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A FINSLER SPACE WITH A SPECIAL METRIC FUNCTION

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ABSTRACT. In this paper, we shall find the conditions that the Finsler space with a special (α, β)-metric be a Riemannian space and a Berwald space.

1. Introduction

A Finsler metric(fundamental function) L in an n-dimensinal differentiable manifold M^n is called an (α, β)-metric if L is positively homogeneous function of degree one of a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a differential 1-form $\beta = b_i(x)y^i$. The notion of (α, β)-metric was introduced by M. Matsumoto [3] and has been studied by many authors. The well-known examples of the (α, β)-metric are the Randers metric $L = \alpha + \beta$, the Kropina metric $L = \alpha^2/\beta$ and the slope (or Matsumoto) metric $L = \alpha^2/(\alpha - \beta)$ which have greatly contributed to the developement of Finsler geometry.

The one of the authors has introduced a special (α, β)-metric $L(\alpha, \beta)$ in [6] satisfying

$$L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2, \tag{1.1}$$

where c_1, c_2 and c_3 are constants.

When the C-tensor of the Finsler space is expressed in the term of the angular metric tensor of the Riemannian metric α, the special (α, β)-metric L satisfying (1.1) is introduced. In case of $c_1 = c_2 = c_3 = 1$ in

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(1.1), the metric L becomes to a Randers metric. Thus the special metric L satisfying (1.1) may be considered as a generalization of the Randers metric.

The purpose of the present paper is to find the condition that the Finsler space with the metric L satisfying (1.1) be a Riemannian space and a Berwald space by simple and peculiar manipulation.

2. The condition to be a Riemannian space

Let $F^n = (M^n, L(\alpha,\beta))$ be a Finsler space with a fundamental function L satisfying (1.1). Then we have

$$\dot{\partial}_i\alpha = \frac{a_{il}y^l}{\alpha}, \qquad \dot{\partial}_j\dot{\partial}_i\alpha = \frac{\alpha^2 a_{ij} - a_{il}a_{jm}y^l y^m}{\alpha^3}, \tag{2.1}$$

$$\dot{\partial}_i\alpha^2 = 2a_{ik}y^k, \qquad \dot{\partial}_j\dot{\partial}_i\alpha^2 = 2a_{ij},$$

$$\dot{\partial}_i\beta = b_i, \quad \dot{\partial}_j\dot{\partial}_i\beta = 0, \quad \dot{\partial}_i\beta^2 = 2b_i b_m y^m, \quad \dot{\partial}_j\dot{\partial}_i\beta^2 = 2b_i b_j,$$

where $\dot{\partial}_i = \partial/\partial y^i$. On the other hand, since

$$\dot{\partial}_j\dot{\partial}_i L^2 = c_1\dot{\partial}_j\dot{\partial}_i\alpha^2 + 2c_2(\dot{\partial}_j\dot{\partial}_i\alpha\beta + \dot{\partial}_i\alpha\dot{\partial}_j\beta + \dot{\partial}_j\alpha\dot{\partial}_i\beta + \alpha\dot{\partial}_j\dot{\partial}_i\beta) + c_3\dot{\partial}_j\dot{\partial}_i\beta^2$$

Using (2.1), we have

$$\begin{aligned} g_{ij} &= \frac{1}{2}\dot{\partial}_j\dot{\partial}_i L^2 \\ &= c_1 a_{ij} + c_2\left(\frac{\alpha^2 a_{ij} - a_{il}a_{jm}y^l y^m}{\alpha^3}\beta + \frac{a_{il}y^l}{\alpha}b_j + \frac{a_{jl}y^l}{\alpha}b_i\right) + c_3 b_i b_j. \end{aligned} \tag{2.2}$$

Now, we assume that F^n is a Riemannian space, that is, g_{ij} is a function of position alone. From (2.2), we have

$$\begin{aligned} g_{ij}(x) - c_1 a_{ij}(x) - c_3 b_i(x) b_j(x) \\ + c_2\frac{\alpha^2\beta a_{ij} - \beta a_{il}a_{jm}y^l y^m + \alpha^2 a_{il}b_j y^l + \alpha^2 a_{jl}b_i y^l}{\alpha^3} = 0, \end{aligned}$$

that is,

$$\alpha^3\{g_{ij}(x) - c_1a_{ij}(x) - c_3b_i(x)b_j(x)\} \\ + c_2\{a_{ij}a_{hk}b_l - a_{ih}a_{jk}b_l + a_{hk}a_{il}b_j + a_{hk}a_{jl}b_i\}y^hy^ky^l = 0.$$

From this, noticing α^3 is irrational with respect to y^i, we have

$$g_{ij} - c_1a_{ij} - c_3b_ib_j = 0, \tag{2.3}$$

$$c_2A_{ijhkl}y^hy^ky^l = 0, \tag{2.4}$$

where $A_{ijhkl} = (a_{ij}b_l + a_{il}b_j + a_{il}b_i)a_{hk} - a_{ih}a_{jk}b_l$. By arbitrariness of y^i, (2.4) is equivalent to

$$c_2 = 0 \quad \text{or} \quad A_{ij(hkl)} = 0, \tag{2.5}$$

where $A_{ij(hkl)}$ denotes the all permutations of indices h, k, l and summation and multiplied by 1/3!. The second equation of (2.5) is equivalent to

$$\begin{aligned} &a_{ij}a_{hk}b_l + a_{il}a_{hk}a_j + a_{jl}a_{hk}b_i - a_{ih}a_{jk}b_l \\ &+a_{ij}a_{kl}b_h + a_{ih}a_{kl}b_j + a_{jh}a_{kl}b_i - a_{ik}a_{jl}b_h \\ &+a_{ij}a_{lh}b_k + a_{ik}a_{lh}b_j + a_{jk}a_{lh}b_i - a_{il}a_{jh}b_k \\ &+a_{ij}a_{hk}a_l + a_{il}a_{hk}b_j + a_{jl}a_{hk}b_i - a_{ih}a_{jk}b_l \\ &+a_{ij}a_{kl}b_h + a_{ih}a_{kl}b_j + a_{jh}a_{lk}b_i - a_{ik}a_{jl}b_h \\ &+a_{ij}a_{lh}b_k + a_{ik}a_{lh}b_j + a_{jk}a_{lh}b_i - a_{il}a_{jh}b_k = 0. \end{aligned} \tag{2.6}$$

Contracting (2.6) by $a^{ij}a^{hk}$, we have

$$[2(n^2 + n) + 4(n + 1)]b_l = 0.$$

Thus we have $b_l = 0$. Hence we have

$$g_{ij} = c_1a_{ij} + c_3b_ib_j \tag{2.7}$$

and

$$(2.8) \qquad c_2 = 0 \quad \text{or} \quad b_l = 0.$$

Conversely, we assume that (2.8) is satisfied. Then, from(2.2), we have

$$g_{ij} = c_1 a_{ij} + c_3 b_i b_j \quad (\text{ in case of } c_2 = 0)$$

or

$$g_{ij} = c_1 a_{ij} \quad (\text{in case of } b_i = 0).$$

This means that F^n is a Riemanian space. Thus we have

THEOREM 2.1. *F^n is a Riemannian space if and only if $c_2 = 0$ or $b_l = 0$. The metric tensor g_{ij} of F^n is given by $g_{ij} = c_1 a_{ij} + c_3 b_i b_j$ (in case of $c_2 = 0$) and $g_{ij} = c_1 a_{ij}$ (in case of $b_i = 0$).*

3.The condition to be a Berwald space

We shall discuss a condition for F^n with (1.1) to be a Berwald space. Differentiating (1.1) h-convariantly with respect to the Berwald connection $B\Gamma = (G^k_{jk}, G^i_j, 0)$, we have

$$(3.1) \qquad (c_1\alpha + c_2\beta)a_{hk|i}y^h y^k + 2\alpha(c_2\alpha + c_3\beta)b_{h|i}y^h = 0.$$

This is rewritten as follows:

$$(3.2) \quad c_2(b_j a_{hk|i} + 2a_{jk}b_{h|i})y^h y^j y^k + \alpha(c_1 a_{hk|i} + 2c_3 b_k b_{h|i})y^h y^k = 0.$$

In the case of $c_2 = 0$, we have $L^2 = (c_1 a_{ij} + c_3 b_i b_j)y^i y^j$. By Theorem 2.1, F^n is a Riemannian space with the metric tensor $g_{ij} = c_1 a_{ij} + c_3 b_i b_j$. In the subsequent cosideration, F^n is the non-Riemannian space, therefore we shall consider only the case of $c_2 \neq 0$.

Now, we assume the F^n is a Berwald space. Then $a_{hk|i}$ and $b_{h|i}$ are functions of position alone. Accordingly, in (3.2), $c_2(b_j a_{hk|i} + 2a_{jk}b_{h|i})y^h$

$y^j y^k$ is a polynomial of degree 3 with respect to y^i and $\alpha(c_1 a_{hk|i} + 2c_3 b_k b_{h|i}) y^h y^k$ is irrational with respect to y^i. Thus we have

$$b_{(j} a_{hk)|i} + 2a_{(jk} b_{h)|i} = 0, \tag{3.3}$$

$$c_1 a_{(hk)|i} + 2c_3 b_{(k} b_{h)|i} = 0, \tag{3.4}$$

where $A_{(hk)} = \frac{1}{2}(A_{hk} + A_{kh})$. We easily show that (3.4) is equivalent to

$$(c_1 a_{hk} + c_3 b_h b_k)_{|i} = 0. \tag{3.5}$$

From (3.5), putting $g_{hk} = c_1 a_{hk} + c_3 b_h b_k$, we have

$$g_{hk|i} = \partial_i g_{hk} - G^r_{hi} g_{rk} - G^r_{hi} g_{hr} = 0.$$

Where $\partial_i = \partial/\partial x_1$. Using the cyclic permutation indices, we get

$$\partial_i g_{hk} + \partial_h g_{ki} - \partial_k g_{ih} = 2G^r_{hj} g_{rk},$$

from which $G^r_{ji} = \gamma^r_{ji}$, where γ^r_{ji} is the Riemannian connection of a Riemannian metric $g_{hk} = c_1 a_{hk} + c_3 b_h b_k$.

Conversely, we assume that (3.3) and (3.4) are satisfied. Then (3.2) is satisfied, and so we have (3.1) with respect to the Riemannian connection of a Riemannian metric $g_{hk} = c_1 a_{hk} + c_3 b_h b_k$. Hence the Riemannian connection is the Berwald connection of our Finsler space. Thus we have

THEOREM 3.1. *The Finsler space with metric $L(x, y)$ given by (1.1) is a Berwald space if and only if (3.3) and (3.4) are satisfied. This is equivalent to that (3.3) is satisfied with respect to the Riemannian connection of the Riemannian metric $g_{hk} = c_1 a_{hk} + c_3 b_h b_k$.*

Next, the equaltion (3.3) is equivalent to

$$(b_j a_{hk|i} + b_h a_{kj|i} + b_k a_{jh|i}) + 2(a_{jk} b_{h|i} + a_{kh} b_{j|i} + a_{hj} b_{k|i}) = 0. \tag{3.6}$$

From (3.5), we have

$$a_{hk|i} = -\frac{c_3}{c_1}(b_k b_{h|i} + b_{k|i} b_h) \qquad (c_1 \neq 0). \tag{3.7}$$

Substituting (3.7) in (3.6), we have

$$a_{jk}b_{h|i} + a_{kh}b_{j|i} + a_{hj}b_{k|i} - \frac{c_3}{c_1}(b_j b_h b_{k|i} + b_j b_k b_{h|i} + b_k b_h b_{j|i}) = 0.$$

Transvecting above the equation with a^{jk}, we obtain

$$(n+2)b_{h|i} - \frac{c_3}{c_1}(b^2 b_{h|i} + 2b_h b^j b_{j|i}) = 0, \tag{3.8}$$

where $b^2 = b^h b_h$, $b^h = a^{hi}b_i$. Furthermore, transvecting (3.8) with b^h, we have

$$[(n+2) - \frac{3c_3}{c_1}b^2]b^k b_{k|i} = 0. \tag{3.9}$$

If we assume that $\frac{c_3}{c_1} \neq \frac{n+2}{3b^2}$, $b^k b_{k|i} = 0$. From this and (3.8), we get $b_{h|i} = 0$ if $\frac{c_3}{c_1} \neq \frac{n+2}{b^2}$. Substituting $b_{h|i} = 0$ in (3.7), we have $a_{hk|i} = 0$, from which $G^i_{jk} = \{^{\,i}_{jk}\}$, where $\{^{\,i}_{jk}\}$ are the Christioffel symbol of a_{hk}. Hence $\nabla_i b_h = 0$, where ∇_i is the convariant derivative with respect to $\{^{\,i}_{jk}\}$.

Conversely, if $G^i_{jk} = \{^{\,i}_{jk}\}$, $\nabla_h b_i = 0$, then (3.3) and (3.4) are satisfied. Therefore, by the Theorem 3.1, F^n is a Berwald space.

Thus we have

THEOREM 3.2. *The Finsler space with (1.1) assumed* $\frac{c_3}{c_1} \neq \frac{n+2}{3b^2}$, $\frac{c_3}{c_1} \neq \frac{n+2}{b^2}$ $(c_1 \neq 0)$, *is a Berwald space if and only if* $G^i_{hk} = \{^{\,i}_{hk}\}$, $\nabla_h b_i = 0$.

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