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FINITELY NORMAL FAMILIES OF INTEGER TRANSLATIONS

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ABSTRACT. For an open set G in the complex plane $\mathbf{C}$, we prove the existence of an entire function f such that its integer translations forms a finitely normal family exactly on G if and only if G is periodic with period 1 and G has no hole.

1. Preliminaries

A family $\mathcal{F}$ of functions $f_\alpha(z)$, holomorphic in a domain D, is said to be *finitely normal* in D if every sequence $\{f_n(z)\}$ from $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of D to a holomorphic function. We say a family of holomorphic functions is *finitely normal at a point* $z \in \mathbf{C}$ if there is an open neighborhood of z such that the family is finitely normal on the neighborhood.

We state P. Montel's well-known criteria for finitely normal families[6].

THEOREM A. *A family $\mathcal{F}$ of functions $f(z)$, holomorphic in a domain D, is finitely normal in D if and only if the family $\mathcal{F}$ is uniformly bounded on every compact subset of D.*

In this paper, we consider the finite normality of the family of integer translations, $\{f(z+n) : n = 0, \pm 1, \pm 2, \cdots\}$, for an entire function f. We say a set G is periodic with period 1 if $z \pm 1 \in G$, for all $z \in G$.

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THEOREM 1.1. *For a nonempty open subset G of $\mathbf{C}$, there is an entire function f such that the set $\{f(z+n); n = 0, \pm 1, \pm 2, \cdots\}$ forms a finitely normal family exactly at z in G if and only if the set G is periodic with period 1 and its complement has no bounded component.*

To prove the existence part of the theorem, we use Arakelian's uniform approximation theorem by entire functions on closed subsets of the complex plane $\mathbf{C}$. We follow the phrasing of [7]. For a set E, we call any bounded component of the complement of E by *a hole of* E.

A closed set E, possibly unbounded, of the complex plane $\mathbf{C}$ is called an Arakelian set if it has no hole and if, for any closed disc D, the union of all holes of the set $E \cup D$ is bounded.

THEOREM B[7]. *Let E be an Arakelian set in the complex plane $\mathbf{C}$. If a function g is continuous on E and is holomorphic in the interior of E, then for any $\epsilon > 0$, there exists an entire function f such that*

$$|f(z) - g(z)| < \epsilon$$

for all $z \in E$.

DEFINITION 1.2. Let f be an entire function. We define the set $\mathbf{FN}(f)$ to be the set of all $z_0 \in \mathbf{C}$ such that the set $\{f(z+n) : n = 0, \pm 1, \pm 2, \cdots\}$ is finitely normal at z_0.

2. Some properties of FN(f)

From definition, the set $\mathbf{FN}(f)$ is a periodic open set with period 1. And the following theorem shows the set $\mathbf{FN}(f)$ need not be connected.

THEOREM 2.1. *There is an entire function f such that $\mathbf{FN}(f)$ is not connected.*

PROOF. For $j = -1, 0, 1$, we let

$$E_j = \{z \in \mathbf{C} : j - \frac{1}{4} \leq \operatorname{Im} z \leq j + \frac{1}{4}\}$$

and

$$E = E_{-1} \cup E_0 \cup E_1.$$

Then E is a closed set without holes. And for any closed disc D, the set $E \cup D$ has no hole. Hence E is an Arakelian set.

We define a function g on the set E by

$$g(z) = \begin{cases} 0, & \text{if } z \in E_{-1} \cup E_1 \\ e^z, & \text{if } z \in E_0. \end{cases}$$

Then by Theorem B, there exists an entire function f such that

$$|f(z) - g(z)| < 1,$$

for all $z \in E$. Thus for all $z \in \text{Int}(E_{-1} \cup E_1)$ and integer n,

$$|f(z+n)| < 1.$$

Hence $\text{Int}(E_{-1} \cup E_1) \subset \mathbf{FN}(f)$ by Theorem A.

But for any $z_0 \in E_0$, we have

$$\begin{aligned} |f(z_0+n)| &> \left|e^{z_0+n}\right| - 1 \\ &\to \infty \end{aligned}$$

as $n \to \infty$. Thus $z_0 \notin \mathbf{FN}(f)$ and $E_0 \cap \mathbf{FN}(f) = \phi$. Therefore the set $\mathbf{FN}(f)$ is not connected.

For every entire function f, the set $\mathbf{FN}(f)$ is a subset of $\mathbf{FN}(f')$, but $\mathbf{FN}(f) \neq \mathbf{FN}(f')$ in general. For example, let $f(z) = z$, then $\mathbf{FN}(f) = \phi$, but $\mathbf{FN}(f') = \mathbf{C}$.

THEOREM 2.2. *If f is an entire function, then* $\mathbf{FN}(f) \subset \mathbf{FN}(f')$.

PROOF. For any $z_0 \in \mathbf{FN}(f)$, we can choose positive numbers ϵ and M such that $\bar{B}(z_0, 2\epsilon) \subset \mathbf{FN}(f)$ and

$$|f(z+n)| < M$$

for all $z \in \bar{B}(z_0, 2\epsilon)$ and integers n. By Cauchy's theorem,

$$\begin{aligned} |f'(z+n)| &= \left| \frac{1}{2\pi i} \int_{|w-(z_0+n)|=\epsilon} \frac{f(w)}{(w-(z+n))^2} dw \right| \\ &\leq \frac{M}{\epsilon} \end{aligned}$$

for all $z \in \bar{B}(z_0, \epsilon)$ and integer n. Hence $z_0 \in \mathbf{FN}(f')$. This completes the proof.

THEOREM 2.3. *For every entire function f, the set* $\mathbf{FN}(f)$ *has no hole.*

PROOF. Let E be a bounded component of the complement of the set $\mathbf{FN}(f)$. We choose a bounded curve $\gamma \subset \mathbf{FN}(f)$ so that it encloses the set E. Since $\gamma \subset \mathbf{FN}(f)$ and it is a compact set, there exist points $z_1, z_2, \cdots, z_s$ and numbers $r_1, r_2, \cdots, r_s$, M_1, $M_2, \cdots, M_s$ such that

$$\gamma \subset \bigcup_{j=1}^{s} B(z_j, r_j)$$

and

$$|f(z+n)| \leq M_j$$

for all $z \in B(z_j, r_j)$ and for every integer n.

Let

$$M = \max\{M_j : 1 \leq j \leq s\};$$

then we obtain

$$|f(z+n)| \leq M$$

for all $z \in \gamma$ and for every integer n. So by the maximum modulus theorem, we have

$$|f(z+n)| \leq M$$

for all $z \in E$ and for every integer n. This means that $E \subset \mathbf{FN}(f)$. This is a contradiction. Hence the complement of $\mathbf{FN}(f)$ has no bounded component.

3. A construction of an Arakelian set

We need some topological facts. We state them as lemmas without proofs.

LEMMA 3.1. *Let Ω be an open set in the complex plane* $\mathbf{C}$. *For each positive integer n, we let*

$$J_n = \{z : d(z, \Omega^c) \geq \frac{1}{n}\} \cap \{z : |Re\, z| \leq n, |\mathrm{Im}\, z| \leq n\},$$

then every component of J_n^c contains a component of Ω^c (See [2]).

LEMMA 3.2. *If A, B are connected sets and if $A \cap B \neq \phi$, then the set $A \cup B$ is connected.*

LEMMA 3.3. *If a connected set A intersects a set B and its complement at the same time, then A intersects the boundary of B.*

LEMMA 3.4. *If Ω is an open set in the complex plane $\mathbf{C}$, then every component of Ω is open in $\mathbf{C}$.*

Now we construct an Arakelian set from a given open set having no hole.

THEOREM 3.5. *Let G be an open set in the complex plane $\mathbf{C}$ without holes. For each positive integer n, we define a compact subset K_n of G by*

$$K_n = \{z : d(z, G^c) \geq \frac{1}{n}\} \cap S_n$$

where

$$S_n = \{z : n-1 \leq |\mathrm{Re}\, z| \leq n,\ |\mathrm{Im}\, z| \leq n\}.$$

Then the set

$$K = \bigcup_{n=1}^{\infty} K_n$$

is an Arakelian set.

PROOF. The set K has no hole: Let

$$S = \bigcup_{n=1}^{\infty} S_n;$$

then $K \subset S$ and S^c has only two components in $\mathbf{C}$, say

$$S_+^c = \{z \in S^c : \mathrm{Im}\, z > 0\}$$

and

$$S_-^c = \{z \in S^c : \mathrm{Im}\, z < 0\},$$

and both of them are unbounded.

First we show, if the set K^c has a bounded component C_K, then C_K is a subset of S. Next we derive a contradiction by showing the component C_K contains a component of G^c which is unbounded.

Suppose that C_K intersects the complement of the set S, say $C_K \cap S_+^c \neq \phi$. Then the set $C_K \cup S_+^c$ is connected by Lemma 3.2. Since $S^c \subset K^c$, $C_K \cup S_+^c \subset K^c$. So the component C_K contains unbounded set S_+^c, this contradicts the boundedness of C_K. We get the same result for the case $C_K \cap S_-^c \neq \phi$. Hence C_K must be contained in S.

By assumption C_K is bounded and by Lemma 3.4 it is open in C. Since $C_K \subset S$, there is a positive number N such that

$$C_K \cap \mathrm{Int}(S_N) \neq \phi \text{ and } C_K \cap \mathrm{Int}(S_{N+1}) = \phi. \tag{3.1}$$

Let J_N be the compact subset of $\mathbf{C}$ which is defined for the open set G as in Lemma 3.1. By the definition of the set K_N, we have $K_N = J_N \cap S_N$. From (3.1), we can choose a point $z_0 \in C_K \cap \mathrm{Int}(S_N)$ and we obtain

$$z_0 \in \mathrm{Int}(S_N) \cap C_K \subset S_N \cap K_N^c = S_N \cap (J_N \cap S_N)^c = S_N \cap J_N^c. \tag{3.2}$$

Here we let C_{J_N} be the component of J_N^c containing z_0 and we shall show $C_{J_N} \subset C_K$. If not, *i.e.*, $C_{J_N} \not\subset C_K$, then the connected set C_{J_N} intersects both of C_K and C_K^c. So there is a point $w_0 \in \partial C_K \cap C_{J_N}$ by Lemma 3.3. Since $C_K \subset S$, there is a number $n_0 (n_0 \leq N)$ such that

$$w_0 \in \partial C_K \cap C_{J_N} \cap S_{n_0}.$$

Because $C_K (\subset K^c)$ is open, $\partial C_K \subset K$ and by the definition of K_{n_0} we have

$$w_0 \in \partial C_K \cap S_{n_0} \subset K \cap S_{n_0} = K_{n_0}$$

and

$$d(w_0, G^c) \geq \frac{1}{n_0} \geq \frac{1}{N}. \tag{3.3}$$

On the other hand,

$$\begin{aligned} w_0 &\in C_{J_N} \cap \{z : |\mathrm{Re}\, z| \leq N,\ |\mathrm{Im}\, z| \leq N\} \\ &\subset J_N^c \cap \{z : |\mathrm{Re}\, z| \leq N,\ |\mathrm{Im}\, z| \leq N\}, \end{aligned}$$

so the definition of J_N gives

$$d(w_0, G^c) < \frac{1}{N}. \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\frac{1}{N} \le d(w_0, G^c) < \frac{1}{N}.$$

This is impossible; thus we can conclude $C_{J_N} \subset C_K$.

But Lemma 3.1 says that the component C_{J_N} of J_N^c contains a component of G^c which is unbounded. This contradicts the boundedness of C_K. Therefore every component of K^c is unbounded.

Now, we shall show that for any closed disc D, the union of all holes of the set $K \cup D$ is bounded. We consider the following two cases.

CASE 1. $D \cap S = \phi$.

Recall that S^c has only two components S_+^c and S_-^c. Let $D = \bar{B}(\hat{w}, r)$ be given closed disc; since $D \subset S^c$, by the connectedness of D, $D \subset S_+^c$ or $D \subset S_-^c$. We assume $D \subset S_+^c$. Since S_+^c is connected(path connected), for any two points z_1, z_2 in $S_+^c \backslash D$, there exists a path $\gamma : [0,1] \to S_+^c$ such that $\gamma(0) = z_1$, $\gamma(1) = z_2$.

If $\gamma([0,1])$ does not intersect D, then γ is a path in $S_+^c \backslash D$. If $\gamma([0,1])$ intersects D, we let

$$\delta = \frac{1}{3}\min\{d(D,S),\, d(z_i, D; i = 1,2)\}$$

and take the circle C_δ centered at $\hat{w}$ of radius $r+\delta$. Then $C_\delta \subset S_+^c \backslash D$ and the path γ intersects the circle C_δ. Let $0 < t_1 \le t_2 < 1$ be the smallest and largest numbers respectively such that $C_\delta \cap \gamma(t_i) \ne \phi$, $i = 1, 2$. Let $\Gamma = \gamma([0,t_1]) \cup \gamma_1 \cup \gamma([t_2, 1])$ where γ_1 is the arc from $\gamma(t_1)$ to $\gamma(t_2)$ along the circle C_δ counterclockwise. Then Γ is a path in $S_+^c \backslash D$ joining z_1 to z_2. Therefore the set $S_+^c \backslash D$ is connected. By the same arguments, we can prove the set $S_-^c \backslash D$ is connected.

Since $S_+^c \backslash D (\subset (S \cup D)^c)$ is connected, unbounded and $(S \cup D)^c \subset (K \cup D)^c$, there is the unbounded component C_{SD} of $(K \cup D)^c$ containing $S_+^c \backslash D$.

Let C_D be a component of $(K \cup D)^c$ which is different from C_{SD}. Since $(K \cup D)^c \subset K^c$, there exists a component C_{KD} of K^c such that $C_D \subset C_{KD}$. Since $C_{KD} \cap S_+^c = \phi$, $C_{KD} \cap D = \phi$ and $C_{KD} \subset (K \cup D)^c$. Thus each component of $(K \cup D)^c$ which is different from C_{SD} is also a component of K^c, so it is unbounded by the first argument of this proof.

Hence, if $D \cap S = \phi$, then $(K \cup D)^c$ has no bounded component and there is nothing to prove.

CASE 2. $D \cap S \neq \phi$.

Let N be the largest integer such that $D \cap S_N \neq \phi$. Let

$$S_{DN} = \{z : |\mathrm{Re}\, z| \leq N, |\mathrm{Im}\, z| \leq N\} \cup D \cup S;$$

then $K \cup D \subset S_{DN}$ and S_{DN}^c has only two components

$$S_{DN}^+ = \{z \in S_{DN}^c : \mathrm{Im}\, z > 0\}$$

and

$$S_{DN}^- = \{z \in S_{DN}^c : \mathrm{Im}\, z < 0\}$$

in $\mathbf{C}$ and they are unbounded.

We carry out the proof in two steps.

LEMMA 3.5.1. *If C_D is a bounded component of $(K \cup D)^c$, then C_D satisfies:*

(a) $C_D \subset \mathrm{Int}(S_{DN})$;

(b) $\partial C_D \cap D \neq \phi$.

PROOF OF LEMMA 3.5.1. (a) If C_D intersects the complement of the S_{DN}, assume $C_D \cap S_{DN}^+ \neq \phi$, then the set $C_D \cup S_{DN}^+$ is connected by Lemma 3.2. Because $S_{DN}^c \subset (K \cup D)^c$, the set $C_D \cup S_{DN}^+$ is a subset of $(K \cup D)^c$. So the component C_D of $(K \cup D)^c$ contains S_{DN}^+ which is unbounded. This contradicts the assumption that C_D is bounded. We obtain the same result for the case $C_D \cap S_{DN}^- \neq \phi$.

Hence, if C_D is a bounded component of $(K \cup D)^c$, then it must be contained in S_{DN}. But by Lemma 3.4, the component C_D is open in $\mathbf{C}$, so $C_D \subset \mathrm{Int}(S_{DN})$. (b) To show $\partial C_D \cap D \neq \phi$, we assume $\partial C_D \cap D = \phi$

and let C_K be the component of K^c containing C_D. We shall show that $C_D = C_K$.

If C_D is a proper subset of C_K, then $\partial C_D \backslash \partial C_K \neq \phi$ and we have

$$\partial C_D \backslash \partial C_K \subset \bar{C}_D \backslash \partial C_K \subset \bar{C}_K \backslash \partial C_K = C_K, \tag{3.5}$$

because C_K is open in $\mathbf{C}$. From the assumption, $\partial C_D \cap D = \phi$, we have $\partial C_D \subset D^c$. So with (3.5), we can choose a point

$$z_1 \in (\partial C_D \backslash \partial C_K) \cap C_K \cap D^c.$$

Since $C_K \cap D^c$ is open, we can choose an open ball $B(z_1, \epsilon)$ such that

$$B(z_1, \epsilon) \subset C_K \cap D^c \ (\subset (K \cup D)^c).$$

But $z_1 \in \partial C_D$, so $B(z_1, \epsilon)$ intersects C_D and the set $C_D \cup B(z_1, \epsilon)$ is a connected subset of $(K \cup D)^c$ by Lemma 3.2. Hence the component C_D contains the open ball $B(z_1, \epsilon)$. This contradicts $z_1 \in \partial C_D \subset C_D^c$.

Thus $C_D = C_K$ and C_D is unbounded because K^c has no bounded component. But we assumed C_D is bounded. This is a contradiction. Therefore every bounded component of $(K \cup D)^c$ must intersect the closed disc D on its boundary. This completes the proof.

LEMMA 3.5.2. *Let C_D^α be a component of $(K \cup D)^c$ with the following properties:*

(a) $C_D^\alpha \subset \mathrm{Int}(S_{DN})$;

(b) C_D^α intersects the line segment,

$$L^+ = \{z : \mathrm{Re}\, z = N_0 + \frac{1}{2}, |\mathrm{Im}\, z| \leq N_0 + 1\}$$

where $N_0 = \max\{N, 3\}$. (Recall, N is the largest integer such that $D \cap S_N \neq \phi$.) Then the number of components such as C_D^α is finite.

PROOF OF LEMMA 3.5.2. By the definition of L^+, $L^+ \subset S_{N_0+1}$. Suppose there are infinitely many C_D^α's with the properties *(a)* and *(b)*. We take a point z_α from each set $C_D^\alpha \cap L^+$. For each z_α, there is a point

$w_\alpha \in G^c$ such that: (1) $d(z_\alpha, w_\alpha) < \frac{1}{N_0+1}$: $z_\alpha \in C_D \cap S_{N_0+1}$;
(2) $B(w_\alpha, \frac{1}{N_0+1}) \subset \mathrm{Int}(S_{N_0+1})$: For any $w \in B(w_\alpha, \frac{1}{N_0+1})$,

$$\mathrm{Re}\, z_\alpha - \frac{2}{N_0+1} < \mathrm{Re}\, w < \mathrm{Re}\, z_\alpha + \frac{2}{N_0+1}$$

and

$$N_0 < \mathrm{Re}\, w < N_0 + 1, \tag{3.6}$$

because $\mathrm{Re}\, z_\alpha = N_0 + \frac{1}{2}$, $N_0 \geq 3$ and w, z_α are in $B(w_\alpha, \frac{1}{N_0+1})$. Since the open set $B(w_\alpha, \frac{1}{N_0+1})$ $(\subset K^c \cap D^c)$ intersects C_D^α, the set $C_D^\alpha \cup B(w_\alpha, \frac{1}{N_0+1})$ is a connected subset of $(K \cup D)^c$. Hence the component C_D^α contains the open ball $B(w_\alpha, \frac{1}{N_0+1})$. But by *(a)*, $C_D^\alpha \subset \mathrm{Int}(S_D)$, so with (3.6) $B(w_\alpha, \frac{1}{N_0+1}) \subset \mathrm{Int}(S_{N_0+1})$;
(3) If $\alpha \neq \beta$, then $B(w_\alpha, \frac{1}{N_0+1}) \cap B(w_\beta, \frac{1}{N_0+1}) = \phi$: It follows from the fact $B(w_\alpha, \frac{1}{N_0+1}) \subset C_D^\alpha$ which was proved in (2), because $C_D^\alpha \cap C_D^\beta = \phi$.

From (1), (2) and (3), we conclude that the set $\mathrm{Int}(S_{N+1})$ contains infinitely many disjoint open balls whose radius is $\frac{1}{N_0+1}$. This is impossible. Hence the number of components with the properties *(a)* and *(b)* is finite. We get the same result for the line segment L^- where

$$L^- = \{z : \mathrm{Re}\, z = -(N_0 + \frac{1}{2}),\ |\mathrm{Im}\, z| \leq N + 1\}.$$

From Lemma 3.5.1, we can say; if C_D is a bounded component of $(K \cup D)^c$ which is not contained in the set

$$\{z : |\mathrm{Re}\, z| \leq N_0 + 1,\ |\mathrm{Im}\, z| \leq N_0 + 1\},$$

then it must intersect the line segment L^+ or L^-.

But the number of such components is finite by Lemma 3.5.2. Therefore the union of all bounded components of $(K \cup D)^c$ is bounded. This completes the proof of Case 2.

Since the set K is closed in $\mathbf{C}$, K is an Arakelian set

4. Proof of Theorem 1.1

To prove our theorem, we need a few lemmas.

LEMMA 4.1. *In Theorem 3.5, if the open set G is periodic with period 1, then for any compact subset Q of $G \cap \{z : -1 < \operatorname{Re} z < 1\}$, there is* an integer $N(\geq 2)$ such that $n \geq N$ implies

$$Q + n \subset (K_{n-1} \cup K_n)$$

where $Q + n = \{z + n : z \in Q\}$.

PROOF. Let $d(Q, G^c) = \delta > 0$; then there exists an integer $N(\geq 2)$ satisfying

$$\max\{|\operatorname{Im} z| : z \in Q\} \leq N - 1 \quad \text{and} \quad \delta \geq \frac{1}{N-1}.$$

For $n \geq N$ and $z \in Q$, $z + n \in (S_{n-1} \cup S_n)$ by the definition of S_n. Now by the periodicity of G, we obtain

$$\begin{aligned} d(z+n, G^c) &\geq d(Q, G^c) = \delta \\ &\geq \frac{1}{N-1} > \frac{1}{n-1} > \frac{1}{n}. \end{aligned}$$

By the definition of K_n, we have $z + n \in K_{n-1} \cup K_n$.

We omit the proof of the following lemma.

LEMMA 4.2. *Let Ω be an open connected set and W be a countable set without finite limit points. Then the set $\Omega \backslash W$ is connected.*

LEMMA 4.3. *Let F be a closed set and W be a countable set without finite limit point. Then each component C_{FW} of $(F \cup W)^c$ has the form*

$$C_{FW} = C_F \backslash W$$

where C_F is the component of F^c containing C_{FW}.

PROOF. Since the set $C_{FW} \cap (C_F \backslash W)$ is non empty and by Lemma 4.2 the set $C_F \backslash W$ is connected, the union $C_{FW} \cup (C_F \backslash W)$ is a connected subset of $(F \cup W)^c$ by Lemma 3.2. So the component C_{FW} of $(F \cup W)^c$ contains the set $C_F \backslash W$.

On the other hand,

$$C_{FW} = C_{FW} \backslash W \subset C_F \backslash W.$$

Hence we can conclude $C_{FW} = C_F \backslash W$.

PROOF OF THEOREM 1.1. Throughout this proof the sets S_n, K_n and K are same as in Theorem 3.5.

Let $\{w_i\}$ be a countable dense subset of $\{z \in G^c : 0 \leq \operatorname{Re} z < 1\}$. For each positive integer i, we let

$$W_i = \{w_{ij} = w_i + j^2 + i : j \geq i\},$$

and

$$W = \bigcup_{i=1}^{\infty} W_i.$$

Then W is a closed subset of G^c. And the set W is countable without finite limit point.

Claim: Let $E = K \cup W$, then E is an Arakelian set.

PROOF OF CLAIM. From Theorem 3.5 we know that K is an Arakelian set. By Lemma 4.3. each component C_E of E^c has the form $C_E = C_K \backslash W$ where C_K is the component of K^c containing C_E. Since K is an Arakelian set, C_K is unbounded. Hence the component C_E of E^c is unbounded.

For a closed disc D, let $\cup_{\alpha \in A} C_{ED}^{\alpha}$ be the union of all bounded components of $(E \cup D)^c$. Then for each $\alpha \in A$, there is the bounded component C_{KD}^{α} of $(K \cup D)^c$ such that $C_{ED}^{\alpha} = C_{KD}^{\alpha} \backslash W$ by Lemma 4.3. Since

$$\bigcup_{\alpha \in A} C_{ED}^{\alpha} \subset \bigcup_{\alpha \in A} C_{KD}^{\alpha}$$

and $\cup_{\alpha \in A} C_{KD}^{\alpha}$ is bounded, the union $\cup_{\alpha \in A} C_{ED}^{\alpha}$ is bounded.

Finally, K and W are closed sets. Therefore $E = K \cup W$ is an Arakelian set.

Now, we define a function g on the set E by

$$g(z) = \begin{cases} 0, & \text{if } z \in K \\ z, & \text{if } z \in W, \end{cases} \tag{4.1}$$

then g is holomorphic on E because $K \cap W = \phi$. So by the Claim and Theorem B, there is an entire function f such that

$$|g(z) - f(z)| < 1 \tag{4.2}$$

for all $z \in E$.

We shall show that $\mathbf{FN}(f) = G$. By the periodicity of G, it is enough to check for z with $\alpha \leq \operatorname{Re} z < \alpha + 1$, α is a real number.

For any $z_0 \in \{z \in G : -\frac{1}{2} \leq \operatorname{Re} z < \frac{1}{2}\}$, we choose a positive number $\epsilon (< \frac{1}{2})$ so that $\bar{B}(z_0, \epsilon) \subset G$. There is a positive integer $N (\geq 2)$ such that $n \geq N$ implies $\bar{B}(z + n, \epsilon) \subset K_{n-1} \cup K_n \, (\subset K)$ for all $z \in \bar{B}(z_0, \epsilon)$ by Lemma 4.1. For $n \geq N$ and $z \in \bar{B}(z_0, \epsilon)$, we have

$$|f(z + n)| < 1$$

by (4.1) and (4.2). So we can conclude that

$$|f(z + n)| \leq \max_{z \in R_{N+1}} \{|f(z)|\} + 1$$

for all $z \in \bar{B}(z_0, \epsilon)$ where

$$R_{N+1} = \{z : |\operatorname{Re} z| \leq N + 1, \ |\operatorname{Im} z| \leq N + 1\}.$$

Since the entire function f bounded on the compact set R_{N+1}, the family $\{f(z + n) : n = 0, \pm 1, \pm 2, \cdots\}$ is uniformly bounded on $\bar{B}(z_0, \epsilon)$. Therefore z_0 belongs to $\mathbf{FN}(f)$.

On the other hand, $\{w_i\}$ is a dense subset of $\{z \in G^c : 0 \leq \operatorname{Re} z < 1\}$, so for any $w_0 \in \{z \in G^c : 0 \leq \operatorname{Re} z < 1\}$ and $\epsilon > 0$, $B(w_0, \epsilon)$ contains a point w_i in $\{w_i\}$. Take a subsequence $\{f(n^2+i+z) : n \geq i\}$; then we have

$$\begin{aligned}|f(w_i+n^2+i)| &= |f(w_{in})| \\ &\geq |w_{in}+n^2+i|-1\end{aligned}$$

which tends to infinity as $n \to \infty$. Hence w_0 does not belong to the set $\mathbf{FN}(f)$.

So the entire function f satisfies desired property $\mathbf{FN}(f)=G$. This completes the proof.

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