An Explicit Solution of the Cubic Spline Interpolation for Polynomials

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Abstract

An algorithm for computing the cubic spline interpolation coefficients for polynomials is presented in this paper. The matrix equation involved is solved analytically so that numerical inversion of the coefficient matrix is not required. For $f(t) = t^m$, a set of constants along with the degree of polynomial m are used to compute the coefficients so that they satisfy the interpolation constraints but not necessarily the derivative constraints. Then, another matrix equation is solved analytically to take care of the derivative constraints. The results are combined linearly to obtain the unique solution of the original matrix equation. This algorithm is tested and verified numerically for various examples.

1 Introduction

There are many recent reports of studies on representing functions by fuzzy systems or neural networks[1,2,3]. Futher works, however, are yet to follow in order to present the characteristics of the particular function in its representation. For the problem of representing a function using its cubic spline interpolation either by a fuzzy system or a neural network, the numerical values of the coefficients obtained from solving the matrix equation do not give any insights on how the coefficients are related to the characteristics of the function.

In this paper, we present an algorithm by which one can compute the coefficients of the cubic spline interpolation for polynomials of the form $f(t) = t^m$

without solving the matrix equation. For an arbitrary polynomial, the coefficients can then be computed by a linear combination of those for $f(t) = t^m$. A survey of literature indicates that related studies on cubic splines are focused on the inverse of the coefficient matrix[4], or on monotone functions[5,6], but none seem to consider the same problem discussed here.

In the following, we assume $f(t) = t^m$ and consider the cubic spline interpolation of $f(t)$ in the interval [-1, 1]. Let $t_j = -1 + jh$, j=-3,-2,-1,0, ... ,2n+4 with $h = \frac{1}{n}$ and let

$$B_i(t) = \frac{1}{6}\begin{cases} (t - t_{i-2})^3, & \text{if } t \in [t_{i-2},\ t_{i-1}] \\ h^3 + 3h^2(t - t_{i-1}) + 3h(t - t_{i-1})^2 - 3(t - t_{i-1})^3, & \text{if } t \in [t_{i-1},\ t_i] \\ h^3 + 3h^2(t_{i+1} - t) + 3h(t_{i+1} - t)^2 - 3(t_{i+1} - t)^3, & \text{if } t \in [t_i,\ t_{i+1}] \\ (t_{i+2} - t)^3, & \text{if } t \in [t_{i+1},\ t_{i+2}] \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

for i=-1,0,1, ... ,2n+1. Consider the problem of finding the coefficients c_j's in

$$S(t) = c_{-1}B_{-1}(t) + c_0 B_0(t) + c_1 B_1(t) + ...c_{2n+1}B_{2n+1}(t).$$

such that $S(t_i) = f(t_i)$, $0 \le i \le 2n$, $S'(t_0) = \alpha$, and $S'(t_{2n}) = \beta$, where B_j's are cubic B-spline functions and α, β are arbitrary real numbers. Using the properties of cubic B-splines [7,p80], the problem can be written as solving the matrix equation

$$Ac = b \tag{2}$$

where $c = (c_{-1}, c_0, c_1, ...c_{2n+1})^T$, $b = (\alpha,\ f(t_0), ... f(t_{2n}), \beta)^T$ and

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & . & . \\ 1 & 4 & 1 & 0 & 0 & . & . \\ 0 & 1 & 4 & 1 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \tag{3}$$

Recall that we must have use $\alpha = \frac{h}{3} f'(t_0)$ and $\beta = \frac{h}{3} f'(t_{2n})$ for $O(h^4)$ accuracy.

2 Interpolation Without Derivative Constraints

In this section, we prove that for polynomials of the form $f(t) = t^m$, the interpolation coefficients c_j, $j = -1, 0, \dots, 2n+1$ can be computed directly without solving the matrix equation (2), even though the constraints $\alpha = \frac{h}{3} f'(t_0)$ and $\beta = \frac{h}{3} f'(t_{2n})$ are satisfied only when $m \le 4$. These constraints will be considered in the next section.

Lemma 1. Let m be a positive integer and let λ_j be defined successively by $\lambda_0 = 1, \lambda_1 = -1,$ and

$$\lambda_{2k} = 1 - \frac{1}{3}\sum_{i=0}^{k-1} {}_{2k}C_{2i}\lambda_{2i}, \quad \lambda_{2k+1} = -1 - \frac{1}{3}\sum_{i=0}^{k-1} {}_{2k+1}C_{2i+1}\lambda_{2i+1}.$$

If $q_0 = \lambda_m$ and $q_j = \sum_{l=0}^{m} {}_mC_l\, j^{m-l}\lambda_l$ for j=$\pm1, \pm2, \dots$, then q_j's satisfy the set of equations

$$q_{j-1} + 4q_j + q_{j+1} = 6(j-1)^m, \quad j = 0, \pm1, \pm2, \pm3, \dots .$$

Proof. By a routine arithmetic, we have

$${}_mC_l\{(j-1)^{m-l} + 4j^{m-l} + (j+1)^{m-l}\} = 6\,{}_mC_l\, j^{m-l} + 2\sum_{k=2,4,6,..}^{m-l} {}_{k+l}C_l\ {}_mC_{k+l}\ j^{m-l-k}.$$

Hence, the sum $q_{j-1} + 4q_j + q_{j+1}$ becomes

$$\sum_{l=0}^{m} \lambda_l\{(j-1)^{m-l} + 4j^{m-l} + (j+1)^{m-l}\}{}_mC_l = 6\sum_{l=0}^{m} {}_mC_l\ \lambda_l\ j^{m-l} + 2\sum_{l=0}^{m-2}\lambda_l \sum_{k=2,4,6,..}^{m-l} {}_{k+l}C_l\ {}_mC_{k+l}\ j^{m-l-k}.$$

We separate the double sum into one part of even $k+l$'s and the other of odd $k+l$'s, and reorder the sum so that the power of j is decreasing. Then we obtain

$$6j^m + 6\,{}_mC_1\lambda_1 j^{m-1} + 2\sum_{k=odd}^{m} {}_mC_k\Big(3\lambda_k + \sum_{i=1,3,5,\dots}^{k-2} {}_kC_i\ \lambda_i\Big) j^{m-k}$$

$$+2\sum_{k=even}^{m} {}_mC_k\Big(3\lambda_k + \sum_{i=0,2,4,\dots}^{k-2} {}_kC_i\ \lambda_i\Big) j^{m-k}.$$

Now, we substitute λ_k's by the successive definition formula, along with $\lambda_0 = 1,\ \lambda_1 = -1$, to obtain

$$6j^m - 6{}_mC_1 j^{m-1} - 6\sum_{k=odd}^{m} {}_mC_k j^{m-k} + 6\sum_{k=even}^{m} {}_mC_k j^{m-k}$$

which is identical to $6(j-1)^m$, and the proof is completed. Note that the above works for the case of $j = \pm 1$, or j=0 with $q_0 = \lambda_m$. Q.E.D

When some of the λ_j's are computed iteratively, we find that $\lambda_2 = \frac{2}{3}, \lambda_3 = 0, \lambda_4 = -\frac{2}{3},\ \lambda_5 = \frac{2}{3}, \lambda_6 = \frac{2}{3}, \lambda_7 = -\frac{10}{3},\ \lambda_8 = \frac{34}{9}, \lambda_9 = 14, \lambda_{10} = -66$. Note that λ_j's do not depend on the value of m which will be used as the degree of polynomial throughout this paper, and that q_j's depend only on the value of m.

Lemma 2. Let λ_k's be defined as in Lemma 1 for k=0,1,2, ... Then we have $|\lambda_k| \le k^k$ for all $k \ge 1$.

Proof. We apply the mathematical induction separately on even k's and odd k's. It is clear that λ_j satisfies the relation for j=1,2,3 and 4. Assume the statement is true for all $i < k$. Then from $|\lambda_{2k}| \le 1 + \frac{1}{3}\sum_{i=0}^{k-1} {}_{2k}C_{2i}|\lambda_{2i}| \le 1 + \frac{1}{3}\sum_{i=0}^{k-1} {}_{2k}C_{2i}(2i)^{2i} \le 1 + \frac{1}{3}(1+2k-1)^{2k}$ and similarly from $|\lambda_{2k+1}| \le 1 + \frac{1}{3}(2k+1)^{2k+1}$, we have $|\lambda_k| \le k^k$ for all $k \ge 1$. Q.E.D ■

Lemma 3. Let m be an even positive integer and q_j's be defined as in Lemma 1. Then the q_j's satisfy $q_{-j+1} = q_{j+1}$ for j=1,2, . . . ,n+1.

Proof. Let $r_j = q_{j+1} - q_{-j+1}$, j=1,2, . . . ,n+1. Then we have $4r_1 + r_2 = 0$ from the two equations $q_{-1} + 4q_0 + q_1 = 6$ and $q_1 + 4q_2 + q_3 = 6$. Also from $q_{-j-1} + 4q_{-j} + q_{-j+1} = 6(-j-1)^m = 6(j+1)^m$, and $q_{j+1} + 4q_{j+2} + q_{j+3} = 6(j+1)^m$, we have $r_j + 4r_{j+1} + r_{j+2} = 0$, for j=1,2,3, . . . ,∞. From the first relation $4r_1 + r_2 = 0$, we have $r_2 = -4r_1$ and hence $|r_2| \ge 4|r_1|$. From the second relation $r_1 + 4r_2 + r_3 = 0$, we have $r_3 = -r_1 - 4r_2 = -17r_1$ from which we obtain $|r_3| \ge 4^2|r_1|$. Continuing the same process, we have $|r_j| \ge 4^{j-1}|r_1|$. On the other hand, we have $|q_j| \le \sum_{l=0}^{m} {}_mC_l |j|^{m-l}|\lambda_l| \le \sum_{l=0}^{m} {}_mC_l |j|^{m-l} m^l \le (|j|+m)^m$ and hence, $|r_j| = |q_{j+1} - q_{-j+1}| \le 2(m+|j|+1)^m$. Now, note that we can have $4^{j-1}|r_1| \le |r_j| \le 2(m+|j|+1)^m$ for all j=1,2, . . . ,∞ only when $r_1 = 0$. Q.E.D ■

Lemma 4. Let m be an odd positive integer and q_j's be defined as in Lemma 1. Then the q_j's satisfy $q_1 = 0$ and $q_{-j+1} = -q_{j+1}$ for j=1,2, ... ,n+1.

The above follows from an argument similar to the proof of Lemma 3 with $s_j = q_{j+1} + q_{-j+1}$ replaced for r_j. Note that when m is even, we have $\frac{q_{-n+2} - q_{-n}}{6n^m} = -\frac{q_{n+2} - q_n}{6n^m}$. Similarly when the degree m is odd, we have $\frac{q_{-n+2} - q_{-n}}{6n^m} = \frac{q_{n+2} - q_n}{6n^m}$, which will be used in Theorem 1.

Theorem 1. *Let $f(t) = t^m$ and let $\phi(t) = \sum_{i=-1}^{2n+1} c_i B_i(t)$ be the cubic spline interpolation of $f(t)$ on $t_j = -1 + \frac{j}{n}$ in the interval [-1,1] with $\beta = \frac{q_{n+2} - q_n}{6n^m}$ and $\alpha = (-1)^{m-1}\beta$. Then $(c_{-1}, c_0, \ldots c_{2n+1}) \equiv (q_{-n}, q_{-n+1}, \ldots, q_{-1}, q_0, q_1, \ldots, q_{n+2})/(6n^m)$ satisfies the equation (2), i.e., $\frac{q_j}{6n^m}$ for j=-n to n+2 are the interpolation coefficients.*

Proof. Recall that c_i's are the unique solution of the (2n+3) equations

$$-c_{-1} + c_1 = \alpha$$

$$c_{j-1} + 4c_j + c_{j+1} = f(t_j) = (-1 + \frac{j}{n})^m, \;\; j = 0, 1, 2, \ldots, 2n$$

$$-c_{2n-1} + c_{2n+1} = \beta.$$

First, we consider the middle equations. By Lemma 1, $q_j's$ satisfy the equations

$$q_{j-1} + 4q_j + q_{j+1} = 6(j-1)^m, \;\; j = 0, \pm 1, \pm 2, \pm 3 \ldots$$

and hence if we define $c_j = q_{-n+j+1}/(6n^m)$, then we have

$$c_{j-1} + 4c_j + c_{j+1} = \frac{q_{-n+j} + 4q_{-n+j+1} + q_{-n+j+2}}{6n^m} = (\frac{-n+j}{n})^m = t_j^m$$

for $j = 0, 1, \ldots, 2n$. Hence, the set of middle equations are satisfied. The last equation is satisfied by the definition of β, and the first equation is satisfied due to Lemmas 3 and 4. Q.E.D ∎

Lemma 5. Let q_j's and λ_j's be defined as in Lemma 1. If $m \leq 4$, then we have $q_{n+2} - q_n = 2mn^{m-1}$, and $q_{-n+2} - q_{-n} = (-1)^{m-1} 2mn^{m-1}$, and hence $\alpha = \frac{h}{3} f'(-1)$ and $\beta = \frac{h}{3} f'(t_1)$ are satisfied, where α and β are as defined in Theorem 1.

Proof. From the definition of $q_j's$, we have

$$q_{n+2} - q_n = \sum_{l=0}^{m-1} {}_mC_l \lambda_l ((n+2)^{m-l} - n^{m-l}),$$

By directly substituting m=1,2,3, and 4 in the left hand side of the above relation, it is trivial to verify that the result is the same as $2mn^{m-1}$ for each m. The second part of Lemma follows from $\frac{h}{3} f'(-1) = \frac{(-1)^{m-1}m}{3n}$ and $\frac{h}{3} f'(1) = \frac{m}{3n}$. Q.E.D ■

3 Corrections of the Solution for Derivative Constraints

In this section, we consider the remainder $r_0 = \frac{h}{3} f'(1) - \beta$ and $s_0 = \frac{h}{3} f'(-1) - \alpha$ in the first and the last equation of (2) respectively, where $\beta = \dfrac{q_{n+2} - q_n}{n^m}$ and $\alpha = (-1)^{m-1}\beta$. If m is even, then q_j's are symmetric in the sense of Lemma 4, while they are antisymmetric when m is odd as in Lemma 5. It is also easy to verify that the cubic spline interpolation coefficients c_j's in (2) for an arbitrary differentiable function $f(t)$ are symmetric when it is even, and antisymmetric when it is odd.

Thus, we may consider only half of the equations in (2). For the case of even m, the middle equation $q_0 + 4q_1 + q_2 = 0$ becomes $4q_1 + 2q_2 = 0$, the next set of equations are $q_j + 4q_{j+1} + q_{j+2} = 6j^m$, j=1,2, ... ,n, and the last equation is $-q_n + q_{n+2} = 0$, so that we have only (n+2)-equations for the (n+2)-unknowns; $q_1, q_2, \ldots, q_{n+2}$. For the case of odd m, we start with the equation $q_1 + 4q_2 + q_3 = 6$ which becomes $4q_2 + q_3 = 6$ from $q_1 = 0$. The rest of the equations are the same as in the case of even m, so that there are (n+1)-equations for the (n+1)-unknowns. Using the following Lemma, one can solve the remaining part of the equation (2) explicitly, i.e., without solving a matrix equation.

Lemma 6. Let A be an $n \times n$ matrix whose entries are the same as in (3) except for the first row which is replaced by $(4, 2, 0, \ldots, 0)$ and let $p = (p_1, p_1, \ldots, p_n)$ be the solution of $Ap = e_n$, where e_n is the unit vector with the last entry of value 1. If $\alpha_1 = 0.5, \alpha_{k+1} = 1/(4 - \alpha_k)$, for k=1,2, ... ,n, then we have $p_n = 1/(1 - \alpha_{n-2}\alpha_{n-1})$ and $p_k = -\alpha_k p_{k+1} = (-1)^{n-k}\alpha_k\alpha_{k+1}\alpha_{k+2}\ldots\alpha_{n-1}p_n$ for k=n-1,n-2, ... ,1. When the first row is $(4, 1, 0, \ldots, 0)$, the same p_k's for $k \neq 1$ and $p_1 = -\frac{1}{4}p_2$ satisfy the equation.

Proof. First, consider the case where the first row is $(4, 2, 0, \ldots, 0)$. It is routine to verify that $4p_1 + 2p_2 = 0$ and $-p_{n-2} + p_n = 1$. For $2 \le k \le n-1$, note that $p_{k-1} + 4p_k + p_{k+1} = (-1)^{n-k+1}(\alpha_{k-1}\alpha_k - 4\alpha_k + 1)\alpha_{k+1}\alpha_{k+2}\ldots\alpha_{n-1}p_n$ which is 0 since α_k is defined by $\alpha_k = 1/(4-\alpha_{k-1})$. A similar proof for the case where the first row is (4, 1, 0, ... , 0) is omitted. Q.E.D ■

Combining the results of Theorem 1 and Lemma 6, we have proved the following Theorem.

Theorem 2. *Let $f(t) = t^m$ where m is an even positive integer and let $q_j = \sum_{l=0}^{m} {}_mC_l\, j^{m-l}\lambda_l$, j=-n,-n+1, ... ,n+2, where λ_l's are as defined in Lemma 1. If p_j's are defined iteratively by $p_{n+2} = 1/(1-\alpha_n\alpha_{n+1})$ and $p_k = -\alpha_k p_{k+1}$, where α_k's are as defined in Lemma 6, then $(q_{-n}+\rho p_{n+2}, q_{-n+1}+\rho p_{n+1}, \ldots, q_0+\rho p_2, q_1+\rho p_1, q_2+\rho p_2, \ldots, q_{n+2}+\rho p_{n+2})$ where $\rho = \dfrac{m}{3n} - \dfrac{q_{n+2}-q_n}{6n^m}$, define the cubic spline interpolation coefficients for $f(t)$. If m is odd, then with $p_{n+1} = 1/(1-\alpha_{n-1}\alpha_n)$, $p_{k+1} = -\alpha_k p_k$, for k=n-1,n-2, ... ,2, and $p_1 = -\frac{1}{4}p_2$, the coefficients become $(q_{-n}-\rho p_{n+1}, q_{-n+1}-\rho p_n, \ldots, q_0-\rho p_1, q_1, q_2+\rho p_1, \ldots, q_{n+2}+\rho p_{n+1})$.*

Example 1. The interpolation coefficients for $f(t) = t^6$ at $t_j = -1 + \dfrac{j}{5}$, j=0,1,2,...,10, computed by the above algorithm using double precision calculations are as follows;

	Original Cubic Spline	*Proposed Algorithm Without Correction*	*Proposed Algorithm With Correction*
c_5	−0.00000322	−0.00000356	−0.00000356
c_6	0.00000643	0.00000711	0.00000643
c_7	0.00004149	0.00003911	0.00004149
c_8	0.00392360	0.00393244	0.00392360
c_9	0.03092012	0.03088711	0.03092012
c_{10}	0.13453998	0.13466311	0.13453994
c_{11}	0.43092025	0.43046044	0.43092011

The differences between the coefficients in the first and the third column are purely computational errors and the differences in the second and third column are due to ρp_j, where $\rho = \dfrac{m}{3n} - \dfrac{q_{n+2}-q_n}{6n^m}$. Note that the magnitude of ρ should decrease as the number of divisions n increases since the second term in ρ is of order $(-1+\frac{1}{n})^m - (-1-\frac{1}{n})^m$. Therefore, the corrections contribute less as the number of divisions becomes larger.

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