\mathbf{u} \in V$.

3. Wilson's nonconforming finite element

Assume now that the domain Ω is the unit square$(0 \le x \le 1, 0 \le y \le 1)$. For the sake of simplicity, we consider a triangulation τ_h of Ω in equal squares with sides $h = 1/I$, for $I = 2^K$ for some integer K. We let

$$x_k = kh, \quad y_l = lh, \quad A_{kl} = (x_k, y_l), \quad 0 \le k, l \le I;$$
$$G_{kl} = ((k + \frac{1}{2})h, (l + \frac{1}{2})h), \quad T_{kl} = [x_k, x_{k+1}] \times [y_l, y_{l+1}], \quad 0 \le k, l \le I - 1.$$

For $0 \le k, l \le I - 1$, we let $F_{kl} \in (P_1)^2$ be the affine transformation mapping the reference square $\hat{T}(-1 \le x \le 1, -1 \le y \le 1)$ onto the square T_{kl}, with $F_{kl} : \hat{T} \to T_{kl}$:

$$x = \frac{1+\xi}{2} x_{k+1} + \frac{1-\xi}{2} x_k, \qquad y = \frac{1+\eta}{2} y_{l+1} + \frac{1-\eta}{2} y_l. \tag{3.1}$$

Definition 3.1. Wilson's brick can be defined on the reference square $\hat{T}$ as follows:

(1) The space of shape functions is $\hat{P} = P_2$,
(2) The degree of freedom $\hat{\Sigma}$ are the values of the functions $\hat{p}$ at the four vertices of the square and the average values of $\partial^2\hat{p}/\partial\xi^2$ and $\partial^2\hat{p}/\partial\eta^2$ on the square $\hat{T}$.

The function $\hat{p} \in \hat{P}$ such that

$$\hat{p}(\hat{a}_i) = p_i, \quad 1 \le i \le 4, \quad \int_{\hat{T}} \frac{\partial^2\hat{p}}{\partial\xi^2} d\xi d\eta = p_\xi, \quad \int_{\hat{T}} \frac{\partial^2\hat{p}}{\partial\eta^2} d\xi d\eta = p_\eta \tag{3.2}$$

can be written as follows:

$$\begin{aligned}\hat{p} = &\frac{(1+\xi)(1+\eta)}{4}p_1 + \frac{(1-\xi)(1+\eta)}{4}p_2 + \frac{(1-\xi)(1-\eta)}{4}p_3 \\ &+ \frac{(1+\xi)(1-\eta)}{4}p_4 + \frac{\xi^2-1}{8}p_\xi + \frac{\eta^2-1}{8}p_\eta.\end{aligned} \tag{3.3}$$

The finite elements $(T, \Sigma, P)_{kl}$ will be the images by the transformations F_{kl} of the element of reference $(\hat{T}, \hat{\Sigma}, \hat{P})$, with

$$P_{kl} = \{p = \hat{p} \circ F_{kl}^{-1} : \forall \hat{p} \in \hat{P}\}, \quad 0 \le k, l \le I - 1. \tag{3.4}$$

The finite dimensional subspace X_h of $L^2(\Omega)$ will be the space of functions whose restriction to each element T_{kl} belongs to $P_{kl}, 0 \le k, l \le I-1$, defined by their values at the vertices of the elements T_{kl} and the averege values of their second derivatives $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ on each element T_{kl}. In general, the inclusion $X_h \subset C^0(\bar{\Omega})$ does not hold.

Definition 3.2. For any function $\varphi \in H^2(\hat{T})$, its interpolation $\Pi\varphi$ will be the unique function of $\hat{P}$ equal to φ at the vertices of $\hat{T}$ and such that

$$\int_{\hat{T}} \frac{\partial^2}{\partial\xi^2}(\varphi - \Pi\varphi) d\xi d\eta = \int_{\hat{T}} \frac{\partial^2}{\partial\eta^2}(\varphi - \Pi\varphi) d\xi d\eta = 0.$$

The following equality is then satisfied:

$$\varphi - \Pi\varphi = 0 \quad \text{for all } \varphi \in P_2. \tag{3.5}$$

Now for all $\mathbf{u} = (u_1, u_2) \in (H^2(\Omega))^2$, we let its $(X_h)^2$-interpolation $\Pi_h\mathbf{u}$ be the unique function of $(X_h)^2$ whose restriction to each element T of τ_h has its components respectively equal to Πu_1 and Πu_2.

Assume that $\Gamma_0 = \cup_{1\le i\le i_0}\Gamma_{0,i}$, where the $\Gamma_{0,i}$ are subsets of Γ. Then space V_h will be the subspace of $(X_h)^2$ of functions equal to zero at the vertices belonging to Γ_0.

Since V_h is nonconforming, i.e., $V_h \not\subset V \cap C^0(\bar{\Omega})$ and the functions of V_h are smooth on each $T \in \tau_h$, we define a new bilinear form $a_h(\cdot,\cdot)$ on $V_h + V$ by

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) &= \sum_{T\in\tau_h} \int_T \sum_{i,j=1}^{2} \sigma_{ij}(\mathbf{u}_h)\epsilon_{ij}(\mathbf{v}_h)dxdy \\ &= \sum_{T\in\tau_h} \left(\lambda \int_T \text{div } \mathbf{u}_h \text{div } \mathbf{v}_h dxdy + 2\mu \int_T \sum_{i,j=1}^{2} \epsilon_{ij}(\mathbf{u}_h)\epsilon_{ij}(\mathbf{v}_h)dxdy \right) \end{aligned} \tag{3.6}$$

The discrete problem of (2.7) will be defined as follows:
Find $\mathbf{u}_h \in V_h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = F(\mathbf{v}_h) \qquad \text{for all } \mathbf{v}_h \in V_h. \tag{3.7}$$

We let $|||\cdot|||_h$ from V_h into $\mathbb{R}$ be defined by

$$|||\mathbf{v}_h|||_h = \left(\sum_{T\in\tau_h} \sum_{i,j=1}^{2} \|\epsilon_{ij}(\mathbf{v}_h)\|_{0,T}^2 \right)^{1/2}. \tag{3.8}$$

Then, by (2.9) and (2.10), there exist two positive constant c and C independent h such that

$$c|||\mathbf{v}_h|||_h^2 \le a_h(\mathbf{v}_h, \mathbf{v}_h) \le C|||\mathbf{v}_h|||_h^2 \tag{3.9}$$

for all $\mathbf{v}_h \in V_h$.

Remark 3.3. *From the definitions (3.6) and (3.8), we note that C in (3.9) depends on the Lamé constant λ.*

Here and thereafter, we assume the smoothness hypothesis for the system of elasticity, i.e., for all $\mathbf{f} = (f_1, f_2) \in (L^2(\Omega))^2$, the system

$$\begin{aligned} -\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\sigma_{ij}(\mathbf{u}) &= f_i \quad \text{in } \Omega, \qquad 1 \le i \le 2, \\ u_i &= 0 \quad \text{on } \Gamma_0, \qquad 1 \le i \le 2, \\ \sum_{j=1}^{2} \sigma_{ij}(\mathbf{u})n_j &= 0 \quad \text{on } \Gamma_1, \qquad 1 \le i \le 2, \end{aligned} \tag{3.10}$$

has a unique solution $\mathbf{u} = (u_1, u_2) \in (H^2(\Omega))^2 \cap V$ and $\|\mathbf{u}\|_{2,\Omega} \le C\|\mathbf{f}\|_{0,\Omega}$.

The next results which concern the error estimation of Wilson's elements for elasticity problems are from [12]:

Lemma 3.4. *Let s be an integer with $2 \le s \le 3$, let $\mathbf{u} \in (H^s(\Omega))^2 \cap V$, and let $\Pi_h\mathbf{u} \in V_h$, the V_h interpolation of $\mathbf{u}$. We have*

$$|\mathbf{u} - \Pi_h\mathbf{u}|_{m,T} \le ch^{s-m}|\mathbf{u}|_{s,T}, \quad 0 \le m \le s, \tag{3.11}$$

for all $\mathbf{u} \in (H^s(\Omega))^2$, all $T \in \tau_h$, the constant $c > 0$ being independent of h.

Theorem 3.5. *Let $\mathbf{u} \in (H^2(\Omega))^2 \cap V$ be the solution of problem (2.7) and let $\mathbf{u}_h \in V_h$ be the solution of problem (3.7). Then we have:*

$$|||\mathbf{u} - \mathbf{u}_h|||_h \le ch|\mathbf{u}|_{2,\Omega} \tag{3.12}$$

where the constant $c > 0$ is independent of h. Moreover we have:

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \le ch^2|\mathbf{u}|_{2,\Omega}. \tag{3.13}$$

Here we describe two equivalence relations which will be used in next sections. For any $\mathbf{v} = (v_1, v_2) \in (X_h)^2$, we define,

$$\begin{gathered} v_x^i(G_{kl}) = 4h^2\left(\frac{\partial^2}{\partial x^2}v_i\right)(G_{kl}), v_y^i(G_{kl}) = 4h^2\left(\frac{\partial^2}{\partial y^2}v_i\right)(G_{kl}), \\ v_{k,l}^i = v_i(A_{kl}), \quad i = 1,2, \quad 0 \le k,l \le I-1. \end{gathered}$$

We let, for $0 \le k, l \le I-1$,

$$\begin{aligned} D_{kl}(\mathbf{v}) = {} & (v_{k,l+1}^2 - v_{k,l}^2)^2 + (v_{k+1,l+1}^2 - v_{k+1,l}^2)^2 + (v_x^2(G_{kl}))^2 + (v_y^2(G_{kl}))^2 \\ & + (v_{k+1,l}^1 - v_{k,l}^1)^2 + (v_{k+1,l+1}^1 - v_{k,l+1}^1)^2 + (v_x^1(G_{kl}))^2 + (v_y^1(G_{kl}))^2 \\ & + (v_{k+1,l+1}^2 + v_{k+1,l}^2 - v_{k,l+1}^2 - v_{k,l}^2 + v_{k+1,l+1}^1 + v_{k,l+1}^1 - v_{k,l}^1 - v_{k+1,l}^1)^2, \end{aligned} \tag{3.14}$$

$$\begin{aligned} B_{kl}(\mathbf{v}) = h^2\sum_{i=1}^{2}\{ & (v_{k,l}^i)^2 + (v_{k+1,l}^i)^2 + (v_{k,l+1}^i)^2 + (v_{k+1,l+1}^i)^2 \\ & + (v_x^i(G_{kl}))^2 + (v_y^i(G_{kl}))^2\}. \end{aligned} \tag{3.15}$$

Lemma 3.6. *There exist two constants c and C, with $0 < c < C$, independent of h, such that*

$$c\sum_{i,j=1}^{2}\|\epsilon_{ij}(\mathbf{v})\|_{0,T_{kl}}^2 \le D_{kl}(\mathbf{v}) \le C\sum_{i,j=1}^{2}\|\epsilon_{ij}(\mathbf{v})\|_{0,T_{kl}}^2 \tag{3.16}$$

for all $\mathbf{v} = (v_1, v_2) \in (X_h)^2$ and for $0 \le k, l \le I-1$.

Proof. On the reference square $\hat{T}$, we let $\varphi(\xi,\eta) = v_1(x,y)$, $\psi(\xi,\eta) = v_2(x,y)$ with $(x,y) = F_{kl}(\xi,\eta)$. Then we have

$$\sum_{i,j=1}^{2}\|\epsilon_{ij}(\mathbf{v}_h)\|_{0,T_{kl}}^2 = \int_{\hat{T}}\left(\left(\frac{\partial\varphi}{\partial\xi}\right)^2 + \left(\frac{\partial\psi}{\partial\eta}\right)^2 + \left(\frac{\partial\varphi}{\partial\eta} + \frac{\partial\psi}{\partial\xi}\right)^2\right)d\xi d\eta. \tag{3.17}$$

To simplicify the calculation, we let $\hat{a}_1 = (-1,-1)$, $\hat{a}_2 = (1,-1)$, $\hat{a}_3 = (1,1)$, and $\hat{a}_4 = (-1,1)$ in ξ-η plane and let $\varphi_i = \varphi(\hat{a}_i)$ and $\psi_i = \psi(\hat{a}_i)$ for $i = 1,2,3,4$, $\varphi_\xi = v_x^1(G_{kl})$, $\varphi_\eta = v_y^1(G_{kl})$, $\psi_\xi = v_x^2(G_{kl})$, and $\psi_\eta = v_y^2(G_{kl})$.

Then according to (3.3), we have

$$
\begin{aligned}
\frac{\partial \varphi}{\partial \xi} &= \frac{1}{4}[(\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4) + \eta(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4) + \varphi_\xi \xi], \\
\frac{\partial \psi}{\partial \eta} &= \frac{1}{4}[(\psi_1 + \psi_2 - \psi_3 - \psi_4) + \xi(\psi_1 - \psi_2 + \psi_3 - \psi_4) + \psi_\eta \eta], \\
\frac{\partial \varphi}{\partial \eta} + \frac{\partial \psi}{\partial \xi} &= \frac{1}{4}[(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 + \psi_1 - \psi_2 - \psi_3 + \psi_4) \\
&\quad + \xi(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4 + \psi_\xi) + \eta(\psi_1 - \psi_2 + \psi_3 - \psi_4 + \varphi_\eta)].
\end{aligned} \tag{3.18}
$$

From (3.18), (3.17) becomes

$$
\begin{aligned}
\sum_{i,j=1}^{2} \|\epsilon_{ij}(\mathbf{v}_h)\|_{0,T_{kl}}^2 = \frac{1}{12}\Big[& 3(\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4)^2 + (\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4)^2 + \varphi_\xi^2 \\
& + 3(\psi_1 + \psi_2 - \psi_3 - \psi_4)^2 + (\psi_1 - \psi_2 + \psi_3 - \psi_4)^2 + \psi_\eta^2 \\
& + 3(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 + \psi_1 - \psi_2 - \psi_3 + \psi_4)^2 \\
& + (\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4 + \psi_\xi)^2 + (\psi_1 - \psi_2 + \psi_3 - \psi_4 + \varphi_\eta)^2 \Big] \\
= \frac{1}{12}\Big[& \frac{25}{9}(\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4)^2 + \frac{2}{9}(\varphi_1 - \varphi_2)^2 + \frac{2}{9}(\varphi_3 - \varphi_4)^2 + \varphi_\xi^2 \\
& + \frac{25}{9}(\psi_1 + \psi_2 - \psi_3 - \psi_4)^2 + \frac{2}{9}(\psi_1 - \psi_4)^2 + \frac{2}{9}(\psi_2 - \psi_3)^2 + \psi_\eta^2 \\
& + 3(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 + \psi_1 - \psi_2 - \psi_3 + \psi_4)^2 + \frac{1}{4}\psi_\xi^2 + \frac{1}{4}\varphi_\eta^2 \\
& + \frac{3}{4}\left(\frac{4}{3}(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4) + \psi_\xi\right)^2 + \frac{3}{4}\left(\frac{4}{3}(\psi_1 - \psi_2 + \psi_3 - \psi_4) + \varphi_\eta\right)^2 \Big].
\end{aligned}
$$

From the above equality and elementary arithemetic calculations, we get (3.16) where $c \geq \frac{1}{54}$ and $C \leq \frac{1}{2}$. $\square$

Lemma 3.7. *There exist two constants c and C, with $0 < c < C$, independent of h, such that*

$$
c\|\mathbf{v}\|_{0,T_{kl}}^2 \leq B_{kl}(\mathbf{v}) \leq C\|\mathbf{v}\|_{0,T_{kl}}^2 \tag{3.19}
$$

for all $\mathbf{v} \in (X_h)^2$ and for $0 \leq k, l \leq I - 1$.

Proof. According to (3.3) and using notations in the proof of Lemma 3.6, we have, for all $v \in P_2(T_{kl})$ and $\varphi(\xi, \eta) = v(x, y)$ with $(x, y) = F_{k,l}(\xi, \eta)$,

$$
\begin{aligned}
\varphi = {} & \frac{1}{4}\left(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 - \frac{1}{2}\varphi_\xi - \frac{1}{2}\varphi_\eta\right) + \frac{1}{4}(\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4)\xi \\
& + \frac{1}{4}(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)\eta + \frac{1}{4}(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4)\xi\eta + \frac{1}{8}\varphi_\xi \xi^2 + \frac{1}{8}\varphi_\eta \eta^2.
\end{aligned} \tag{3.20}
$$

Here we let, for $i = 1, 2$,

$$\begin{aligned}
A_i &= \frac{1}{4}(v^i_{k+1,l+1} + v^i_{k,l+1} + v^i_{k,l} + v^i_{k+1,l}),\\
B_i &= \frac{1}{4}(-v^i_{k+1,l+1} + v^i_{k,l+1} + v^i_{k,l} - v^i_{k+1,l}),\\
C_i &= \frac{1}{4}(-v^i_{k+1,l+1} - v^i_{k,l+1} + v^i_{k,l} + v^i_{k+1,l}),\\
D_i &= \frac{1}{4}(v^i_{k+1,l+1} - v^i_{k,l+1} + v^i_{k,l} - v^i_{k+1,l}),\\
E_i &= \frac{1}{8}v^i_x(G_{kl}), \qquad F_i = \frac{1}{8}v^i_y(G_{kl}).
\end{aligned}$$

Then we get

$$\begin{aligned}
\|\mathbf{v}\|^2_{0,T_{kl}} = h^2 \sum_{i=1}^{2} \Bigg(& 4A_i^2 + \frac{4}{3}(B_i^2 + C_i^2) + \frac{4}{9}D_i^2 - \frac{16}{3}A_i(E_i + F_i)\\
& + \frac{16}{9}(E_i + F_i)^2 + \frac{16}{45}(E_i^2 + F_i^2)\Bigg).
\end{aligned} \tag{3.21}$$

From (3.21), we have

$$\begin{aligned}
&\|\mathbf{v}\|^2_{0,T_{kl}} \le h^2 \sum_{i=1}^{2} \left(\frac{28}{3}(A_i^2 + B_i^2 + C_i^2 + D_i^2) + \frac{76}{9}(E_i^2 + F_i^2) \right)\\
&\le Ch^2 \sum_{i=1}^{2} \left((v^i_{k,l})^2 + (v^i_{k+1,l})^2 + (v^i_{k,l+1})^2 + (v^i_{k+1,l+1})^2 + (v^i_x(G_{kl}))^2 + (v^i_y(G_{kl}))^2 \right)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathbf{v}\|^2_{0,T_{kl}} &\ge h^2 \sum_{i=1}^{2} \Bigg(\frac{4}{17}A_i^2 + \frac{4}{3}(B_i^2 + C_i^2) + \frac{4}{9}D_i^2\\
&\qquad + \frac{17}{9}\left(\frac{24}{17}A_i - E_i - F_i\right)^2 + \frac{6}{45}(E_i^2 + F_i^2)\Bigg)\\
&\ge ch^2 \sum_{i=1}^{2} \left(\frac{4}{17}(A_i^2 + B_i^2 + C_i^2 + D_i^2) + \frac{6}{45}(E_i^2 + F_i^2) \right)\\
&\ge ch^2 \sum_{i=1}^{2} \Big((v^i_{k,l})^2 + (v^i_{k+1,l})^2 + (v^i_{k,l+1})^2 + (v^i_{k+1,l+1})^2\\
&\qquad + (v^i_x(G_{kl}))^2 + (v^i_y(G_{kl}))^2 \Big). \quad \square
\end{aligned}$$

From the Lemma 3.7, we know that

$$h^2 \sum_{i=1}^{2} \left(\sum_{k,l=0}^{I} v_i(A_{kl})^2 + \sum_{k,l=0}^{I-1} (v^i_x(G_{kl})^2 + v^i_y(G_{kl})^2) \right)$$

is a norm in V_h which is equivalent to $\|\cdot\|_{0,\Omega}^2$. Therefore we can define the inner product $(\cdot,\cdot)_h$ as follows

$$\begin{aligned}(\mathbf{u},\mathbf{v})_h =& h^2 \sum_{i=1}^{2} \left(\sum_{k,l=0}^{I} u_i(A_{kl}) v_i(A_{kl}) \right.\\ &\left. + \sum_{k,l=0}^{I-1} (u_x^i(G_{kl}) v_x^i(G_{kl}) + u_y^i(G_{kl}) v_y^i(G_{kl})) \right).\end{aligned}$$

4. Intergrid transfer operator I_H^h

In this and next section, we define two types of intergrid transfer operators $I_H^h : V_H \to V_h$ and $J_H^h : V_H \to V_h$ where V_H and V_h are defined in section 3 for $H = 2h$.

We consider $h = 1/I_h$ and $H = 1/I_H$ for $I_h = 2^m$ and $I_H = 2^{m-1}$ in section 3 and let

$$\begin{aligned}
&x_k^h = kh, \quad y_l^h = lh, \quad A_{kl}^h = (x_k^h, y_l^h), \quad 0 \le k,l \le I_h;\\
&G_{kl}^h = \left(\left(k+\frac{1}{2}\right)h, \left(l+\frac{1}{2}\right)h \right), \quad 0 \le k,l \le I_h - 1,\\
&T_{kl}^h = [x_k^h, x_{k+1}^h] \times [y_l^h, y_{l+1}^h],\\
&x_k^H = kH, \quad y_l^H = lH, \quad A_{kl}^H = (x_k^H, y_l^H), \quad 0 \le k,l \le I_H;\\
&G_{kl}^H = \left(\left(k+\frac{1}{2}\right)H, \left(l+\frac{1}{2}\right)H \right), \quad 0 \le k,l \le I_H - 1.\\
&T_{kl}^H = [x_k^H, x_{k+1}^H] \times [y_l^H, y_{l+1}^H],
\end{aligned}$$

For $u_H \in X_H$, we define $u_h = I_H^h u_H \in X_h$ as follows:

$$\begin{aligned}
&u_h(A_{2k2l}^h) = u_H(A_{kl}^H),\\
&u_h(A_{(2k+1)2l}^h) = \frac{u_H(A_{kl}^H) + u_H(A_{(k+1)l}^H)}{2},\\
&u_h(A_{2k(2l+1)}^h) = \frac{u_H(A_{kl}^H) + u_H(A_{k(l+1)}^H)}{2},\\
&u_h(A_{(2k+1)(2l+1)}^h) = \frac{u_H(A_{kl}^H) + u_H(A_{(k+1)(l+1)}^H) + u_H(A_{(k+1)l}^H) + u_H(A_{k(l+1)}^H)}{4},\\
&u_{h,x}(G_{2k,2l}^h) = u_{h,x}(G_{2k+1,2l}^h) = u_{h,x}(G_{2k,2l+1}^h) = u_{h,x}(G_{2k+1,2l+1}^h) = \frac{u_{H,x}(G_{k,l}^H)}{4},\\
&u_{h,y}(G_{2k,2l}^h) = u_{h,y}(G_{2k+1,2l}^h) = u_{h,y}(G_{2k,2l+1}^h) = u_{h,y}(G_{2k+1,2l+1}^h) = \frac{u_{H,y}(G_{k,l}^H)}{4}.
\end{aligned}$$

We define $I_H^h : V_H \to V_h$ as follows: for $\mathbf{v}_H = (v_1, v_2) \in V_H$,

$$I_H^h \mathbf{v}_H = (I_H^h v_1, I_H^h v_2).$$

Lemma 4.1. *There exists a constant $C > 0$, independent of h, such that*

$$\|\mathbf{v}_H - I_H^h \mathbf{v}_H\|_{0,\Omega} \le Ch|||\mathbf{v}_H|||_H \quad \textit{for all } \mathbf{v}_H \in V_H. \tag{4.1}$$

Proof. Since $\mathbf{v}_H|_{T_{kl}^h} \in (P_2(T_{kl}^h))^2$, we have

$$\|\mathbf{v}_H - I_H^h \mathbf{v}_H\|_{0,\Omega}^2 = \sum_{k,l=0}^{I_h-1} \|\mathbf{v}_H - I_H^h \mathbf{v}_H\|_{0,T_{kl}^h}^2 \le C \sum_{k,l=0}^{I_h-1} B_{kl}^h(\mathbf{v}_H|_{T_{kl}^h} - I_H^h \mathbf{v}_H).$$

From (3.3), we have, for $0 \le k, l \le I_H - 1$,

$$\begin{aligned}
B_{2k2l}^h(\mathbf{v}_H|_{T_{2k2l}^h} - I_H^h \mathbf{v}_H) &= h^2 \sum_{i=1}^{2} \{(v_{i,H}|_{T_{2k2l}^h}(A_{2k2l}^h) - I_H^h v_{i,H}(A_{2k2l}^h))^2 \\
&+ (v_{i,H}|_{T_{2k2l}^h}(A_{(2k+1)2l}^h) - I_H^h v_{i,H}(A_{(2k+1)2l}^h))^2 \\
&+ (v_{i,H}|_{T_{2k2l}^h}(A_{2k(2l+1)}^h) - I_H^h v_{i,H}(A_{2k(2l+1)}^h))^2 \\
&+ (v_{i,H}|_{T_{2k2l}^h}(A_{(2k+1)(2l+1)}^h) - I_H^h v_{i,H}(A_{(2k+1)(2l+1)}^h))^2 \\
&+ (v_x^i(G_{2k2l}^h) - (I_H^h v)_x^i(G_{2k2l}^h))^2 + (v_y^i(G_{2k2l}^h) - (I_H^h v)_y^i(G_{2k2l}^h))^2\} \\
&= h^2 \sum_{i=1}^{2} \left\{ \left(\frac{v_x^i(G_{kl}^H)}{8} \right)^2 + \left(\frac{v_y^i(G_{kl}^H)}{8} \right)^2 + \left(\frac{v_x^i(G_{kl}^H)}{8} + \frac{v_y^i(G_{kl}^H)}{8} \right)^2 \right. \\
&\left. + \left(\frac{3}{4} v_x^i(G_{kl}^H) \right)^2 + \left(\frac{3}{4} v_y^i(G_{kl}^H) \right) \right\} \\
&\le ch^2 \sum_{i=1}^{2} \left\{ v_x^i(G_{kl}^H) + v_y^i(G_{kl}^H) \right\} \le Ch^2 D_{kl}^H(\mathbf{v}_H).
\end{aligned}$$

By the same argument, we have

$$\begin{aligned}
B_{(2k+1)2l}^h(\mathbf{v}_H|_{T_{(2k+1)2l}}^h - I_H^h \mathbf{v}_H) &\le Ch^2 D_{kl}^H(\mathbf{v}_H), \\
B_{2k(2l+1)}^h(\mathbf{v}_H|_{T_{2k(2l+1)}}^h - I_H^h \mathbf{v}_H) &\le Ch^2 D_{kl}^H(\mathbf{v}_H), \\
B_{(2k+1)(2l+1)}^h(\mathbf{v}_H|_{T_{(2k+1)(2l+1)}}^h - I_H^h \mathbf{v}_H) &\le Ch^2 D_{kl}^H(\mathbf{v}_H).
\end{aligned}$$

From the above and (3.16), we have (4.1). $\square$

Lemma 4.2. *There exists a constant $C > 0$, independent of h, such that*

$$|||I_H^h \mathbf{v}_H|||_h \le C|||\mathbf{v}_H|||_H \quad \textit{for all } \mathbf{v}_H \in V_H. \tag{4.2}$$

Proof. Since $\mathbf{v}_H|_{T_{kl}^h} \in (P_2(T_{kl}^h))^2$ and from (3.16), we have

$$\begin{aligned}
|||I_H^h \mathbf{v}_H|||_h^2 &= \sum_{k,l=0}^{I_h-1} \sum_{i,j=1}^{2} \|\epsilon_{ij}(I_H^h \mathbf{v}_H - \mathbf{v}_H) + \epsilon_{ij}(\mathbf{v}_H)\|_{0,T_{kl}^h}^2 \\
&\le 2 \sum_{k,l=0}^{I_h-1} \sum_{i,j=1}^{2} \|\epsilon_{ij}(I_H^h \mathbf{v}_H - \mathbf{v}_H)\|_{0,T_{kl}^h}^2 + 2 \sum_{k,l=0}^{I_h-1} \sum_{i,j=1}^{2} \|\epsilon_{ij}(\mathbf{v}_H)\|_{0,T_{kl}^h}^2 \\
&\le 2 \sum_{k,l=0}^{I_h-1} \sum_{i,j=1}^{2} \|\epsilon_{ij}(\mathbf{v}_H)\|_{0,T_{kl}^h}^2 + C \sum_{k,l=0}^{I_h-1} D_{kl}^h(\mathbf{v}_H|_{T_{kl}^h} - I_H^h \mathbf{v}_H).
\end{aligned}$$

Since

$$\sum_{k,l=0}^{I_h-1}\sum_{i,j=1}^{2}\|\epsilon_{ij}(\mathbf{v}_H)\|_{0,T_{kl}^h}^2=\sum_{k,l=0}^{I_H-1}\sum_{i,j=1}^{2}\|\epsilon_{ij}(\mathbf{v}_H)\|_{0,T_{kl}^H}^2, \tag{4.3}$$

we only need to show that

$$\sum_{k,l=0}^{I_h-1} D_{kl}^h(\mathbf{v}_H|_{T_{kl}^h}-I_H^h\mathbf{v}_H)\le C\sum_{k,l=0}^{I_H-1} D_{kl}^H(\mathbf{v}_H). \tag{4.4}$$

By using the arguments in proof of Lemma 4.1,

$$\begin{aligned}
&D_{2k2l}^h(\mathbf{v}_H|_{T_{2k2l}^h}-I_H^h\mathbf{v}_H)\\
&=2\left(\frac{v_y^2(G_{kl}^H)}{8}\right)^2+2\left(\frac{v_x^1(G_{kl}^H)}{8}\right)^2+\left(\frac{v_x^2(G_{kl}^H)}{4}+\frac{v_y^1(G_{kl}^H)}{4}\right)^2\\
&+\left(\frac{3}{4}v_x^1(G_{kl}^H)\right)^2+\left(\frac{3}{4}v_x^2(G_{kl}^H)\right)^2+\left(\frac{3}{4}v_y^1(G_{kl}^H)\right)^2+\left(\frac{3}{4}v_y^2(G_{kl}^H)\right)^2\\
&\le\frac{19}{32}\left(v_x^1(G_{kl}^H)\right)^2+\frac{11}{16}\left(v_x^2(G_{kl}^H)\right)^2+\frac{11}{16}\left(v_y^1(G_{kl}^H)\right)^2+\frac{19}{32}\left(v_y^2(G_{kl}^H)\right)^2\\
&\le CD_{kl}^H(\mathbf{v}_H).
\end{aligned}$$

By the same argument, we have

$$\begin{aligned}
D_{(2k+1)2l}^h(\mathbf{v}_H|_{T_{(2k+1)2l}^h}-I_H^h\mathbf{v}_H)&\le CD_{kl}^H(\mathbf{v}_H),\\
D_{2k(2l+1)}^h(\mathbf{v}_H|_{T_{2k(2l+1)}^h}-I_H^h\mathbf{v}_H)&\le CD_{kl}^H(\mathbf{v}_H),\\
D_{(2k+1)(2l+1)}^h(\mathbf{v}_H|_{T_{(2k+1)(2l+1)}^h}-I_H^h\mathbf{v}_H)&\le CD_{kl}^H(\mathbf{v}_H).
\end{aligned}$$

From the above and (3.16), we have (4.4). □

Corollary 4.3. *Let $P_h^H: V_h\to V_H$ be the operator such that*

$$a_h(\mathbf{u}_h, I_H^h\mathbf{v}_H)=a_H(P_h^H\mathbf{u}_h,\mathbf{v}_H)\quad \textit{for all } \mathbf{v}_H\in V_H, \mathbf{u}_h\in V_h.$$

Then we have

$$|||P_h^H\mathbf{u}_h|||_H\le C|||\mathbf{u}_h|||_h\quad \textit{for all } \mathbf{u}_h\in V_h. \tag{4.5}$$

Lemma 4.4. *There exists a positive constant C, independent of h, such that*

$$|||I_H^h(\Pi_H\mathbf{u})-\Pi_h\mathbf{u}|||_h\le Ch|\mathbf{u}|_{2,\Omega} \tag{4.6}$$

and

$$\|I_H^h(\Pi_H\mathbf{u})-\Pi_h\mathbf{u}\|_{0,\Omega}\le Ch^2|\mathbf{u}|_{2,\Omega} \tag{4.7}$$

for all $\mathbf{u} \in (H^2(\Omega))^2 \cap V$.

Proof. By using the arguments in the proof of Lemma 4.1 and Lemma 4.2, we have, for all $\mathbf{u} \in (H^2(\Omega))^2 \cap V$,

$$
\begin{aligned}
|||I_H^h(\Pi_H\mathbf{u}) - \Pi_h\mathbf{u}|||_h^2 &= \sum_{k,l=0}^{I_h-1}\sum_{i,j=1}^{2} \|\epsilon_{ij}(I_H^h(\Pi_H\mathbf{u}) - \Pi_h\mathbf{u})\|_{0,T_{kl}^h}^2 \\
&\le 3\sum_{k,l=0}^{I_h-1}\sum_{i,j=1}^{2}\Big\{\|\epsilon_{ij}(I_H^h\Pi_H\mathbf{u} - \Pi_H\mathbf{u})\|_{0,T_{kl}^h}^2 + \|\epsilon_{ij}(\Pi_H\mathbf{u} - \mathbf{u})\|_{0,T_{kl}^h}^2 \\
&+\|\epsilon_{ij}(\mathbf{u} - \Pi_h\mathbf{u})\|_{0,T_{kl}^h}^2\Big\} \\
&\le C\sum_{k,l=0}^{I_h-1}\sum_{i=1}^{2} D_{kl}^h(I_H^h(\Pi_H\mathbf{u}) - (\Pi_H\mathbf{u})|_{T_{kl}^h}) \\
&+ 3\sum_{k,l=0}^{I_H-1}\sum_{i,j=1}^{2}\|\epsilon_{ij}(\Pi_H\mathbf{u} - \mathbf{u})\|_{0,T_{kl}^H}^2 + 3\sum_{k,l=0}^{I_h-1}\sum_{i,j=1}^{2}\|\epsilon_{ij}(\mathbf{u} - \Pi_h\mathbf{u})\|_{0,T_{kl}^h}^2 \\
&\le C\sum_{k,l=0}^{I_H-1}\sum_{i=1}^{2}\Big\{\big((\Pi_H u^i)_x(G_{kl}^H)\big)^2 + \big((\Pi_H u_i)_y(G_{kl}^H)\big)^2\Big\} \\
&+ 3|||\Pi_H\mathbf{u} - \mathbf{u}|||_H^2 + 3|||\mathbf{u} - \Pi_h\mathbf{u}|||_h^2.
\end{aligned}
$$

For all $v \in H^2(\Omega)$ and $\varphi = v \circ F_{kl}^H$, from the definition and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
(\Pi_H v)_x(G_{kl}) &= \int_{\hat{T}} \frac{\partial^2}{\partial\xi^2}(\Pi_H\varphi)d\xi d\eta = \int_{\hat{T}} \frac{\partial^2}{\partial\xi^2}\varphi d\xi d\eta \\
&= \int_{T_{kl}^H} \frac{\partial^2}{\partial x^2} v dxdy \le Ch\left(\int_{T_{kl}^H}\left(\frac{\partial^2}{\partial x^2}v\right)^2 dxdy\right)^{1/2}, \\
(\Pi_H v)_y(G_{kl}) &= \int_{\hat{T}} \frac{\partial^2}{\partial\eta^2}(\Pi_H\varphi)d\xi d\eta = \int_{\hat{T}} \frac{\partial^2}{\partial\eta^2}\varphi d\xi d\eta \\
&= \int_{T_{kl}^H} \frac{\partial^2}{\partial y^2} v dxdy \le Ch\left(\int_{T_{kl}^H}\left(\frac{\partial^2}{\partial y^2}v\right)^2 dxdy\right)^{1/2}.
\end{aligned}
$$

From the above inequality and Lemma 3.4, we have

$$
\begin{aligned}
&|||I_H^h(\Pi_H\mathbf{u}) - \Pi_h\mathbf{u}|||_h^2 \\
&\qquad\le Ch^2\sum_{k,l=0}^{I_H-1}\sum_{i=1}^{2}\left(\int_{T_{kl}^H}\left(\frac{\partial^2}{\partial x^2}u^i\right)^2 dxdy + \int_{T_{kl}^H}\left(\frac{\partial^2}{\partial y^2}u^i\right)^2 dxdy\right) \\
&\qquad+ CH^2|\mathbf{u}|_{2,\Omega} + Ch^2|\mathbf{u}|_{2,\Omega} \\
&\qquad\le Ch^2\left(\sum_{k,l=0}^{I_H-1}|\mathbf{u}|_{2,T_{kl}^H}^2 + |\mathbf{u}|_{2,\Omega}^2\right) \le Ch^2|\mathbf{u}|_{2,\Omega}^2.
\end{aligned}
$$

By the same way, we get

$$
\begin{aligned}
&\|I_H^h(\Pi_H\mathbf{u}) - \Pi_h\mathbf{u}\|_{0,\Omega}^2 = \sum_{k,l=0}^{I_h-1} \|I_H^h(\Pi_H\mathbf{u}) - \Pi_h\mathbf{u}\|_{0,T_{kl}^h}^2 \\
&\le 3\sum_{k,l=0}^{I_h-1} \left\{\|I_H^h(\Pi_H\mathbf{u}) - \Pi_H\mathbf{u}\|_{0,T_{kl}^h}^2 + \|\Pi_H\mathbf{u} - \mathbf{u}\|_{0,T_{kl}^h}^2 + \|\Pi_h\mathbf{u} - \mathbf{u}\|_{0,T_{kl}^h}^2\right\} \\
&\le C\sum_{k,l=0}^{I_h-1} (B_{kl}^h(I_H^h(\Pi_H\mathbf{u}) - (\Pi_H\mathbf{u})|_{T_{kl}^h}) + 3\sum_{k,l=0}^{I_H-1} \|\Pi_H\mathbf{u} - \mathbf{u}\|_{0,T_{kl}^H}^2 \\
&\quad + 3\sum_{k,l=0}^{I_h-1} \|\Pi_h\mathbf{u} - \mathbf{u}\|_{0,T_{kl}^h}^2 \\
&\le Ch^2\sum_{k,l=0}^{I_h-1}\sum_{i=1}^{2}\left\{\big((\Pi_H u^i)_x(G_{kl}^H)\big)^2 + \big((\Pi_H u_i)(G_{kl}^H)_y\big)^2\right\} \\
&+ \|\Pi_H\mathbf{u} - \mathbf{u}\|_{0,\Omega}^2 + \|\mathbf{u} - \Pi_h\mathbf{u}\|_{0,\Omega}^2 \\
&\le ch^4|\mathbf{u}|_{2,\Omega}^2. \quad \square
\end{aligned}
$$

Theorem 4.5. *There exists a positive constant C, independent of h, such that*

$$\|(I - I_H^h P_h^H)\mathbf{v}_h\|_{0,\Omega} \le Ch|||\mathbf{v}_h|||_h \quad \textit{for all } \mathbf{v}_h \in V_h. \tag{4.8}$$

Proof. Given $\mathbf{v}_h \in V_h$, let $\mathbf{w} \in V$ be the solution of (3.10) where $\mathbf{f} = (I - I_H^h P_h^H)\mathbf{v}_h$. Then we have $\|\mathbf{w}\|_{2,\Omega} \le \|\mathbf{f}\|_{0,\Omega}$.

Let

$$
\begin{aligned}
a_h(\mathbf{w}_h, \mathbf{u}_h) &= F(\mathbf{u}_h) \qquad \forall \mathbf{u}_h \in V_h, \\
a_H(\mathbf{w}_H, \mathbf{u}_H) &= F(\mathbf{u}_H) \qquad \forall \mathbf{u}_H \in V_H.
\end{aligned} \tag{4.9}
$$

From Theorem 3.5, we have

$$|||\mathbf{w} - \mathbf{w}_h|||_h \le Ch|\mathbf{w}|_{2,\Omega} \quad \text{and} \quad |||\mathbf{w} - \mathbf{w}_H|||_H \le Ch|\mathbf{w}|_{2,\Omega}. \tag{4.10}$$

Let, for all $\mathbf{v}, \mathbf{u} \in (H^1(T))^2$,

$$a_T(\mathbf{v}, \mathbf{u}) = \int_T \sum_{i,j=1}^{2} \sigma_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{u})dxdy$$

Then we have

$$\|\mathbf{f}\|_{0,\Omega}^2 = \left\{\|\mathbf{f}\|_{0,\Omega}^2 - \sum_{T\in\tau_h} a_T(\mathbf{w}, \mathbf{v}_h - P_h^H\mathbf{v}_h)\right\} + \sum_{T\in\tau_h} a_T(\mathbf{w}, \mathbf{v}_h - P_h^H\mathbf{v}_h). \tag{4.11}$$

Since $P_h^H\mathbf{v}_h \in V_H$, we can rewrite the first term on the right-hand side of (4.11) as follows:

$$
\begin{aligned}
&\|\mathbf{f}\|_{0,\Omega}^2 - \sum_{T\in\tau_h} a_T(\mathbf{w}, \mathbf{v}_h - P_h^H\mathbf{v}_h) \\
&= (\mathbf{f}, \mathbf{v}_h - I_H^h P_h^H\mathbf{v}_h)_{(L^2(\Omega))^2} - \sum_{T\in\tau_h} a_T(\mathbf{w}, \mathbf{v}_h) + \sum_{T\in\tau_h} a_T(\mathbf{w}, P_h^H\mathbf{v}_h) \\
&= a_h(\mathbf{w}_h, \mathbf{v}_h) - (\mathbf{f}, I_H^h P_h^H\mathbf{v}_h)_{(L^2(\Omega))^2} - a_h(\mathbf{w}, \mathbf{v}_h) + a_H(\mathbf{w}, P_h^H\mathbf{v}_h) \\
&= a_h(\mathbf{w}_h - \mathbf{w}, \mathbf{v}_h) - (\mathbf{f}, P_h^H\mathbf{v}_h)_{(L^2(\Omega))^2} \\
&+ (\mathbf{f}, (I - I_H^h)P_h^H\mathbf{v}_h)_{(L^2(\Omega))^2} + a_H(\mathbf{w}, P_h^H\mathbf{v}_h) \\
&= a_h(\mathbf{w}_h - \mathbf{w}, \mathbf{v}_h) + a_H(\mathbf{w} - \mathbf{w}_H, P_h^H\mathbf{v}_h) + (\mathbf{f}, (I - I_H^h)P_h^H\mathbf{v}_h)_{(L^2(\Omega))^2}.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, (4.10), Lemma 4.1, Corollary 4.3 and Theorem 3.5, we have

$$
\begin{aligned}
\|\mathbf{f}\|_{0,\Omega}^2 - \sum_{T\in\tau_h} a_T(\mathbf{w}, \mathbf{v}_h - P_h^H\mathbf{v}_h) &\le |||\mathbf{w} - \mathbf{w}_h|||_h |||\mathbf{v}_h|||_h \\
&+ |||\mathbf{w} - \mathbf{w}_H|||_H |||P_h^H\mathbf{v}_h|||_H + \|\mathbf{f}\|_{0,\Omega}\|(I - I_H^h)P_h^H\mathbf{v}_h\|_{0,\Omega} \\
&\le Ch\|\mathbf{f}\|_{0,\Omega}|||\mathbf{v}_h|||_h.
\end{aligned} \tag{4.12}
$$

The remaining term on the right-hand side of (4.11) can be rewritten as follows:

$$
\begin{aligned}
\sum_{T\in\tau_h} a_T(\mathbf{w}, \mathbf{v}_h - P_h^H\mathbf{v}_h) &= a_h(\mathbf{w}, \mathbf{v}_h) - a_H(\mathbf{w}, P_h^H\mathbf{v}_h) \\
&= a_h(\mathbf{w} - \Pi_h\mathbf{w}, \mathbf{v}_h) + a_h(\Pi_h\mathbf{w} - I_H^h\Pi_H\mathbf{w}, \mathbf{v}_h) \\
&+ a_h(I_H^h\Pi_H\mathbf{w}, \mathbf{v}_h) - a_H(\mathbf{w}, P_h^H\mathbf{v}_h) \\
&= a_h(\mathbf{w} - \Pi_h\mathbf{w}, \mathbf{v}_h) + a_h(\Pi_h\mathbf{w} - I_H^h\Pi_H\mathbf{w}, \mathbf{v}_h) + a_H(\Pi_H\mathbf{w} - \mathbf{w}, P_h^H\mathbf{v}_h).
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, Lemma 4.4, Lemma 3.4 and Corollary 4.3, we have

$$
\begin{aligned}
\sum_{T\in\tau_h} a_T(\mathbf{w}, \mathbf{v}_h - P_h^H\mathbf{v}_h) &\le |||\mathbf{w} - \Pi_h\mathbf{w}|||_h |||\mathbf{v}_h|||_h \\
&+ |||\Pi_h\mathbf{w} - I_H^h\Pi_H\mathbf{w}|||_h |||\mathbf{v}_h|||_h + |||\Pi_H\mathbf{w} - \mathbf{w}|||_H |||P_h^H\mathbf{v}_h|||_H \\
&\le Ch\|\mathbf{w}\|_{2,\Omega}|||\mathbf{v}_h|||_h \le Ch\|\mathbf{f}\|_{0,\Omega}|||\mathbf{v}_h|||_h.
\end{aligned} \tag{4.13}
$$

Inequality (4.8) follows from (4.11), (4.12) and (4.13). □

5. Intergrid transfer operator J_H^h

In this section, we define other intergrid transfer operator $J_H^h : V_H \to V_h$. The notation are the same in section 4.

For $u_H \in X_H$, we define $u_h = J_H^h u_H \in X_h$ as follows:

$$u_h(A^h_{2k2l}) = u_H(A^H_{kl}),$$

$$u_h(A^h_{(2k+1)2l}) = \frac{u_H(A^H_{kl}) + u_H(A^H_{(k+1)l})}{2} - \frac{(u_{H,x})(G^H_{kl}) + u_{H,x}(G^H_{k(l-1)})}{16},$$

$$u_h(A^h_{2k(2l+1)}) = \frac{u_H(A^H_{kl}) + u_H(A^H_{k(l+1)})}{2} - \frac{u_{H,y}(G^H_{kl}) + u_{H,y}(G^H_{(k-1)l})}{16},$$

$$u_h(A^h_{(2k+1)(2l+1)}) = \frac{u_H(A^H_{kl}) + u_H(A^H_{(k+1)(l+1)}) + u_H(A^H_{(k+1)l}) + u_H(A^H_{k(l+1)})}{4} - \frac{u_{H,x}(G^H_{kl}) + u_{H,y}(G^H_{kl})}{8},$$

$$u_{h,x}(G^h_{2k,2l}) = u_{h,x}(G^h_{2k+1,2l}) = u_{h,x}(G^h_{2k,2l+1}) = u_{h,x}(G^h_{2k+1,2l+1}) = \frac{u_{H,x}(G^H_{k,l})}{4},$$

$$u_{h,y}(G^h_{2k,2l}) = u_{h,y}(G^h_{2k+1,2l}) = u_{h,y}(G^h_{2k,2l+1}) = u_{h,x}(G^h_{2k+1,2l+1}) = \frac{u_{H,y}(G^H_{k,l})}{4}.$$

We define $J_H^h : V_H \to V_h$ as follows: for $\mathbf{v}_H = (v_1, v_2) \in V_H$,

$$J_H^h \mathbf{v}_H = (J_H^h v_1, J_H^h v_2).$$

Lemma 5.1. *There exists a constant $C > 0$, independent of h, such that*

$$\|\mathbf{v}_H - J_H^h \mathbf{v}_H\|_{0,\Omega} \le Ch|||\mathbf{v}_H|||_H \quad \text{for all } \mathbf{v}_H \in V_H. \tag{5.1}$$

Proof. Since $\mathbf{v}_H|_{T^h_{kl}} \in (P_2(T^h_{kl}))^2$, we have

$$\|\mathbf{v}_H - J_H^h \mathbf{v}_H\|^2_{0,\Omega} = \sum_{k,l=0}^{I_h-1} \|\mathbf{v}_H - J_H^h \mathbf{v}_H\|^2_{0,T^h_{kl}} \le C \sum_{k,l=0}^{I_h-1} B^h_{kl}(\mathbf{v}_H|_{T^h_{kl}} - J_H^h \mathbf{v}_H).$$

From (3.3) and the argument in Lemma 4.1, we have, for $0 \le k, l \le I_H - 1$,

$$\begin{aligned} B^h_{2k2l}(\mathbf{v}_H|_{T^h_{2k2l}} - J_H^h \mathbf{v}_H) &= h^2 \sum_{i=1}^{2} \left\{ \left(\frac{v^i_x(G^H_{kl}) - v^i_x(G^H_{k(l-1)})}{16} \right)^2 \right. \\ &\quad \left. + \left(\frac{v^i_y(G^H_{kl}) - v^i_y(G^H_{(k-1)l})}{16} \right)^2 + \left(\frac{3}{4} v^i_x(G^H_{kl}) \right)^2 + \left(\frac{3}{4} v^i_y(G^H_{kl}) \right) \right\} \\ &\le Ch^2 \{ D^H_{kl}(\mathbf{v}_H) + D^H_{k(l-1)}(\mathbf{v}_H) + D^H_{(k-1)l}(\mathbf{v}_H) \}. \end{aligned}$$

By the same argument, we have

$$\begin{aligned} &B^h_{(2k+1)2l}(\mathbf{v}_H|_{T^h_{(2k+1)2l}} - J_H^h \mathbf{v}_H) \\ &\qquad \le Ch^2 \{ D^H_{kl}(\mathbf{v}_H) + D^H_{k(l-1)}(\mathbf{v}_H) + D^H_{(k+1)l}(\mathbf{v}_H) \}, \\ &B^h_{2k(2l+1)}(\mathbf{v}_H|_{T^h_{2k(2l+1)}} - J_H^h \mathbf{v}_H) \\ &\qquad \le Ch^2 \{ D^H_{kl}(\mathbf{v}_H) + D^H_{k(l+1)}(\mathbf{v}_H) + D^H_{(k-1)l}(\mathbf{v}_H) \}, \\ &B^h_{(2k+1)(2l+1)}(\mathbf{v}_H|_{T^h_{(2k+1)(2l+1)}} - J_H^h \mathbf{v}_H) \\ &\qquad \le Ch^2 \{ D^H_{kl}(\mathbf{v}_H) + D^H_{(k+1)l}(\mathbf{v}_H) + D^H_{k(l+1)}(\mathbf{v}_H) \}. \end{aligned}$$

From the above and (3.16), we have (5.1). $\square$

Lemma 5.2. *There exists a constant $C > 0$, independent of h, such that*

$$|||J_H^h \mathbf{v}_H|||_h \leq C|||\mathbf{v}_H|||_H \quad \text{for all } \mathbf{v}_H \in V_H. \tag{5.2}$$

Proof. Following the arguments in the proof of Lemma 4.2, we only need to show that

$$\sum_{k,l=0}^{I_h-1} D_{kl}^h(\mathbf{v}_H|_{T_{kl}^h} - J_H^h\mathbf{v}_H) \leq C \sum_{k,l=0}^{I_H-1} D_{kl}^H(\mathbf{v}_H) \tag{5.3}$$

By using the arguments in the proof of Lemma 5.1,

$$\begin{aligned}
D_{2k2l}^h&(\mathbf{v}_H|_{T_{2k2l}^h} - I_H^h\mathbf{v}_H)\\
&= \left(\frac{v_y^2(G_{kl}^H) - v_y^2(G_{(k-1)l})}{16}\right)^2 + \left(\frac{v_x^2(G_{kl}^H) - v_x^2(G_{k(l-1)})}{16}\right)^2\\
&+ \left(\frac{v_x^1(G_{kl}^H) - v_x^1(G_{k(l-1)})}{16}\right)^2 + \left(\frac{v_y^1(G_{kl}^H) - v_y^1(G_{(k-1)l})}{16}\right)^2\\
&+ \Bigg(\frac{1}{16}\{v_x^2(G_{kl}^H) - v_x^2(G_{k(l-1)}^H) - v_y^2(G_{kl}^H) + v_y^2(G_{(k-1)l}^H)\\
&\qquad + v_y^1(G_{kl}^H) - v_y^1(G_{(k-1)l}^H) - v_x^1(G_{kl}^H) + v_x^1(G_{k(l-1)}^H)\}\Bigg)^2\\
&+ \left(\frac{3}{4}v_x^1(G_{kl}^H)\right)^2 + \left(\frac{3}{4}v_x^2(G_{kl}^H)\right)^2 + \left(\frac{3}{4}v_y^1(G_{kl}^H)\right)^2 + \left(\frac{3}{4}v_y^2(G_{kl}^H)\right)^2\\
&\leq C\{D_{kl}^H(\mathbf{v}_H) + D_{k(l-1)}^H(\mathbf{v}_H) + D_{(k-1)l}^H(\mathbf{v}_H)\}.
\end{aligned}$$

By the same argument, we have

$$\begin{aligned}
D_{(2k+1)2l}^h&(\mathbf{v}_H|_{T_{(2k+1)2l}^h} - J_H^h\mathbf{v}_H)\\
&\leq C\{D_{kl}^H(\mathbf{v}_H) + D_{k(l+1)}^H(\mathbf{v}_H) + D_{(k-1)l}^H(\mathbf{v}_H)\},\\
D_{2k(2l+1)}^h&(\mathbf{v}_H|_{T_{2k(2l+1)}^h} - J_H^h\mathbf{v}_H)\\
&\leq C\{D_{kl}^H(\mathbf{v}_H) + D_{k(l-1)}^H(\mathbf{v}_H) + D_{(k+1)l}^H(\mathbf{v}_H)\},\\
D_{(2k+1)(2l+1)}^h&(\mathbf{v}_H|_{T_{(2k+1)(2l+1)}^h} - J_H^h\mathbf{v}_H)\\
&\leq C\{D_{kl}^H(\mathbf{v}_H) + D_{k(l+1)}^H(\mathbf{v}_H) + D_{(k+1)l}^H(\mathbf{v}_H)\}.
\end{aligned}$$

From the above and (3.16), we have (5.3). $\square$

Corollary 5.3. *Let $Q_h^H : V_h \to V_H$ be the operator such that*

$$a_h(\mathbf{u}_h, J_H^h\mathbf{v}_H) = a_H(Q_h^H\mathbf{u}_h, \mathbf{v}_H) \quad \text{for all } \mathbf{v}_H \in V_H, \mathbf{u}_h \in V_h.$$

Then we have

$$|||Q_h^H\mathbf{u}_h|||_H \leq C|||\mathbf{u}_h|||_h \quad \text{for all } \mathbf{u}_h \in V_h. \tag{5.4}$$

Lemma 5.4. *There exists a positive constant C, independent of h, such that*

$$|||J_H^h(\Pi_H \mathbf{u}) - \Pi_h \mathbf{u}|||_h \leq Ch|\mathbf{u}|_{2,\Omega} \tag{5.5}$$

and

$$\|J_H^h(\Pi_H \mathbf{u}) - \Pi_h \mathbf{u}\|_{0,\Omega} \leq Ch^2|\mathbf{u}|_{2,\Omega} \tag{5.6}$$

for all $\mathbf{u} \in (H^2(\Omega))^2 \cap V$.

Proof. By using arguments in proof of Lemma 5.1, Lemma 5.2 and Lemma 4.4, we have, for all $\mathbf{u} \in (H^2(\Omega))^2 \cap V$,

$$
\begin{aligned}
&|||J_H^h(\Pi_H \mathbf{u}) - \Pi_h \mathbf{u}|||_h^2 = \sum_{k,l=0}^{I_h-1} \sum_{i,j=1}^{2} \|\epsilon_{ij}(J_H^h(\Pi_H \mathbf{u}) - \Pi_h \mathbf{u})\|_{0,T_{kl}^h}^2 \\
&\leq C \sum_{k,l=0}^{I_h-1} \sum_{i=1}^{2} D_{kl}^h(J_H^h(\Pi_H \mathbf{u}) - (\Pi_H \mathbf{u})|_{T_{kl}^h}) \\
&\quad + 3 \sum_{k,l=0}^{I_H-1} \sum_{i,j=1}^{2} \|\epsilon_{ij}(\Pi_H \mathbf{u} - \mathbf{u})\|_{0,T_{kl}^H}^2 + 3 \sum_{k,l=0}^{I_h-1} \sum_{i,j=1}^{2} \|\epsilon_{ij}(\mathbf{u} - \Pi_h \mathbf{u})\|_{0,T_{kl}^h}^2 \\
&\leq C \sum_{k,l=0}^{I_H-1} \sum_{i=1}^{2} \left\{ \left((\Pi_H u^i)_x(G_{kl}^H)\right)^2 + \left((\Pi_H u^i)_y(G_{kl}^H)\right)^2 \right\} \\
&\quad + 3|||\Pi_H \mathbf{u} - \mathbf{u}|||_H^2 + 3|||\mathbf{u} - \Pi_h \mathbf{u}|||_h^2 \\
&\leq Ch^2 \sum_{k,l=0}^{I_H-1} \sum_{i=1}^{2} \left(\int_{T_{kl}^H} \left(\frac{\partial^2}{\partial x^2} u^i \right)^2 dxdy + \int_{T_{kl}^H} \left(\frac{\partial^2}{\partial y^2} u^i \right)^2 dxdy \right) \\
&\quad + CH^2|\mathbf{u}|_{2,\Omega}^2 + Ch^2|\mathbf{u}|_{2,\Omega}^2 \\
&\leq Ch^2 \sum_{k,l=0}^{I_H-1} |\mathbf{u}|_{2,T_{kl}}^2 + |\mathbf{u}|_{2,\Omega}^2 \leq Ch^2|\mathbf{u}|_{2,\Omega}^2.
\end{aligned}
$$

By the same way, we get

$$
\begin{aligned}
&\|J_H^h(\Pi_H \mathbf{u}) - \Pi_h \mathbf{u}\|_{0,\Omega}^2 = \sum_{k,l=0}^{I_h-1} \|J_H^h(\Pi_H \mathbf{u}) - \Pi_h \mathbf{u}\|_{0,T_{kl}^h}^2 \\
&\leq C \sum_{k,l=0}^{I_h-1} (B_{kl}^h(J_H^h(\Pi_H \mathbf{u}) - (\Pi_H \mathbf{u})|_{T_{kl}^h}) + 3 \sum_{k,l=0}^{I_H-1} \|\Pi_H \mathbf{u} - \mathbf{u}\|_{0,T_{kl}^H}^2 \\
&\quad + 3 \sum_{k,l=0}^{I_h-1} \|\Pi_h \mathbf{u} - \mathbf{u}\|_{0,T_{kl}^h}^2 \\
&\leq Ch^2 \sum_{k,l=0}^{I_H-1} \sum_{i=1}^{2} \left\{ \left((\Pi_H u^i)_x(G_{kl}^H)\right)^2 + \left((\Pi_H u_i)_y(G_{kl}^H)\right)^2 \right\} \\
&\quad + 3\|\Pi_H \mathbf{u} - \mathbf{u}\|_{0,\Omega}^2 + 3\|\mathbf{u} - \Pi_h \mathbf{u}\|_{0,\Omega}^2 \\
&\leq ch^4|\mathbf{u}|_{2,\Omega}^2. \quad \square
\end{aligned}
$$

Theorem 5.5. *There exists a positive constant C such that*

$$\|(I - J_H^h Q_h^H)\mathbf{v}_h\|_{0,\Omega} \le Ch|||\mathbf{v}_h|||_h \quad \textit{for all } \mathbf{v}_h \in V_h. \tag{5.7}$$

Proof. The proof is same as the proof of Theorem 4.5. □

6. Multigrid Algorithm

In this section, we describe the multigrid algorithm, its convergence proof and numerical results. To define the multigrid algorithm, we let $h_k = \frac{1}{2^k}$ for $k = 1, \ldots, K$ and denote V_{h_k} and $a_{h_k}(\cdot,\cdot)$ in section 3 as V_k and $a_k(\cdot,\cdot)$, respectively. And we also define $(\cdot,\cdot)_k$ as in section 3. Here we denote

$$|||\cdot|||_k = a_k(\cdot,\cdot)^{1/2} \quad \text{and} \quad ||\cdot||_{0,k} = (\cdot,\cdot)_k^{1/2}.$$

Then $||\cdot||_{0,k}$ is a discretize norm which is equivalent to L^2-norm on V_k.

The symmertric positive definite operator $A_k : V_k \to V_k$ is defined by

$$(A_k\mathbf{v}, \mathbf{w})_k = a_k(\mathbf{v}, \mathbf{w}) \quad \forall \mathbf{v}, \mathbf{w} \in V_k. \tag{6.1}$$

By a standard inverse estimate,

$$a_k(\mathbf{v}, \mathbf{v}) \le Ch_k^{-2}(\mathbf{v}, \mathbf{v})_{(L^2(\Omega))^2} \quad \forall \mathbf{v} \in V_k. \tag{6.2}$$

Then (6.1) and (6.2) imply that the largest eigenvalue λ_k of A_k is bounded by

$$\lambda_k \le Ch_k^{-2}. \tag{6.3}$$

Here we define coarse-to-fine intergrid transfer operator I_{k-1}^k by either $I_{h_{k-1}}^{h_k}$ or $J_{h_{k-1}}^{h_k}$ in previous sections. The fine-to-coarse intergrid transfer operator $I_k^{k-1} : V_k \to V_{k-1}$ is defined by

$$(\mathbf{v}, I_k^{k-1}\mathbf{w})_{k-1} = (I_{k-1}^k\mathbf{v}, \mathbf{w})_k \quad \forall \mathbf{v} \in V_{k-1}, \mathbf{w} \in V_k, \tag{6.4}$$

and P_k^{k-1} be either $P_{h_k}^{h_{k-1}}$ or $Q_{h_k}^{h_{k-1}}$.

Also, we require a sequence of linear smoothing operators $R_k : V_k \to V_k$ for $k = 2, \ldots, K$. We shall always take $R_1 = A_1^{-1}$. Let R_k^T denote the adjoint of R_k with respect to the $(\cdot,\cdot)_k$ inner product and define

$$R_k^{(l)} = \begin{cases} R_k & \text{if } l \text{ is odd,} \\ R_k^T & \text{if } l \text{ is even.} \end{cases}$$

We define the multigrid operator $B_k : V_k \to V_k$ in terms of an iterative process as follows.

Multigrid Algorithm. Set $B_1 = A_1^{-1}$. Assume that B_{k-1} has been defined and define $B_k\mathbf{g}$ for $\mathbf{g} \in V_k$ as follows;

(1) Set $\mathbf{v}^0 = 0$ and $\mathbf{q}^0 = 0$.

(2) Define $\mathbf{v}^i$ for $i = 1, 2, \ldots, m(k)$ by

$$\mathbf{v}^i = \mathbf{v}^{i-1} + R_k^{(i+m(k))}(\mathbf{g} - A_k\mathbf{v}^{i-1}). \tag{6.5}$$

(3) Define $\mathbf{w}^{m(k)} = \mathbf{v}^{m(k)} + I_{k-1}^k \mathbf{q}^p$, where $\mathbf{q}^i$ for $i = 1, \ldots, p$ is defined by

$$\mathbf{q}^i = \mathbf{q}^{i-1} + B_{k-1}[I_k^{k-1}(\mathbf{g} - A_k\mathbf{v}^{m(k)}) - A_{k-1}\mathbf{q}^{i-1}]. \tag{6.6}$$

(4) Define $\mathbf{w}^i$ for $i = m(k)+1, \ldots, 2m(k)$ by

$$\mathbf{w}^i = \mathbf{w}^{i-1} + R_k^{(i+m(k))}(\mathbf{g} - A_k\mathbf{w}^{i-1}). \tag{6.7}$$

(5) Set $B_k\mathbf{g} = \mathbf{w}^{2m(k)}$.

In the algorithm, $m(k)$ is the number of pre- and post-smoothing iterations and can vary as a function of k. If $p = 1$, we have a $\mathcal{V}$-cycle multigrid algorithm. If $p = 2$, we have a $\mathcal{W}$-cycle algorithm. A variable $\mathcal{V}$-cycle algorithm is one in which the number of smoothings $m(k)$ increase exponentially as k decreases (i.e., $p = 1$ and $m(k) = m2^{K-k}$ for fixed integer m). The smoothings are alternated following [5] and are put together so that the resulting multigrid preconditioner B_k is symmertric in the $(\cdot,\cdot)_k$ inner product for each k.

Here we show the regularity and approximation property.

Proposition 6.1. *There exists a positive constant C_A such that*

$$|a_k((I - I_{k-1}^k P_k^{k-1})\mathbf{v}, \mathbf{v})| \le C_A \left(\frac{(A_k\mathbf{v}, A_k\mathbf{v})_k}{\lambda_k}\right)^{1/2} a_k(\mathbf{v}, \mathbf{v})^{1/2}, \quad \forall \mathbf{v} \in V_k, \tag{6.8}$$

for $k = 1, \ldots, j$.

Proof. From (4.8) or (5.6) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} |a_k((I - I_{k-1}^k P_k^{k-1})\mathbf{v}, \mathbf{v})| &= |((I - I_{k-1}^k P_k^{k-1})\mathbf{v}, A_k\mathbf{v})_k| \\ &\le ||(I - I_{k-1}^k P_k^{k-1})\mathbf{v}||_{0,k}||A_k\mathbf{v}||_{0,k} \\ &\le Ch_k|||\mathbf{v}|||_k \cdot ||A_k\mathbf{v}||_{0,k}. \end{aligned}$$

From (6.3), we get (6.8). □

Now we can apply the theory in [5] to have following results.

Theorem 6.2. *Define B_k by $p = 2$ and $m(k) = m$ for all k in the multigrid algorithm. Then there exists $C > 0$, independent on k, such that for m large enough,*

$$\delta_k \le \delta \equiv \frac{C}{C + m^{1/4}}. \tag{6.9}$$

where

$$|a_k((I - B_kA_k)\mathbf{v}, \mathbf{v})| \le \delta_k a_k(\mathbf{v}, \mathbf{v}) \quad \forall \mathbf{v} \in V_k. \tag{6.10}$$

The condition number of B_kA_k for the preconditioner B_k is $K(B_kA_k) = \eta_1/\eta_0$ where η_0 and η_1 satisfy

$$\eta_0 a_k(\mathbf{v}, \mathbf{v}) \le a_k(B_kA_k\mathbf{v}, \mathbf{v}) \le \eta_1 a_k(\mathbf{v}, \mathbf{v}) \quad \forall \mathbf{v} \in V_k. \tag{6.11}$$

Theorem 6.3. *Define B_k by $p = 1$ and $m(k) = m2^{K-k}$ for $k = 1, \ldots, K$ in the multigrid algorithm. Then the constants η_0 and η_1 in (6.11) satisfy*

$$\eta_0 \geq \frac{m(k)^{1/4}}{C + m(k)^{1/4}} \quad \textit{and} \quad \eta_1 \leq \frac{C + m(k)^{1/4}}{m(k)^{1/4}}.$$

Remark 6.4. *The constants C in Theorem 6.2 and 6.3 depend on the Lamé constant λ because the constants C in Theorem 4.5 and 5.5 depend on the Lamé constant.*

From Theorem 6.2 and Theorem 6.3, we have an optimal convergence property of the $\mathcal{W}$-cycle and a uniform condition number estimate for the variable $\mathcal{V}$-cycle preconditioner if the Lamé constant λ is small. If the Lamé constant is big then the $\mathcal{W}$-cycly multigrid algorithm needs large m and the variable $\mathcal{V}$-cycle multigrid preconditioner have a large condition number.

Next we give the numerical results for various λ and fixed $\mu = 1.0$ for variable $\mathcal{V}$-cycle and $\mathcal{W}$-cycle multigrid algorithms with Gauss-Seidel smoother.

$K\backslash\lambda$	0.0	0.5	1.0	2.0	10.0	50.0
2	1.9029	1.8774	1.9910	2.3871	7.5843	52.7875
3	1.9055	1.8775	1.9818	2.3439	6.3980	41.3589
4	1.9198	1.8863	1.9881	2.3457	6.1564	34.4257
5	1.9252	1.8890	1.9894	2.3464	6.1591	30.4515

Table I. Condition Number of $B_k A_k$ for variable $\mathcal{V}$-cycle with I_H^h and $m(k) = 2^{K-k}$

$K\backslash\lambda$	0.0	0.5	1.0	2.0	10.0	50.0
2	1.9508	1.9807	2.1369	2.5893	7.4635	61.9335
3	1.9908	1.9984	2.1402	2.6012	7.7160	66.6534
4	2.0416	2.0519	2.2036	2.6844	8.1370	70.7702
5	2.0611	2.0540	2.2318	2.7301	8.2283	72.6752

Table II. Condition Number of $B_k A_k$ for variable $\mathcal{V}$-cycle with J_H^h and $m(k) = 2^{K-k}$

$K\backslash\lambda$	0.0	0.5	1.0	2.0	10.0	50.0
2	0.45734	0.44849	0.47450	0.54631	0.81387	divergent
3	0.46150	0.45316	0.47852	0.55097	0.81991	divergent
4	0.46161	0.45335	0.47868	0.55037	0.81952	divergent
5	0.46161	0.45335	0.47868	0.55040	0.81953	divergent

Table III. Convergence factor δ_k in (6.10) with I_H^h and $m(k) = 1$

$K\backslash\lambda$	0.0	1.0	2.0	10.0	50.0	100.0
2	0.09083	0.08934	0.10976	0.45255	0.83602	0.91310
3	0.07339	0.07810	0.11216	0.46232	0.84035	0.91554
4	0.07134	0.07782	0.11313	0.46271	0.84088	0.91589
5	0.07134	0.07783	0.11213	0.46274	0.84091	0.91591

Table IV. Convergence factor δ_k in (6.10) with I_H^h and $m(k) = 4$

$K\backslash\lambda$	0.0	0.5	1.0	2.0	10.0	50.0
2	0.43707	0.42467	0.44722	0.51554	0.79072	divergent
3	0.43420	0.42127	0.44396	0.51450	0.79527	divergent
4	0.43425	0.42128	0.44401	0.51462	0.79562	divergent
5	0.43426	0.42128	0.44402	0.51462	0.79563	divergent

Table V. Convergence factor δ_k in (6.10) with J_H^h and $m(k) = 1$

$K\backslash\lambda$	0.0	1.0	2.0	10.0	50.0	100.0
2	0.08121	0.08560	0.09496	0.42649	0.81839	0.90164
3	0.05837	0.05221	0.08590	0.44118	0.82373	0.90425
4	0.05800	0.05204	0.08536	0.44130	0.82347	0.90416
5	0.05799	0.05494	0.08536	0.44124	0.82347	0.90416

Table VI. Convergence factor δ_k in (6.10) with J_H^h and $m(k) = 4$

References

1. R.E. Bank and C.C. Douglas, *Sharp estimates for multigrid rates of convergence with general smoothing and acceleration*, SIAM J. Numer. Anal. **22** (1985), 617–633.
2. R.E. Bank and T. Dupont, *An optimal order process for solving finite element equations*, Math. Comp. **36** (1981), 35–51.
3. D. Braess and W. Hackbush, *A new convergence proof for the multigrid method including the V-cycle*, SIAM J. Numer. Anal. **20** (1983), 967–975.
4. J.H. Bramble and J. E. Pasciak, *New convergence estimates for multigrid algorithms*, Math. Comp. **49** (1987), 311–329.
5. J.H. Bramble, J. E. Pasciak, and J. Xu, *The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms*, Math. Comp. **56** (1991), 1–34.
6. S.C. Brenner, *An optimal-order multigrid method for $P1$ nonconforming finite elements*, Math. Comp. **52** (1988), 1–15.
7. S.C. Brenner, *A nonconforming multigrid method for the stationary Stokes Equations*, Math. Comp. **55** (1990), 411–437.
8. Z. Chen and D.Y. Kwak, *The analysis of multigrid algorithms for nonconforming and mixed method for second order elliptic problems*, Preprint.
9. P.G. Ciarlet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, New York, and Oxford, 1978.
10. W. Hackbush, *Multi-grid methods and applications*, Springer-Verlag, NewYork, 1985.
11. S. McCormick (Ed.), *Multigrid methods*, SIAM, Philadelphia, PA, 1987.
12. P. Lesaint, *On the convergence of Wilson's nonconforming element for solving the elastic problems*, Computer Methods in Applied Mechanics and Engineering **7** (1976), 1–16.

Department of Mathematics,
Korea Advanced Institute of Science and Technology
Taejon, Korea 305-701
E-mail address: kks002@math.kaist.ac.kr

Department of Mathematics,
Korea Advanced Institute of Science and Technology
Taejon, Korea 305-701
E-mail address: dykwak@math.kaist.ac.kr