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THE REIDEMEISTER NUMBERS ON TRANSFORMATION GROUPS

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ABSTRACT. In this paper we study the Reidemeister number $R(f_G)$ for a self-map $f_G : (X, G) \to (X, G)$ of the transformation group (X, G), as an extension of the Reidemeister number $R(f)$ for a self-map $f : X \to X$ of a topological space X.

1. Introduction

It is observed that the number of the fixed point classes for a self-map $f : X \to X$ of a connected finite polyhedron could be calculated by defining an equivalence relation on the fundamental group $\pi_1(X, x_0)$. The relation depends on the endomorphism of $\pi_1(X, x_0)$ induced by f. The number of equvalence classes of $\pi_1(X, x_0)$, the Reidemeister number $R(f)$, equals the number of the fixed point classes of f.

In paper [3], F. Rhodes introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G), a group G of homeomorphisms of a space X, as a generalization of the fundamental group $\pi_1(X, x_0)$ of a topological space X. He also showed that if the transformation group (X, G) admits a family of preferred paths, then $\sigma(X, x_0, G)$ can be represented as a group extension of $\pi_1(X, x_0)$ by G.

The purpose of this paper is to define the Reidemeister number $R(f_G)$ for a self-map $f_G : (X, G) \to (X, G)$ of the transformation group (X, G) and investigate its properties. We also give the algebraic formulation of the definition of $R(f_G)$ in the same way as in [2].

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2. Preliminaries

In this paper, a transformation group is a pair (X,G), where X is a path connected space with x_0 as base point and G is a group of homeomorphisms of X. A map $(\varphi,\psi):(X,G)\to(Y,H)$ between two transformation groups (X,G) and (Y,H) consists of a continuous map $\varphi:X\to Y$ and a homomorphism $\psi:G\to H$ such that $\varphi(g(x))=\psi(g)\varphi(x)$ for every pair (x,g).

Given any element g of G, a *path* α *of order* g with base point x_0 is a continuous map $\alpha:I\to X$ such that $\alpha(0)=x_0$ and $\alpha(1)=gx_0$. A path α of order g_1 and a path β of order g_2 form a new path $\alpha+g_1\beta$ of order g_1g_2 defined by the following equations

$$(\alpha+g_1\beta)(t)=\begin{cases}\alpha(2t), & 0\le t\le 1/2\\ g_1\beta(2t-1), & 1/2\le t\le 1.\end{cases}$$

Two paths α and β of the same order g are said to be *homotopic* if there is a continuous map $F:I\times I\to X$ such that

$$\begin{aligned} F(t,0)&=\alpha(t), & 0\le t\le 1,\\ F(t,1)&=\beta(t), & 0\le t\le 1,\\ F(0,s)&=x_0, & 0\le s\le 1,\\ F(1,s)&=gx_0, & 0\le s\le 1.\end{aligned}$$

The equivalence relation $\alpha\sim\beta$ denotes that α and β are homotopic paths of the same order. Denote the equivalence class containing a path α of order g by $[\alpha;g]$. Two homotopic classes of paths of different orders g_1 and g_2 are distinct, even if $g_1x_0=g_2x_0$. F. Rhodes [3] showed that the set of homotopy classes of paths of prescribed order with the rule of composition $*$ is a group, where $*$ is defined by $[\alpha;g_1]*[\beta;g_2]=[\alpha+g_1\beta;g_1g_2]$. This group was called the fundamental group of (X,G) with base point x_0, and was denoted by $\sigma(X,x_0,G)$.

3. Main Results

DEFINITION 3.1. Let $f_\sigma:\sigma(X,x_0,G)\to\sigma(X,x_0,G)$ be a self-homomorphism on a fundamental group $\sigma(X,x_0,G)$. Two elements

$[\alpha; g_1]$ and $[\beta; g_2]$ in $\sigma(X, x_0, G)$ are said to be the f_σ-*equivalent*, $[\alpha; g_1] \overset{f_\sigma}{\sim} [\beta; g_2]$, if there exists $[\gamma; g] \in \sigma(X, x_0, G)$ such that $[\alpha; g_1] = [\gamma; g][\beta; g_2] f_\sigma([\gamma; g]^{-1})$.

It is easy to see that the relation $\overset{f_\sigma}{\sim}$ is an equivalence relation on $\sigma(X, x_0, G)$, and partitions $\sigma(X, x_0, G)$ into disjoint equivalence classes. Let $\sigma(X, x_0, G)'(f_\sigma)$ be the set of equivalence classes of $\sigma(X, x_0, G)$ under f_σ-equivalence, and we shall write $\langle\alpha; g\rangle$ for the equivalence class of $[\alpha; g]$, that is,

$$\sigma(X, x_0, G)'(f_\sigma) = \sigma(X, x_0, G)/ \underset{\sim}{f_\sigma} = \{\langle\alpha; g\rangle \,|g \in G\}.$$

Let G be an abelian group. If $\sigma(X, x_0, G)$is an abelian group, then we define a product operation in $\sigma(X, x_0, G)'(f_\sigma)$ by $\langle\alpha; g_1\rangle \langle\beta; g_2\rangle = \langle\alpha + g_1\beta; g_1 g_2\rangle$. This is a well-defined operation, and $\sigma(X, x_0, G)'\ (f_\sigma)$ forms an abelian group.

DEFINITION 3.2. For a self-map $f_G : (X, G) \to (X, G)$, we define the *Reidemeister number* $R(f_G)$ of f_G to be the cardinality of $\sigma(X, x_0, G)'(f_\sigma)$ denoted by $\#\sigma(X, x_0, G)'(f_\sigma)$. In symbols,

$$R(f_G) = \#\sigma(X, x_0, G)'(f_\sigma).$$

Notice that $R(f_G)$ may be either a natural number or infinity.

REMARK. If the acting group G on X is trivial, then $\sigma(X, x_0, G) \cong \pi_1(X, x_0)$. Hence $R(f_G) = \#\sigma(X, x_0, G)'(f_\sigma) = \#\pi_1(X, x_0)'(f_\pi) = R(f)$, where $f_\pi : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the homomorphism induced by a self-map $f : X \to X$, and $R(f)$ is the Reidemeister number of f. For undefined terminology and notation, see [1]. From this fact, we can see that $R(f_G)$ is an extended concept of $R(f)$.

Let $(\varphi, \psi) : (X, G) \to (Y, H)$ be a mapping. It is easy to see that if α is a path in X of order g with base point x_0 then $\varphi\alpha$ is a path in Y of order $\psi(g)$ with base point $\varphi(x_0) = y_0$. Furthermore, if $\alpha \sim \beta$ then $\varphi\alpha \sim \varphi\beta$. Thus (φ, ψ) induces a homomorphism $(\varphi, \psi)_* : \sigma(X, x_0, G) \to \sigma(Y, y_0, H)$ defined by $(\varphi, \psi)_*[\alpha; g] = [\varphi\alpha; \psi(g)]$.

Two transformation groups (X, G) and (Y, H) are said to be of the *same homotopy type* if there exist mappings

$$(\varphi, \psi) : (X, G) \to (Y, H) \text{ and } (\varphi', \psi') : (Y, H) \to (X, G)$$

such that ψ and ψ' are isomorphisms and $\varphi'\varphi$ an $\varphi\varphi'$ are homotopic to the identity maps of X and Y, respectively.

THEOREM 3.3. *The Reidemeister number $R(f_G)$ is an invariant of the homotopy type of the transformation group.*

PROOF. Let the mapping $(\varphi, \psi) : (X, G) \to (Y, H)$ be a homotopy equivalence. Then we obtain the following commutative diagram;

$$\begin{array}{ccc} \sigma(X, x_0, G) & \xrightarrow{f_\sigma} & \sigma(X, x_0, G) \\ {\scriptstyle (\varphi,\psi)_*}\downarrow & & \uparrow{\scriptstyle (\varphi',\psi')_*} \\ \sigma(Y, y_0, H) & \xrightarrow{h_\sigma} & \sigma(Y, y_0, H) \end{array} \tag{3.1}$$

where $(\varphi, \psi)_*$ and $(\varphi', \psi')_*$ are isomorphisms.

Using the diagram (3.1), we have

$$\begin{aligned} &[\alpha; g_1] \overset{f_\sigma}{\sim} [\beta; g_2] \leftrightarrow [\alpha; g_1] = [\gamma; g][\beta; g_2] f_\sigma([\gamma; g]^{-1}) \\ &\quad \leftrightarrow (\varphi, \psi)_*[\alpha; g_1] = (\varphi, \psi)_*[\gamma; g](\varphi, \psi)_*[\beta; g_2](\varphi, \psi)_* f_\sigma([\gamma; g]^{-1}) \\ &\quad \leftrightarrow (\varphi, \psi)_*[\alpha; g_1] \overset{h_\sigma}{\sim} (\varphi, \psi)_*[\beta; g_2]. \end{aligned}$$

Similarly, we also obtain

$$[\alpha'; g_1'] \overset{h_\sigma}{\sim} [\beta'; g_1'] \leftrightarrow (\varphi', \psi')_*[\alpha'; g_1'] \overset{f_\sigma}{\sim} (\varphi', \psi')_*[\beta'; g_2'].$$

Thus $R(f_G) = R(h_G)$.

Let $\sigma(X, x_0, G)'$ be a commutator subgroup of $\sigma(X, x_0, G)$ generated by the set

$$\{[\alpha; g_1][\beta; g_2][\alpha; g_1]^{-1}[\beta; g_2]^{-1} \mid [\alpha; g_1][\beta; g_2] \in \sigma(X, x_0, G)\}.$$

For a convenient notation, we shall write $\bar{\sigma}(X, x_0, G)$ for the quotient group $\sigma(X, x_0, G)/\sigma(X, x_0, G)'$.

THEOREM 3.4. *If $f_G : (X,G) \to (X,G)$ is a self-mapping, then $R(f_G) \geq \#\mathrm{Coker}(1 - \bar{f}_\sigma) \geq 1$, where 1 and $\bar{f}_\sigma$ denote respectively the identity isomorphism and the endomorphism of $\bar{\sigma}(X, x_0, G)$ induced by f_G. Furthermore, if $\sigma(X, x_0, G)$ is abelian, then*

$$R(f_G) = \#\mathrm{Coker}(1 - \bar{f}_\sigma).$$

PROOF. Obiously, there exists a cannonical homomorphism

$$\theta_\sigma : \sigma(X, x_0, G) \to \bar{\sigma}(X, x_0, G) \tag{3.2}$$

such that $\mathrm{Ker}\theta_\sigma = \sigma(X, x_0, G)'$. Hence the following diagram is commutative :

$$\begin{array}{ccc} \sigma(X, x_0, G) & \xrightarrow{f_\sigma} & \sigma(X, x_0, G) \\ {\scriptstyle \theta_\sigma}\downarrow & & {\scriptstyle \theta_\sigma}\downarrow \\ \bar{\sigma}(X, x_0, G) & \xrightarrow{\bar{f}_\sigma} & \bar{\sigma}(X, x_0, G) \end{array} \tag{3.3}$$

For $[\gamma; g] \in \sigma(X, x_0, G)$, any element of the f_σ-equivalent class containing $[\beta; g_2]$ may be expressed in the form

$$[\alpha; g_1] = [\gamma; g][\beta; g_2] f_\sigma([\gamma; g]^{-1}).$$

From the diagram (3.3), we can easily obtain

$$\begin{aligned} \theta_\sigma([\alpha; g_1]) &= \theta_\sigma\big([\gamma; g][\beta; g_2] f_\sigma([\gamma; g]^{-1})\big) \\ &= \theta_\sigma([\gamma; g]) + \theta_\sigma([\beta; g_2]) - \theta_\sigma f_\sigma([\gamma; g]) \\ &= \theta_\sigma([\gamma; g]) + \theta_\sigma([\beta; g_2]) - \bar{f}_\sigma \theta_\sigma([\gamma; g]) \\ &= \theta_\sigma([\beta; g_2]) + (1 - \bar{f}_\sigma)\big(\theta_\sigma([\gamma; g])\big). \end{aligned}$$

Thus there exists an element $[\gamma; g] \in \sigma(X, x_0, G)$ such that

$$\theta_\sigma([\alpha; g_1]) - \theta_\sigma([\beta; g_2]) = (1 - \bar{f}_\sigma)\big(\theta_\sigma([\gamma; g])\big) \in (1 - \bar{f}_\sigma)\big(\bar{\sigma}(X, x_0, G)\big).$$

Let $\eta_{\bar{\sigma}} : \bar{\sigma}(X, x_0, G) \to \mathrm{Coker}(1 - \bar{f}_\sigma)$ be the natural projection. Now consider

$$\sigma(X, x_0, G) \xrightarrow{\theta_\sigma} \bar{\sigma}(X, x_0, G) \xrightarrow{\eta_{\bar{\sigma}}} \mathrm{Coker}(1 - \bar{f}_\sigma).$$

Since both θ_σ and $\eta_{\bar{\sigma}}$ are epimorphism, $\eta_{\bar{\sigma}}\theta_\sigma$ is also epimorphism. Moreover, the $\eta_{\bar{\sigma}}\theta_\sigma$ images of all element of a f_σ-equivalent class are the same element of $\mathrm{Coker}(1 - \bar{f}_\sigma)$. This completes the proof of the first result.

If $\sigma(X, x_0, G)$ is abelian, the homomorphism θ_σ of (3.2) is an isomorphism. From Definition 3.1, $[\alpha; g_1] \overset{f_\sigma}{\sim} [\beta; g_2]$ if and olny if there exists $[\gamma; g] \in \sigma(X, x_0, G)$ such that

$$\theta_\sigma([\alpha; g_1]) - \theta_\sigma([\beta, g_2]) = (1 - \bar{f}_\sigma)(\theta_\sigma([\gamma; g])),$$

or, equivalently,

$$\eta_{\bar{\sigma}}\theta_\sigma([\alpha; g_1]) - \eta_{\bar{\sigma}}\theta_\sigma([\beta; g_2]) = 0.$$

This completes the proof of theorem.

The following lemma plays a central role for the Theorem.

LEMMA 3.5. *Let $f_\sigma : \sigma(X, x_0, G) \to \sigma(X, x_0, G)$ be a homomorphism and let G be abelian. Then, for any $[\alpha; g_1]$, $[\beta; g_2] \in \sigma(X, x_0, G)$,*

(1) $[\alpha; g_1][\beta; g_2] \overset{f_\sigma}{\sim} [\beta; g_2] f_\sigma([\alpha; g_1])$.
(2) $[\alpha; g_1] \overset{f_\sigma}{\sim} f_\sigma([\alpha; g_1])$.

PROOF. (1) By Definition 3.1, we have

$$\begin{aligned}[\alpha; g_1][\beta; g_2] &\overset{f_\sigma}{\sim} [\alpha; g_1]^{-1}([\alpha; g_1][\beta; g_2]) f_\sigma([\alpha; g_1]) \\ &\overset{f_\sigma}{\sim} [\beta; g_2] f_\sigma([\alpha; g_1]).\end{aligned}$$

(2) By taking $[\beta; g_2] = [x_0'; e]$ in (1), we get

$$[\alpha; g_1] \overset{f_\sigma}{\sim} f_\sigma([\alpha; g_1]).$$

THEOREM 3.6. *Let G be abelian. Then the following statements are equivalent:*

(1) *The f_σ-equivalence relation $[\alpha; g_1] \overset{f_\sigma}{\sim} [\beta; g_2]$ implies $[\alpha; g_1][\gamma; g] \overset{f_\sigma}{\sim} [\beta; g_2][\gamma; g]$ for any $[\gamma; g] \in \sigma(X, x_0, G)$.*
(2) *For any $[\alpha; g_1]$, $[\beta; g_2]$, $[\gamma; g_3] \in \sigma(X, x_0, G)$, the relation $[\alpha; g_1][\beta; g_2][\gamma; g_3] \overset{f_\sigma}{\sim} [\beta; g_2][\alpha; g_1][\gamma; g_3]$ holds.*
(3) *In Theorem 3.4, the epimorphism $\eta_{\bar{\sigma}}\theta_\sigma : \sigma(X, x_0, G) \to$ Coker $(1-\bar{f}_\sigma)$ induces a one-to-one correspondence between $\sigma(X, x_0, G)'(f_\sigma)$ and Coker$(1-\bar{f}_\sigma)$. That is, $R(f_G) = \#$ Coker$(1-\bar{f}_\sigma)$.*

PROOF. It can be proved by the same method as in the proof of Theorem 2.3 in [2] by using Lemma 3.5.

NOTATION. For a positive integer n, let f_G^k denote the n-th iteration $f_G \circ f_G \circ \cdots f_G$ (n times) of f_G, and f_G^n the homomorphism $(f_G^n)_\sigma = (f_\sigma)^n : \sigma(X, x_0, G) \to \sigma(X, x_0, G)$.

Let G be an abelian group. A self-map $f_G : (X, G) \to (X, G)$ is said to be *eventually commutative* if there exists a positive integer n such that

$$f_\sigma^n([\alpha; g_1][\beta; g_2]) = f_\sigma^n([\beta; g_2][\alpha; g_1])$$

for each $[\alpha; g_1]$, $[\beta; g_2] \in \sigma(X, x_0, G)$. This means that $f_\sigma^n(\sigma(X, x_0, G))$ is a commutative subgroup of $\sigma(X, x_0, G)$.

THEOREM 3.7. *Let G be abelian. If $f_G : (X, G) \to (X, G)$ is eventually commutative, then $R(f_G) = \#$Coker$(1-\bar{f}_\sigma)$.*

PROOF. It suffices to prove condition (2) of Theorem 3.6. By the assumption, there exists a positive integer n such that $f_\sigma^n([\alpha; g_1][\beta; g_2]) = f_\sigma^n([\beta; g_2][\alpha; g_1])$. From Lemma 3.5,

$$\begin{aligned}
[\alpha; g_1][\beta; g_2][\gamma; g_3] &\overset{f_\sigma}{\sim} f_\sigma^n([\alpha; g_1][\beta; g_2][\gamma; g_3]) \\
&= f_\sigma^n([\alpha; g_1][\beta; g_2]) f_\sigma^n([\gamma; g_3]) \\
&= f_\sigma^n([\beta; g_2][\alpha; g_1]) f_\sigma^n([\gamma; g_3]) \\
&= f_\sigma^n([\beta; g_2][\alpha; g_1][\gamma; g_3]) \\
&\overset{f_\sigma}{\sim} [\beta; g_2][\alpha; g_1][\gamma; g_3].
\end{aligned}$$

DEFINITION 3.8. For a subgroup $\rho(X, x_0, G)$ of $\sigma(X, x_0, G)$, we define

$$\begin{aligned}
&\rho(X, x_0, G)'(f_\sigma) \\
&= \{\langle\alpha; g_1\rangle \in \sigma(X, x_0, G)'(f_\sigma) | \langle\alpha; g_1\rangle \cap \rho(X, x_0, G) \neq \phi\} \\
&= \{\langle\alpha; g_1\rangle \in \sigma(X, x_0, G)'(f_\sigma) | [\alpha; g_1] \overset{f_\sigma}{\sim} [\beta; g_2] \\
&\qquad \text{for some } [\beta; g_2] \in \rho(X, x_0, G)\}.
\end{aligned}$$

THEOREM 3.9. *Let $Z(\sigma(X, x_0, G))$ be the center of $\sigma(X, x_0, G)$. If $f_\sigma : \sigma(X, x_0, G) \to \sigma(X, x_0, G)$ be a homomorphism such that $f_\sigma(\sigma(X, x_0, G)) \subseteq Z(\sigma(X, x_0, G))$, then*

(1) *$Z(\sigma(X, x_0, G))'(f_\sigma)$ is an abelian group.*
(2) *$R(f_G) = \# Z(\sigma(X, x_0, G))'(f_\sigma)$.*

PROOF. (1) Since $Z(\sigma(X, x_0, G))$ is a subgroup of $\sigma(X, x_0, G)$, by Definition 3.8,

$$\begin{aligned}
Z(\sigma(X, x_0, G))'(f_\sigma) = \{&\langle\alpha; g_1\rangle \in \sigma(X, x_0, G)'(f_\sigma) | \\
&[\alpha; g_1] \overset{f_\sigma}{\sim} [\alpha'; g_1'] \text{ for some } [\alpha'; g_1'] \in Z(\sigma(X, x_0, G))\}.
\end{aligned}$$

If $\langle\alpha; g_1\rangle$ and $\langle\beta; g_2\rangle$ are elements in $Z(\sigma(X, x_0, G))'(f_\sigma)$, then the product $\langle\alpha; g_1\rangle \langle\beta; g_2\rangle$ is defined as follows:

$$\langle\alpha; g_1\rangle \langle\beta; g_2\rangle = \langle[\alpha; g_1][\beta; g_2]\rangle = \langle\alpha + g_1\beta; g_1 g_2\rangle.$$

Then the product is a well-defined in $Z(\sigma(X, x_0, G))'(f_\sigma)$, and $\langle\alpha + g_1\beta; g_1 g_2\rangle \in Z(\sigma(X, x_0, G))'(f_\sigma)$. Indeed, we see that $[\alpha; g_1] \overset{f_\sigma}{\sim} [\alpha'; g_1']$ and $[\beta; g_2] \overset{f_\sigma}{\sim} [\beta'; g_2']$ for elements $[\alpha'; g_1'], [\beta'; g_2'] \in Z(\sigma(X, x_0, G))$. Then $[\alpha; g_1][\beta; g_2] \overset{f_\sigma}{\sim} [\alpha'; g_1'][\beta'; g_2']$ and $\langle\alpha; g_1\rangle \langle\beta; g_2\rangle = \langle\alpha + g_1\beta; g_1 g_2\rangle = \langle\alpha' + g_1'\beta'; g_1' g_2'\rangle = \langle\alpha'; g_1'\rangle \langle\beta'; g_2'\rangle \in Z(\sigma(X, x_0, G))'(f_\sigma)$. Since $[\alpha'; g_1'] [\beta'; g_2'] = [\beta'; g_2'][\alpha'; g_1']$, $\langle\alpha' + g_1'\beta'; g_1' g_2'\rangle = \langle\beta' + g_2'\alpha'; g_2' g_1'\rangle$. Thus $\langle\alpha; g_1\rangle \langle\beta; g_2\rangle = \langle\beta; g_2\rangle \langle\alpha; g_1\rangle$. Let $\langle\alpha; g\rangle \in Z(\sigma(X, x_0, G))'(f_\sigma)$. Then $\langle[\alpha; g][\alpha; g]^{-1}\rangle = \langle x_0'; e\rangle = \langle[\alpha; g]^{-1}[\alpha; g]\rangle$. Hence $\langle[\alpha; g]^{-1}\rangle = \langle\alpha; g\rangle^{-1}$ in $Z(\sigma(X, x_0, G))$.

(2) In order to prove our result, it is sufficient to show that $\sigma(X, x_0, G)'(f_\sigma) = Z(\sigma(X, x_0, G))'(f_\sigma)$. Clearly,

$$Z(\sigma(X, x_0, G))'(f_\sigma) \subset \sigma(X, x_0, G)'(f_\sigma).$$

On the other hand, if $\langle \alpha; g \rangle \in \sigma(X, x_0, G)'(f_\sigma)$, then

$$[\alpha; g] = [\alpha; g] f_\sigma([\alpha; g]) f_\sigma([\alpha; g]^{-1}).$$

Since $f_\sigma([\alpha; g]) \in Z(\sigma(X, x_0, G))$, $\langle \alpha; g \rangle \in Z(\sigma(X, x_0, G))'(f_\sigma)$. Thus $\sigma(X, x_0, G)'(f_\sigma) \subset Z(\sigma(X, x_0, G))'(f_\sigma)$. This completes our proof.

In [3], a transformation group (X, G) is said to admit *a family K of preferred paths at x_0* if it is possible to associate with every element g of G a path k_g from gx_0 to x_0 such that the path k_e associated with the identity element e of G is x_0' which is the constant map such that $x_0'(t) = x_0$ for each $t \in I$ and for every pair of elements g_1, g_2 the path $k_{g_1 g_2}$ from $g_1 g_2 x_0$ to x_0 is homotopic to $g_1 k_{g_2} + k_{g_1}$.

In [6], a continuous map $H : X \times I \to X$ is called a *homotopy of order* g if $H(x, 0) = x$, $H(x, 1) = gx$, where $g \in G$. If H is a homotopy of order g, then the path $\alpha : I \to X$ such that $\alpha(t) = H(x_0, t)$ is a called the *trace of H*. The subgroup $E(X, x_0, G)$ of $\sigma(X, x_0, G)$ was defined by the set of all elements $[\alpha; g] \in \sigma(X, x_0, G)$ such that $\alpha(t)$ is the trace of a homotopy of order g, where $g \in G$.

In [4], a family K of preferred paths at x_0 is called *a family of preferred traces at x_0* if for every preferred path k_g in K, $k_g \rho$ is the trace of a homotopy of order g, where $\rho(t) = 1 - t$. It is known (see [4]) that if (G, G) admits a family of preferred paths at e, then (X, G) admits a family of preferred traces at x_0.

THEOREM 3.10. *Let (X, G) admits a family K of preferred traces at x_0. If G is abelian and $E(X, x_0, G) = \sigma(X, x_0, G)$, then*

(1) *$\sigma(X, x_0, G)$ is an abelian group.*
(2) *$\sigma(X, x_0, G)'(f_\sigma) \cong Coker(1 - f_\sigma)$. Thus $R(f_G) = \#Coker\ (1 - f_\sigma)$.*

PROOF. (1) It is similar to [5, Theorem5]. In fact, let $[\alpha; g_1]$ and $[\beta; g_2]$ be elements in $E(X, x_0, G)$. Since G is abelian, it is enough to show that $\alpha + g_1\beta$ is homotopic to $\beta + g_2\alpha$. Since $k_{g_1}\rho$ is a trace of a homotopy of order g_1, the loop $\beta + k_{g_2}$ at x_0 is homotopic to $k_{g_1}\rho + g_1(\beta + k_{g_2}) + k_{g_1}$ [4, Lemma2]. Thus we have

$$\begin{aligned}\alpha + g_1\beta &\sim \alpha + k_{g_1} + k_{g_1}\rho + g_1\beta + k_{g_1 g_2} + k_{g_1 g_2}\rho \\ &\sim \alpha + k_{g_1} + k_{g_1}\rho + g_1(\beta + k_{g_2}) + k_{g_1} + k_{g_1 g_2}\rho \\ &\sim \alpha + k_{g_1} + \beta + k_{g_2} + k_{g_1 g_2}\rho\end{aligned}$$

and similarly, we also obtain

$$\beta + g_2\alpha \sim \beta + k_{g_2} + \alpha + k_{g_1} + k_{g_1 g_2}\rho.$$

Now we must show $\alpha + k_{g_1} + \beta + k_{g_2} \sim \beta + k_{g_2} + \alpha + k_{g_1}$. Since $[\alpha; g_1] \in E(X, x_0, G)$ and $k_{g_1} \in K$, there exists a homotopy $H_1 : X \times I \to X$ of order g_1 such that $H_1(x_0, t) = \alpha(t)$ and a homotopy $H_2 : X \times I \to X$ of order g_1 such that $H_2(x_0, t) = k_{g_1}\rho(t)$. Define $F : X \times I \to X$ by

$$F(x,t) = \begin{cases} H_1(x, 2t), & 0 \le t \le 1/2 \\ H_2(x, 2-2t), & 1/2 \le t \le 1. \end{cases}$$

Then F is a homotopy of order e such that $F(x_0, t) = (\alpha + k_{g_1})(t)$. Hence the loop $\beta + k_{g_2}$ at x_0 is homotopic to $(\alpha + k_{g_1})\rho + (\beta + k_{g_2}) + (\alpha + k_{g_1})$ [4, Lemma2]. So

$$\begin{aligned}\alpha + k_{g_1} + \beta + k_{g_2} &\sim \alpha + k_{g_1} + (\alpha + k_{g_1})\rho + (\beta + k_{g_2}) + (\alpha + k_{g_1}) \\ &\sim \beta + k_{g_2} + \alpha + k_{g_1}.\end{aligned}$$

(2) By Definition 3.1, two elements $[\alpha; g_1]$ and $[\beta; g_2]$ of $\sigma(X, x_0, G)$ are f_σ-equivalent if and only if, for some $[\gamma; g] \in \sigma(X, x_0, G)$, $[\alpha; g_1] = [\gamma; g] + [\beta; g_2] - f_\sigma([\gamma; g])$ so that $[\alpha; g_1] - [\beta; g_2] = [\gamma; g] - f_\sigma([\gamma; g]) = (1 - f_\sigma)([\gamma; g]) \in (1 - f_\sigma)(\sigma(X, x_0, G))$. Hence the f_σ-equivalence classes are the cosets of $\sigma(X, x_0, G)/(1 - f_\sigma)(\sigma(X, x_0, G)) = \text{Coker }(1 - f_\sigma)$. Thus we have $\sigma(X, x_0, G)'(f_\sigma) \cong \text{Coker}(1 - f_\sigma)$.

As a direct consequence of Theorem 3.10, we have the following:

COROLLARY 3.11 [6]. *If (X, G) admits a family of preferred traces at x_0 and G is abelian, then $E(X, x_0, G)$ is an abelian subgroup of $\sigma(X, x_0, G)$.*

References

1. R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, Illinois, 1971.
2. B. J. Jiang, *Lectures on Nielsen fixed point theory*, Comtemporary Math., 14 Amer. Math. Soc. Providence, R. I. (1983), 1–99.
3. F. Rhodes, *On the fundamental group of a transformation group*, Proc. London Math. Soc. **16** (1966), 635–650.
4. M. H. Woo, *A representation of* $E(X, x_0, G)$ *in terms of* $G(X, x_0)$, J. Korean Math. Soc. **23** (1986), 101–108.
5. ———, *On the extended Jiang subgroup*, J. Korean Math. Soc. **31** (1994), 609–618.
6. ——— and Y. S. Yoon, *Certain subgroups of homotopy groups of a transformation group*, J. Korean Math. Soc. **20** (1983), 223–233.

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