Journal of Hydrospace Technology, Vol. 2, No. 2, 1996, pp.80-87

Finite Difference Analysis of Safe Load and Critical Time in a Four-Parameter Viscoelastic Column

Jong Gye Shin*and Jae Yeul Lee†

Abstract

A creep-buckling analysis is studied for a simply-supported viscoelastic column. The fluid-type four-parameter model is employed because of its general applicability to creep materials. Using the imperfection-based incremental approach, a nonlinear load-deflection equation is derived. Safe load and critical (or life) time which characterize the stability of the viscoelastic column are obtained mathematically and interpreted physically. A finite difference algorithm is applied to solve the second-order differential equation of the viscoelastic stress-strain relation. Numerical calculation has been made and discussed for a SUS316 stainless steel column.

1 Introduction

Some materials, such as concrete, polymers, and metals at high temperature, continue to increase their deformation while the applied load is kept constant. This phenomenon is called creep. In design applications to slender structures of those materials, the coupled creep-buckling analysis should be required. Among typical examples are gas uptakes in ships and exhaustion pipes in nuclear reactor made of stainless steels.

There are many publications on creep and buckling analyses. However, characteristics of the coupled creep-buckling responses has not been studied intensively. It is recognized that conventional critical load of an elastic structure is no longer valid for the creep-buckling problems.

Song and Simitses[1] investigated the elastoviscoplastic buckling behavior of a simply-supported beam with incremental approach. For viscoplastic case, unified Bodner-Partom? constitutive relations were used. In a practical view, mechanical constants of Bodner-Partom? model are hardly determined for a specific material, especially for elevated temperature environments. Stubstad and Simitses[2] analyzed creep response using both the Laplace transform and a finite difference method and compared results. Dost and Glockner[3] used the Laplace transform to a perfect column of a three-parameter solid

*Member, Seoul National University
†Member, Samsung Heavy Industries

model. Szyszkowski and Glockner[4] used simplest one-degree-of-freedom rigid bar-spring-dashpot structural models to clarify creep stability problems. He suggested a convenient graphical method which helped to understand the safe load and the safe-service time for the solid-type viscoelastic materials. Vinogradov[5] treated the creep-buckling behavior as a quasi-elastic state. He analyzed beam-columns of a three-parameter solid and four-parameter model under an axial load, axial-lateral load, and axial-bending moment. However, all the previous papers have treated simple viscoelastic models or the inefficient Laplace transform method.

This paper presents the coupled creep-buckling analysis of a column. Among many viscoelastic models (Bazant and Cedolin[6]), the fluid-type four-parameter rheological creep model, shown in Figure 1, is adopted since it is useful engineering material model for concrete, polymers, metals at high temperature, sea ice, and others. Also, this model can describe the simple viscoelastic models like the Kelvin and Maxwell models. The incremental formulation of an initially imperfect column is employed. This imperfection-based incremental method results in exploring two key quantities of the creep-buckling phenomenon; the safe load and the critical (or safe) time. A finite difference algorithm is applied to solve the coupled governing equations under initial conditions. Mathematical and physical concepts of the safe load and the critical creep or safe time are discussed. For numerical calculations, a SUS316 stainless steel column at high temperature is studied.

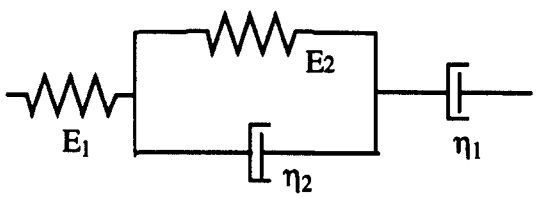

Figure 1: Four-parameter viscoelastic model

2 MATHEMATICAL FORMULATION

2.1 Incremental governing equation

A nonlinear constitutive relation between the incremental stress($\Delta\sigma$) and the incremental strain($\Delta\epsilon$) has the form of

$$\Delta\sigma = C\Delta\epsilon - \Delta\zeta \tag{1}$$

where C and $\Delta\zeta$ are functions of temperature, deformation, and other internal state variables. The incremental form of the adjacent equilibrium equation of a simply supported beam with initial imperfections in Figure 2 is given by

$$\Delta M - (P_a + \Delta P)\Delta w = \Delta P(w_a + w^0) \tag{2}$$

where P is axial load and w is lateral deflection. The subscript 'a' denotes the corresponding quantity at time t_a, and w^0 is the prescribed initial imperfection. Taking sinusoidal functions of w and ζ in Eq.(1) and (2) gives

$$\Delta w_c = \frac{\Delta P(w_{ac} + w_c^0) + \int \Delta\zeta_c z\, dA}{(\frac{n^2\pi^2}{l^2}) \int Cz^2\, dA - (P_a + \Delta P)} \tag{3}$$

where l is the beam span, and n is an integer. The subscript 'c' means the quantity at the center of the beam.

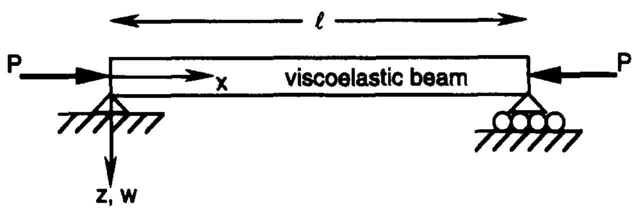

Figure 2: Coordinate system

2.2 Creep-buckling analysis of a four-parameter viscoelastic model

The generalized 4-parameter viscoelastic model has the following differential constitutive relation.

$$\alpha_2\ddot{\sigma} + \alpha_1\dot{\sigma} + \alpha_0\sigma = \beta_2\ddot{\epsilon} + \beta_1\dot{\epsilon} + \beta_0\epsilon \tag{4}$$

where α_i and β_i are material constants.

Thus the relation between the incremental stress and strain in terms of the finite difference components can be obtained by

$$\begin{aligned}\Delta\sigma_i = &\frac{\beta_2 + \Delta t \cdot \beta_1}{\alpha_2 + \Delta t \cdot \alpha_1}\Delta\epsilon_i - \frac{1}{2\alpha_2 + \Delta t \cdot \alpha_1}[-\alpha_2(\sigma_i - \sigma_{i-1}) \\ &+ \beta_2(\epsilon_i - \epsilon_{i-1}) + (\Delta t)^2(\alpha_0\sigma_i - \beta_0\epsilon_i)]\end{aligned} \tag{5}$$

Comparing Eq.(5) with Eq.(1) gives, at the i-th increment,

$$\begin{aligned} C &= \frac{\beta_2 + \Delta t \cdot \beta_1}{\alpha_2 + \Delta t \cdot \alpha_1}, \\ \zeta_i &= \frac{1}{2\alpha_2 + \Delta t \cdot \alpha_1}[-\alpha_2(\sigma_i - \sigma_{i-1}) + \beta_2(\epsilon_i - \epsilon_{i-1}) + (\Delta t)^2(\alpha_0\sigma_i - \beta_0\epsilon_i)]\end{aligned} \tag{6}$$

Using Eqs.(3) and (6), post-buckling responses and critical loads of a four-parameter model can be calculated by increasing incremental deflection Δw_c. Also, in Eq.(3), a creep response can be obtained under constant axial load P_a. Special discussions can be made for the following material models:

(i) Pure elastic model:
For a pure elastic material case, $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 0$ and $\beta_0 = 0, \beta_1 = E_1, \beta_2 = 0$. Then Eq.(3) is converted to:

$$\Delta w_c = \frac{\Delta P(w_{ac} + w_c^0)}{P_E - (P_a - \Delta P)} \tag{7}$$

where P_E is the Euler critical load. As shown in Eq.(7), the total load is converged to the Euler critical load and the deflection increases infinitely.

(ii) Three-Parameter solid model:
Material constants are $\alpha_0 = 0, \alpha_1 = (E_1 + E_2)/E_1, \alpha_2 = \eta_2/E_1$ and $\beta_0 = 0, \beta_1 = E_2, \beta_2 = \eta_2$. In this case, one can easily find that deflection is finite under a certain load, which is called a safe load.

(iii) Three-parameter fluid model:
Material constants are $\alpha_0 = E_2/\eta_1, \alpha_1 = (\eta_1 + \eta_2)/\eta_1, \alpha_2 = 0$ and $\beta_0 = 0, \beta_1 = E_2, \beta_2 = \eta_2$. Initial conditions of this model are $\sigma_{(0)} = \epsilon_{(0)} = \dot{\epsilon}_{(0)} = 0$. During the loading process, the creep deformation increases unboundedly.

(iv) Four-parameter model:
Material constants in Eq.(4) are $\alpha_0 = 0, \alpha_1 = 1 + E_2/E_1 + \eta_2/\eta_1, \alpha_2 = \eta_2/E_1$ and $\beta_0 = 0, \beta_1 = E_2, \beta_2 = \eta_2$. Integrating Eq.(4) twice and letting $t \to 0$ gives:

$$\sum_{i=j}^{2} \left\{ \alpha_i \frac{\partial^{i-j} \sigma_{(0)}}{\partial\, t^{i-j}} - \beta_i \frac{\partial^{i-j} \epsilon_{(0)}}{\partial\, t^{i-j}} \right\} = 0, \quad j = 1, 2 \tag{8}$$

From initial conditions, that is, $\sigma_{(0)} = \epsilon_{(0)} = 0, \dot{\sigma}_{(0)} = \dot{\epsilon}_{(0)} = 0$, it can be derived that $\sigma_{(1)} = \sigma_{(-1)}$ and $\epsilon_{(1)} = \epsilon_{(-1)}$. Similar to the three-parameter fluid model, creep response increases unboundedly during loading process.

2.3 Definition of safe load and critical time

When $\Delta P = 0$, the limit creep load can be obtained. However, there exists an incremental deflection when $\int \Delta\zeta z\, dA > 0$ in Eq.(3). Thus, the safe load can be defined by

$$\int \Delta\zeta z\, dA = 0 \tag{9}$$

For a constant P_a there would exist deflection increment Δw_c if $\Delta P = 0$ and $\Delta\zeta > 0$. In that case, a certain state where a structure lost its function is reached. It is called the critical creep deformation and the time duration to reach the state is called the critical creep time or life time. Time-dependent creeping materials have one of these characteristics; safe load or critical time.

Applying the finite difference form of Eq.(6) to Eq.(9) and expressing in terms of loads and deflection gives:

$$\begin{aligned}\int \Delta\zeta_i z\, dA = &\frac{1}{2\alpha_2+\delta t\alpha_1}[(\delta t\alpha_1 - 2\alpha_2)\{P_i(w_{ic}-w_c^0)\\ &- P_{i-1}(w_{i-1,c}-w_c^0)\} - (\frac{n\pi}{l})^2(\Delta t\beta_1 - 2\beta_2)I(w_{i,c}-w_{i-1,c})\\ &+ 2(\Delta t)^2\{\alpha_0 P_i(w_{ic}+w_c^0) + (\frac{n\pi}{l})^2\beta_0 I w_{i,c}\}]\end{aligned} \qquad (10)$$

Since the initial stress and strain are zero, $\Delta\zeta_0 = 0$. Thus, we have:

$$\begin{aligned}\int \Delta\zeta_0 z\, dA = &\frac{1}{2\alpha_2+\delta t\alpha_1}[(\delta t\alpha_1 - 2\alpha_2)P_1(w_{1c}+w_c^0)\\ &- (\frac{n\pi}{l})^2(\Delta t\beta_1 - 2\beta_2)I w_{1,c}] = 0\end{aligned} \qquad (11)$$

To satisfy the criterion of a safe load in Eq.(9), the incremental form of Eq.(10) should be vanished at every incremental step. The first incremental time step (i=1) can be written as:

$$\begin{aligned}\int \Delta\zeta_1 z\, dA = &\frac{1}{2\alpha_2+\delta t\alpha_1}[(\delta t\alpha_1 - 2\alpha_2)P_1(w_{1c}+w_c^0)\\ &- (\frac{n\pi}{l})^2(\Delta t\beta_1 - 2\beta_2)I w_{1,c}\\ &+ 2(\Delta t)^2\{\alpha_0 P_1(w_{1c}+w_c^0) + (\frac{n\pi}{l})^2\beta_0 I w_{1,c}\}]\end{aligned} \qquad (12)$$

Applying the initial condition of Eq.(11) to Eq.(12) and letting $P_1 = P$ since there is no load increment gives:

$$\int \Delta\zeta_1 z\, dA = \frac{2(\Delta t)^2}{2\alpha_2+\delta t\alpha_1}\{\alpha_0 P(w_{1c}+w_c^0) + (\frac{n\pi}{l})^2\beta_0 I w_{1,c}\} \qquad (13)$$

Eq.(13) takes an important role to mathematical description of a safe load.

(i) For the pure elastic model : Since $\alpha_0 = \beta_0 = 0$, Eq.(13) vanishes. Thus the initial condition of Eq.(11) governs the system and the safe load is identical to the Euler buckling load.

(ii) For the three-parameter solid model : Similar to the elastic case, the safe load can be obtained by:

$$P_s = \frac{(\frac{n\pi}{l})^2 E_1 E_2 I}{E_1+E_2}\cdot\frac{w_c}{w_c+w_c^0} = \frac{E_1}{E_1+E_2}\cdot\frac{w_c}{w_c+w_c^0}n^2 P_E \qquad (14)$$

where $P_E = (\pi^2 E_1 I)/l^2$.

(iii) For the three-parameter fluid model: Since $\alpha_0 \neq 0$ and $\beta_0 = 0$, we have $\int \Delta\zeta z\, dA > 0$ in Eq.(13). Therefore, under a constant load, the deflection increases with time. No safe load can be defined. The critical time duration concept should be considered.

(iv) For the four-parameter model: Similar to the 3-parameter fluid mode, we can derive

$$\int \Delta\zeta z \, dA = \frac{(\Delta t)^2}{\alpha_2 + \delta t \alpha_1}\{\alpha_0 P(w_{1c} + w_c^0) + (\frac{n\pi}{l})^2 \beta_0 I w_{1,c}\} \tag{15}$$

Therefore, since $\int \Delta\zeta z \, dA > 0$ under constant loading, the deflection increases with time. No safe load can be defined, and the critical time should be considered for this model.

3 NUMERICAL CALCULATIONS AND DISCUSSIONS

Numerical calculations are performed for a SUS316 stainless steel column at 1,100°F(593 °C). Material properties and structural dimensions are $l = 1\text{m}, I = 2.5\times10^{-7}\text{m}^4$, $E = 1.572\times10^{11}\text{Pa}$, and $\eta = 1.7208\times10^{15}\text{Pa}\cdot\text{min}$. The material is verified to behave as the four-parameter creep model (Lee[7]).

Figure 3 shows creep responses for various loading conditions; $P/P_E = 0.94, 0.96, 0.98$ About 4% decrease in loading gives two times longer time duration to reach the same creep deflection. Since the critical time is a main parameter for the four-parameter model, slight change of loading results in longer critical time.

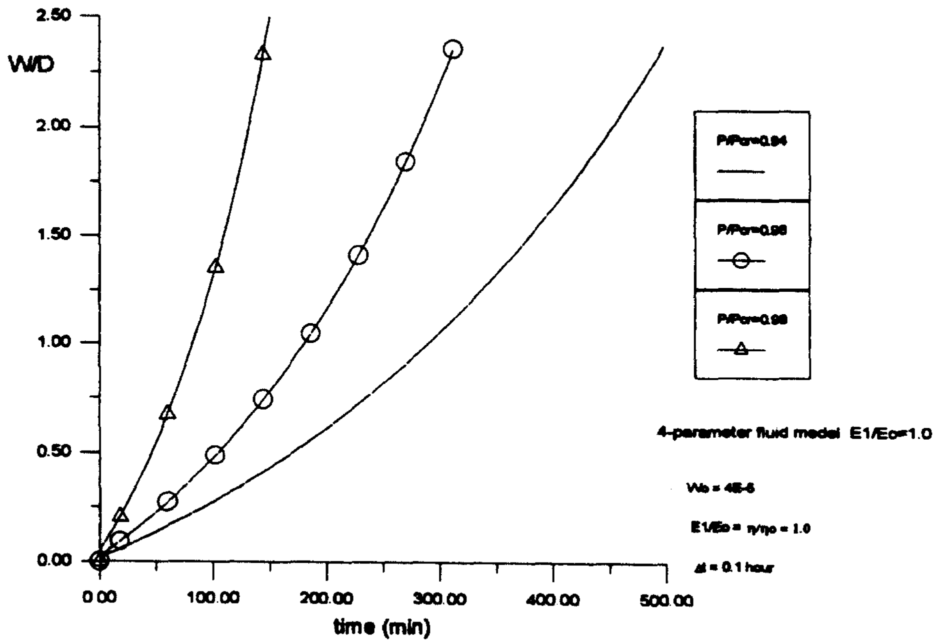

Figure 3: Creep behavior of the four-parameter viscoelastic model for various loadings

Figure 4 shows the effect of the magnitude of initial imperfections. For $E_2 = E_1$ and $\eta_2 = \eta_1$, variations of P/P_E are plotted for different $w_c^o = 4 \times 10^{-5}, 2 \times 10^{-4}, 4 \times 10^{-4} m$. This figure is very similar to elastic case.

Since the safe load is defined for a three-parameter solid model, Figure 5 shows the safe load for $E_2/E_1 = 0.5, 1, 1.5$. The safe load is higher for larger E_2/E_1. That is due to the constrained effect of the spring E_2 for applied loads.

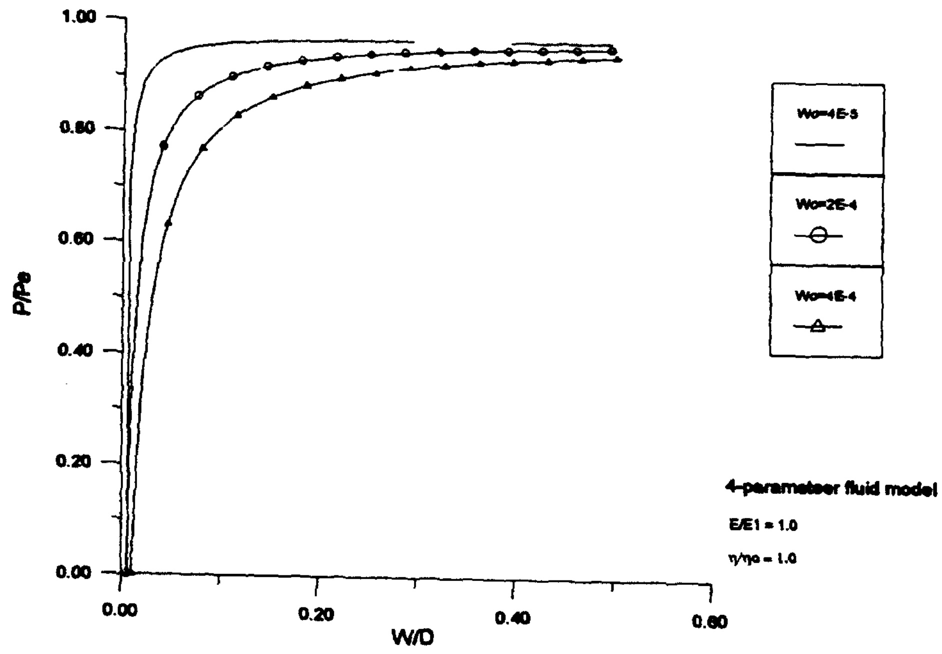

Figure 4: Creep-buckling responses for various imperfections

4 Conclusion

A creep-buckling characteristics of a simply-supported column is analyzed for the fluid-type four-parameter viscoelastic model. The nonlinear adjacent equilibrium equation is derived using the imperfection-based incremental approach. Safe load and critical (or life) time which characterize the stability of the viscoelastic column are obtained mathematically and interpreted physically. An efficient finite difference algorithm is developed for the second-order differential equation of the viscoelastic stress-strain relation. For creep-buckling responses, the safe load and critical time concepts should be distinguished for different viscoelastic models. In this paper, it is found that for a four-parameter model, the magnitude of applied loads affects significantly the critical time duration. Since the creep-buckling of the four-parameter model is formulated in this paper, the results can be expanded to other simple viscoelastic columns.

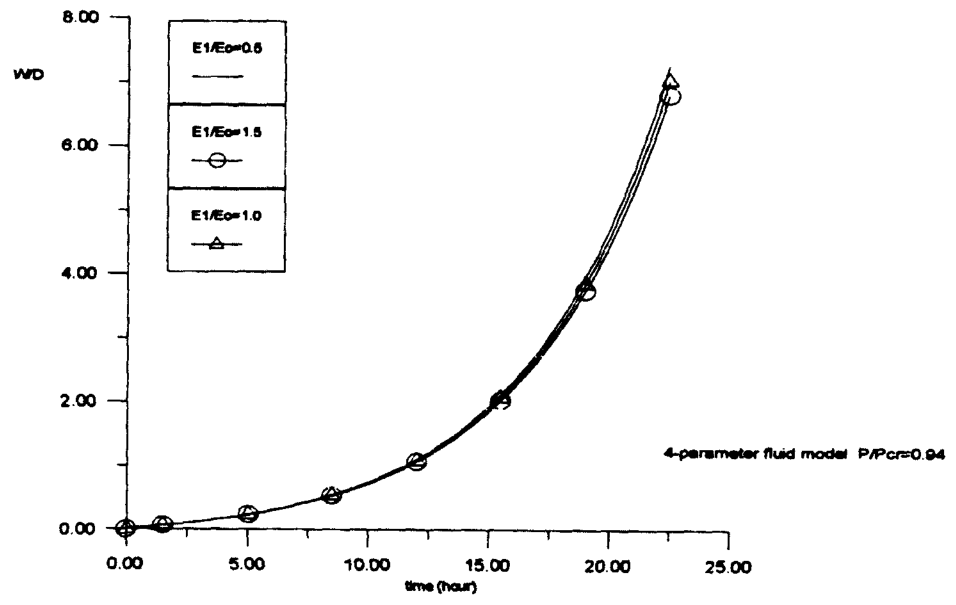

Figure 5: Safe load of a three-parameter solid model for various E_2

5 Acknowledgments

This research was sponsored by the Seoul National University POSCO Research Fund.

References

[1] Song, Y, and Smitses, G.J., "Elastoviscoplastic buckling behavior of simply supported columns," *AIAA J*, Vol. 30, No. 1, 1992, pp.261-265.

[2] Stubstad, JM, and Smitses, GJ., "Solution method for one-dimensional viscoelastic problems,"*AIAA J*, Vol. 26, No. 9, 1988, pp 1127-1134.

[3] Dost, S., and Glockner, P.G., "On the dynamic stability of viscoelastic perfect columns," *Int'J Solids and Structures*, Vol. 18, No. 7, 1982, pp.587-596.

[4] Szyszkowski, W., and Glockner, P.G., "Finite creep deformation and stability analysis of simple structures," *Int'J of Non-linear Mechanics*, Vol. 27, No. 2, 1992, pp 173-196.

[5] Vinogradov, A.M., "Buckling of viscoelastic beam columns," *AIAA J*, Vol. 25, No. 3, 1987, pp.479-483.

[6] Bazant, Z.P., and Cedolin, L., Stability of Structures,*Oxford University Press*, 1991.