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ON COMPLETE LIE SUPERALGEBRAS

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ABSTRACT. This paper is concerned with the equivalent conditions, the decomposition and the uniqueness of complete Lie superalgebras and a complete Lie superalgebra Der$\mathfrak{g}$.

1. Introduction

Interest in Lie superalgebras, motivated mainly by problems in homotopy theory and particle physics, dates back to the early 60′s. As a natural generalization of Lie algebras, Lie superalgebra appears to be a promising ingredient of unified field theories currently under investigation. They are also interesting from a purely mathmatical point of view, and their enveloping algebras provide various rich classes of associative algebras. The definition of a complete Lie algebra was given by N.Jacobson in 1962 [1]. In recent years some theories of complete Lie algebras have been delveloping by Dae Ji Meng [2].

In this paper, we announce and prove some results on complete Lie superalgebras.

Notations used in this paper are as follows :

$\mathfrak{g}$	finite dimensional Lie superalgebra over a field of characteristic 0
$C(\mathfrak{g})$	the centre of a Lie superalgebra $\mathfrak{g}$
Der$\mathfrak{g}$	the Lie superalgebra of superderivations of $\mathfrak{g}$
ad$\mathfrak{g}$	the inner derivation algebra of $\mathfrak{g}$

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$C_{\mathfrak{g}}(A)$ the centralizer of an ideal A in $\mathfrak{g}$

$A \lhd \mathfrak{g}, \mathfrak{g} \rhd A$ A is a graded ideal of $\mathfrak{g}$

All ideals are graded ideals. Other definitions are referred to [5].

2. Complete Lie superalgebras

DEFINITION 2.1 [5]. Let $\mathfrak{g} = \mathfrak{g}_o \oplus \mathfrak{g}_1$ be a superalgebra whose multipication is denoted by a pointed bracket $<,>$. We call $\mathfrak{g}$ a Lie superalgebra if the multiplication satisfies the following identities :

$$< x, y >= -(-1)^{\alpha\beta} < y, x > \text{ and}$$
$$\text{(graded skew-symmetry)}$$
$$(-1)^{\alpha\gamma} < x, < y, z >> +(-1)^{\beta\alpha} < y, < z, x >>$$
$$+ (-1)^{\gamma\beta} < z, < x, y >>= 0$$
$$\text{(graded Jacobi identity)}$$

for all $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta, z \in \mathfrak{g}_\gamma; \alpha, \beta, \gamma \in \mathbb{Z}_2$. Then $\mathfrak{g}_0$ is a Lie algebra.

DEFINITION 2.2 [5]. Let $S = S_0 \oplus S_1$ be a superalgebra.

$\mathrm{Hom}(S) = \{f : S \longrightarrow S$ is a linear mapping$\}$ is a Lie superalgebra if one defines the $\mathbb{Z}_2$-gradation by $\mathrm{Hom}(S)_\alpha = \{f \in \mathrm{Hom}(S) \mid f(S_\beta) \subset S_{\alpha+\beta}\}$. The Lie superalgebra associated with $\mathrm{Hom}(S)$ will be denoted by $\mathrm{pl}(S)$. Let $(\mathrm{Der}S)_\alpha, \alpha \in \mathbb{Z}_2$ be the subspace of all $D \in \mathrm{pl}(S)_\alpha$ such that $D(xy) = (Dx)y + (-1)^{\alpha\zeta} x(Dy)$ for all $x \in S_\zeta,\ y \in S$, where $\alpha, \zeta \in \mathbb{Z}_2$. Then $\mathrm{Der}S = (\mathrm{Der}S)_0 + (\mathrm{Der}S)_1$ is a graded subalgebra of $\mathrm{pl}(S)$. The elements of $\mathrm{Der}S$ are called superderivations, and $\mathrm{Der}S$ is called the Lie superalgebra of superderivations of S.

THEOREM 2.1 [5]. *Let the base field F of $\mathfrak{g}$ be algebraically closed. Each of the following statements is strictly stonger than the foregoing one.*

1) *$\mathfrak{g}$ does not contain non-zero solvable graded ideals.*
2) *$\mathfrak{g}$ is the direct product of finitely many simple Lie superalgebras.*
3) *The Killing form of $\mathfrak{g}$ is non-degenerate.*
4) *All finite-dimensional graded representations of $\mathfrak{g}$ are completely reducible.*

It is well known that in the Lie algebra case these statements are mutually equivalent, and characterize the semi-simple Lie algebras.

DEFINITION 2.3. A Lie superalgebra $\mathfrak{g}$ is called a complete Lie superalgebra if C($\mathfrak{g}$)=0 and Der$\mathfrak{g}$=ad$\mathfrak{g}$.

EXAMPLES. 1) It is remarkable that a simple Lie superalgebra may have outer superderivations ; in fact, it is known that simple Lie algebras have only inner derivations. But classical simple Lie superalgebras osp$(n, 2r)(n, r \geq 1)$, spl$(n, m)(n \neq m)$, $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, Γ_2 and Γ_3 are complete Lie superalgebras by [5].

2) Though $\mathfrak{g}_0$ is a complete Lie algebra, $\mathfrak{g}$ need not be a complete Lie superalgebra. For example, let $\mathfrak{g}_0$ be a semisimple Lie algebra and $\mathfrak{g}_1$ be a finite dimensional vector space. Define $< x, y >= 0$ for all $x \in \mathfrak{g}_1$ and $y \in \mathfrak{g}$, and $< x, y >= [x, y]$ for all $x, y \in \mathfrak{g}_0$, where [,] is the braket operation of the Lie algebra. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra. Since C($\mathfrak{g}$)$\neq$ 0, $\mathfrak{g}$ is not complete. But $\mathfrak{g}_0$ is complete.

The completeness of $\mathfrak{g}_0$ and $\mathfrak{g}$ are equivalent under the following condition.

THEOREM 2.2. *If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ has the nondegenerate Killing form over an algebraically closed field, then $\mathfrak{g}_0$ and $\mathfrak{g}$ are complete.*

PROOF. Since $\mathfrak{g}$ has the nondegenerate Killing form, Der$\mathfrak{g}$=ad$\mathfrak{g}$ by [5]. Since C($\mathfrak{g}$) is abelian, C($\mathfrak{g}$) is solvable. Then C($\mathfrak{g}$)=0 by Theorem 2.1. Hence $\mathfrak{g}$ is complete.

Let K be the nondegenerate Killing form of $\mathfrak{g}$. Then its restriction to $\mathfrak{g}_0$, $K\mid_{\mathfrak{g}_0 \times \mathfrak{g}_0}$, which is the Killing form of $\mathfrak{g}_0$, is nondegenerate by [1]. Therefore $\mathfrak{g}_0$ is a semisimple Lie algebra and Der$\mathfrak{g}$=ad$\mathfrak{g}$ by [2]. Since $\mathfrak{g}_0$ has no nonzero solvable ideals and C($\mathfrak{g}_0$)=0, $\mathfrak{g}_0$ is complete.

LEMMA 2.3. *If A is an ideal of g, then $C_{\mathfrak{g}}(A)$ is also an ideal of $\mathfrak{g}$. In particular, $C(\mathfrak{g}) = C_{\mathfrak{g}}(\mathfrak{g})$ and $C(A) = C_A(A)$ are ideals.*

PROOF. If $x \in C_{\mathfrak{g}}(A)$ and $a \in A = A_1 \oplus A_2$, then $< x, a >= 0$, where $x = x_0 + x_1$, $a = a_0 + a_1$, $x_0 \in \mathfrak{g}_0$, $x_1 \in \mathfrak{g}_1$, $a_0 \in A_0$ and $a_1 \in A_1$. Since A is graded, $a_0, a_1 \in A$. Then $< x_0 + x_1, a_0 >=< x_0 + x_1, a_1 >= 0$ and $< x_0, a_0 > + < x_1, a_0 >=< x_0, a_1 > + < x_1, a_1 >= 0$. Hence $< x_0, a_0 >, < x_1, a_1 >, < x_0, a_1 >, < x_1, a_0 > \in A_0 \cap A_1$ and $< x_0, a_0 >=<$

$x_0, a_1 >=< x_1, a_0 >=< x_1, a_1 >= 0$. Hence $< x_0, a >=< x_1, a >= 0$. Therefore $x_0, x_1 \in \mathrm{C}_{\mathfrak{g}}(A)$ and $\mathrm{C}_{\mathfrak{g}}(A)$ is graded.

Let $x \in \mathrm{C}_{\mathfrak{g}}(A)$ and $y \in \mathfrak{g}$. Since $a = a_0 + a_1 \in A$, $a_0, a_1 \in A$ by the gradation of A. Then

$$<< x, y >, a_0 >=<< x_0 + x_1, y_0 + y_1 >, a_0 >= 0 \text{ and}$$
$$<< x, y >, a_1 >=<< x_0 + x_1, y_0 + y_1 >, a_1 >= 0$$

by the graded Jacobi identity and $<< x, y >, a >= 0$. Hence $\mathrm{C}_{\mathfrak{g}}(A)$ is an ideal. The second and the third are similar.

THEOREM 2.4. *If A is a complete ideal of $\mathfrak{g}$, there exists an ideal B such that $\mathfrak{g} = A \oplus B$.*

PROOF. Let $B = \mathrm{C}_{\mathfrak{g}}(A)$. By Lemma 2.3, $\mathrm{C}_{\mathfrak{g}}(A)$ is an ideal of $\mathfrak{g}$. For each $a \in \mathfrak{g}$, $\mathrm{ad}a \in \mathrm{Der}A$ since $A \lhd \mathfrak{g}$. Since $\mathrm{Der}A = \mathrm{ad}A$, there exists a derivation D in $\mathrm{Der}A$ such that $\mathrm{ad}a = D$. Therefore there exists $r \in A$ such that $Dx =< a, x >=< r, x >$ for $\forall x \in A$. Then $< a - r, x >= 0$ and $a - r \in \mathrm{C}_{\mathfrak{g}}(A) = B$. Hence $a = b + r$ for some $b \in B$. $A \cap B = A \cap \mathrm{C}_{\mathfrak{g}}(A) = \mathrm{C}(A) = 0$ since A is complete. Therefore $\mathfrak{g} = A \oplus B$.

DEFINITION 2.4. Let $\mathfrak{g}$ be a Lie superalgebra and $\mathrm{h}(\mathfrak{g}) = \mathfrak{g} \oplus \mathrm{Der}\mathfrak{g}$. Define the bracket in $\mathrm{h}(\mathfrak{g})$ by $< x + D, y + E >=< x, y > + Dy - (-1)^{\alpha\beta} Ex + < D, E >$, where $E \in (\mathrm{Der}\ \mathfrak{g})_{\alpha}, x \in \mathfrak{g}_{\beta}$. Then $\mathrm{h}(\mathfrak{g})$ is a Lie superalgebra. We call $\mathrm{h}(\mathfrak{g})$ a holomorph Lie superalgebra.

REMARK.

1) If $\mathrm{C}(\mathfrak{g})=0$, then $\mathrm{C}(\mathrm{Der}\mathfrak{g})=0$, since $\mathrm{C}(\mathrm{Der}\mathfrak{g}) \subset \mathrm{Der}\mathfrak{g}$ and $\mathrm{C}(\mathrm{Der}\mathfrak{g}) \subset \mathrm{C}_{\mathrm{Der}\mathfrak{g}}(\mathrm{ad}\mathfrak{g}) = 0$.
2) $\mathfrak{g} \lhd \mathrm{h}(\mathfrak{g})$ and $\mathrm{h}(\mathfrak{g})/\mathfrak{g} \simeq \mathrm{Der}\mathfrak{g}$.
3) $\mathfrak{g} \cap \mathrm{C}_{\mathrm{h}(\mathfrak{g})}(\mathfrak{g}) = \mathrm{C}(\mathfrak{g})$.

LEMMA 2.5. *If $\mathfrak{g}$ is a Lie superalgebra, then*

1) $C_{h(\mathfrak{g})}(\mathfrak{g}) = \{x - adx \mid x \in \mathfrak{g}\}$,
2) *the map θ of $h(\mathfrak{g})$ defined by $\theta(x + D) = adx - x + D$ ($x \in \mathfrak{g}, D \in Der\mathfrak{g}$) is an automorphism of $h(\mathfrak{g})$ and $\theta(\mathfrak{g}) = C_{h(\mathfrak{g})}(\mathfrak{g})$.*

PROOF. 1) If $x + D \in C_{h(\mathfrak{g})}(\mathfrak{g})$, then $< x + D, y >=< x, y > + < D, y >=< x, y > + Dy = (\mathrm{ad}x + D)y = 0$ for all $x \in \mathfrak{g}$. Then $\mathrm{ad}x + D = 0$ implies D=-ad$x$. Conversely, $< x - \mathrm{ad}x, y >=< x, y > -\mathrm{ad}x(y) =< x, y > - < x, y >= 0$. Therefore, $x - \mathrm{ad}x \in C_{h(\mathfrak{g})}(\mathfrak{g})$.

2) Since θ is a linear mapping and $\theta^2(x + D) = \theta(\mathrm{ad}x - x + D) = \mathrm{ad}(-x) + x + \mathrm{ad}x + D = x + D$, θ^2 is the identity.

For any $x \in \mathfrak{g}, \theta(x) = \mathrm{ad}x - x \in C_{h(\mathfrak{g})}(\mathfrak{g})$ by 1) and $\theta(\mathfrak{g}) \subset C_{h(\mathfrak{g})}(\mathfrak{g})$. On the other hand, for $x - \mathrm{ad}x \in C_{h(\mathfrak{g})}\mathfrak{g}$, $x - \mathrm{ad}x = \theta(-x) \in \theta(\mathfrak{g})$. Therefore, $C_{h(\mathfrak{g})}\mathfrak{g} \subset \theta(\mathfrak{g})$. Hence $\theta(\mathfrak{g}) = C_{h(\mathfrak{g})}(\mathfrak{g})$. To show that $\theta \in \mathrm{Aut}(h(\mathfrak{g}))$,

$$\begin{aligned}
< \theta(x + D), \theta(y + E) > &=< \mathrm{ad}x - x + D, \mathrm{ad}y - y + E > \\
&= < -x, -y > + (\mathrm{ad}x + D)(-y) - (-1)^{\alpha\beta}(\mathrm{ad}y + E)(-x) \\
&\quad + < \mathrm{ad}x + D, \mathrm{ad}y + E > \\
&= - Dy + (-1)^{\alpha\beta} < y, x > + (-1)^{\alpha\beta} Ex + < \mathrm{ad}x, \mathrm{ad}y > \\
&\quad + < D, \mathrm{ad}y > + < \mathrm{ad}x, E > + < D, E > \\
&= - Dy + (-1)^{\alpha\beta} < y, x > + (-1)^{\alpha\beta} Ex + \mathrm{ad} < x, y > \\
&\quad + \mathrm{ad}D(y) - (-1)^{\alpha\beta}\mathrm{ad}E(x) + < D, E > \\
&= \theta(< x, y > + Dy - (-1)^{\alpha\beta} Ex + < D, E >) \\
&= \theta < x + D, y + E > \quad \text{for all } x \in \mathfrak{g}_\alpha, \\
&\quad \text{where } D \in (\mathrm{Der}\mathfrak{g})_\alpha, y \in \mathfrak{g}_\beta \text{ and } E \in (\mathrm{Der}\mathfrak{g})_\beta.
\end{aligned}$$

Hence θ is a homomorphism. Since θ^2 is the identity, $\theta \in \mathrm{Aut}(h(\mathfrak{g}))$.

THEOREM 2.6. *The following conditions are equivalent :*

1) *$\mathfrak{g}$ is a complete Lie superalgebra.*
2) *Any splitting extension $\mathfrak{a}$ by $\mathfrak{g}$ is a trivial extension and $\mathfrak{a} = \mathfrak{g} \oplus C_{\mathfrak{a}}(\mathfrak{g})$.*
3) *$h(\mathfrak{g}) = \mathfrak{g} \oplus C_{h(\mathfrak{g})}(\mathfrak{g})$.*

PROOF. 1)$\Longrightarrow$ 2) Let $\mathfrak{a}$ be a splitting extension by $\mathfrak{g}$. Then $\mathfrak{g} \lhd \mathfrak{a}$ and $C_{\mathfrak{a}}(\mathfrak{g}) \lhd \mathfrak{a}$. Since $C(\mathfrak{g}) = 0$ by 1), $\mathfrak{g} \cap C_{\mathfrak{a}}(\mathfrak{g}) = 0$. Let $x \in \mathfrak{a}$. Since $\mathfrak{g} \lhd \mathfrak{a}, \mathrm{ad}x\mathfrak{g} \subset \mathfrak{g}$. Then the restriction $\mathrm{ad}x \mid_{\mathfrak{g}}$ on $\mathfrak{g}$ is a derivation of $\mathfrak{g}$. But $\mathfrak{g}$ is complete, so $\mathrm{ad}x \mid_{\mathfrak{g}}$ is an inner derivation of $\mathfrak{g}$. We define π by $\pi(x) = \mathrm{ad}x \mid_{\mathfrak{g}}$ for $\forall x \in \mathfrak{a}$. Since Der$\mathfrak{g}$=ad$\mathfrak{g} \simeq \mathfrak{g}$, the map π is a homomorphism from $\mathfrak{a}$ onto Der$\mathfrak{g}$ with kernal $C_{\mathfrak{a}}(\mathfrak{g})$. Hence dim$\mathfrak{a}$=dim$\mathfrak{g}$+dim$C_{\mathfrak{a}}(\mathfrak{g})$.

2)$\Longrightarrow$3) It is clear by setting $\mathfrak{a} = \mathrm{h}(\mathfrak{g})$.

3) $\Longrightarrow$1) $\mathrm{C}(\mathfrak{g}) = \mathfrak{g} \cap \mathrm{C}_{\mathrm{h}(\mathfrak{g})}(\mathfrak{g}) = 0$ by Remark. $\mathrm{C}_{\mathrm{h}(\mathfrak{g})}(\mathfrak{g}) \simeq \mathrm{h}(\mathfrak{g})/\mathfrak{g}$ by 3) and $\mathrm{h}(\mathfrak{g})/\mathfrak{g} \simeq \mathrm{Der}\mathfrak{g} \simeq \mathrm{C}_{\mathrm{h}(\mathfrak{g})}(\mathfrak{g}) \simeq \mathfrak{g}$ by Remark. Since $\mathrm{C}(\mathfrak{g}) = 0$, $\mathfrak{g} \simeq \mathrm{ad}\mathfrak{g}$. Hence $\mathrm{Der}\mathfrak{g} \simeq \mathrm{ad}\mathfrak{g}$.

THEOREM 2.7. *Let $\mathfrak{g}$ be a Lie superalgebra over a field F and K be an extension of F. Then $\mathfrak{g} \otimes_F K$ is complete if and only if $\mathfrak{g}$ is complete.*

PROOF. By a result of [4], $\mathrm{Der}(\mathfrak{g} \otimes_F K) = \mathrm{Der}\mathfrak{g} \otimes_F K$.

3. The decomposition of the complete Lie superalgebra and uniqueness

LEMMA 3.1. *Let the Lie superalgebra $\mathfrak{g}$ be decomposed into the direct sum of two ideals. i.e., $\mathfrak{g} = A \oplus B$.*
Then we have

1) *$C(\mathfrak{g})$ has the decomposition $C(\mathfrak{g}) = C(A) \oplus C(B)$.*
2) *If $C(\mathfrak{g})=0$, then*

$$ad\mathfrak{g} = adA \oplus adB \text{ and}$$

$$Der\mathfrak{g} = DerA \oplus DerB.$$

3) *$\mathfrak{g}$ is complete if and only if A and B are complete.*

PROOF. 1) By Lemma 2.3, $\mathrm{C}(A)$ and $\mathrm{C}(B)$ are ideals of $\mathfrak{g}$. $\mathrm{C}(A) \cap \mathrm{C}(B) = 0$, since $A \cap B = 0$. Let $a + b \in \mathrm{C}(A) \oplus \mathrm{C}(B)$, where $a \in \mathrm{C}(A)$ and $b \in \mathrm{C}(B)$. Then $< a, A >= 0$ and $< b, B >= 0$.

Let $p + q \in A \oplus B$, where $p \in A$ and $q \in B$. Then $< a + b, p + q >=< a + b, p > + < a + b, q >= 0$, since $< b, p >\in A \cap B$. Hence $a + b \in \mathrm{C}(\mathfrak{g})$ and $\mathrm{C}(A) \oplus \mathrm{C}(B) \subset \mathrm{C}(\mathfrak{g})$. Let $x = a + b \in \mathrm{C}(\mathfrak{g})$, where $a \in A$ and $b \in B$. Then $< a + b, \mathfrak{g} >=< a + b, A + B >= 0$ and $< a, A >=< x - b, A >= 0$ since $x \in \mathrm{C}(\mathfrak{g})$ and $< b, A >\subset< B, A >= 0$. Hence $a \in \mathrm{C}(A)$. Similarly, $b \in \mathrm{C}(B)$. Therefore $\mathrm{C}(\mathfrak{g}) \subset \mathrm{C}(A) \oplus \mathrm{C}(B)$.

2) For $D \in \mathrm{Der}A$, extend it to a linear transformation on $\mathfrak{g}$ by setting $D(a+b) = Da$ for $a \in A$ and $b \in B$. Obviously, $D \in \mathrm{Der}\mathfrak{g}$ and $\mathrm{Der}A \subset \mathrm{Der}\mathfrak{g}$. Similarly, $\mathrm{Der}B \subset \mathrm{Der}\mathfrak{g}$. Let $a \in A_\alpha, b \in B$ and $D \in (\mathrm{Der}\mathfrak{g})_\beta$. Then $< Da, b >= \mathrm{D} < a, b > -(-1)^{\alpha\beta} < a, Db >= -(-1)^{\alpha\beta} < a, Db > \in A \cap B$. Since $A \cap B = 0, < Da, b >=< a, Db >= 0$. Suppose $Da = a' + b'$, where $a' \in A$ and $b' \in B$. Then $< Da, b >=< a', b > + < b', b >= 0$ for all $b \in B$ and $b' \in \mathrm{C}(B)$. Since $\mathrm{C}(\mathfrak{g}) = \mathrm{C}(A) \oplus \mathrm{C}(B) = 0, b' = 0$. Hence $Da = a' \in A$. Therefore $D(A) \subset A$. Similarly, $D(B) \subset B$.

Let $D \in \mathrm{Der}\mathfrak{g}$ and $a + b \in A + B$, where $a \in A$ and $b \in B$. Define E and F by $E(a+b) = D(a)$ and $F(a+b) = D(b)$. Then $E \in \mathrm{Der}A$ and $F \in \mathrm{Der}B$. Hence $D = E + F \in \mathrm{Der}A + \mathrm{Der}B$. Since $\mathrm{Der}A \cap \mathrm{Der}B = 0, \mathrm{Der}\mathfrak{g} = \mathrm{Der}A \oplus \mathrm{Der}B$ as a vector space.

Let $D \in (\mathrm{Der}A)_\alpha$, $\mathrm{E} \in (\mathrm{Der}\mathfrak{g})_\beta$ and $b \in B$. Then $< E, D > b = EDb - (-1)^{\alpha\beta} DEb = 0$. Hence $\mathrm{Der}A \lhd \mathrm{Der}\mathfrak{g}$. Similarly $\mathrm{Der}B \lhd \mathrm{Der}\mathfrak{g}$.

3) ($\Longrightarrow$) Since $\mathfrak{g}$ is complete, $\mathrm{C}(\mathfrak{g})=0$. And $\mathrm{C}(A) = \mathrm{C}(B)=0$ by 1).

Since $\mathrm{ad}\mathfrak{g} = \mathrm{Der}\mathfrak{g}$, $\mathrm{ad}\mathfrak{g} = \mathrm{ad}A \oplus \mathrm{ad}B$ and $\mathrm{Der}\mathfrak{g} = \mathrm{Der}A \oplus \mathrm{Der}B$ by 2). Since $\mathrm{ad}A \subset \mathrm{Der}A$ and $\mathrm{ad}B \subset \mathrm{Der}B$, $\mathrm{ad}A = \mathrm{Der}A$ and $\mathrm{ad}B = \mathrm{Der}B$. Hence A and B are complete Lie superalgebras.

($\Longleftarrow$) $\mathrm{C}(\mathfrak{g})=\mathrm{C}(A)\oplus \mathrm{C}(B)=0$ by 1). $\mathrm{Der}\mathfrak{g} = \mathrm{Der}A \oplus \mathrm{Der}B = \mathrm{ad}A \oplus \mathrm{ad}B = \mathrm{ad}\mathfrak{g}$ by 2).

Definition 3.1. A complete Lie superalgebra $\mathfrak{g}$ is called a simply complete Lie superalgebra if any non-trivial ideal of $\mathfrak{g}$ is not complete.

Example. A simple and complete Lie superalgebra is a simply complete Lie superalgebra.

Theorem 3.2.

1) *Any complete Lie superalgebra can be decomposed into the direct sum of simply complete ideals.*
2) *A complete Lie superalgebra is simply complete if and only if it is indecomposable.*

Proof. 1) If $\mathfrak{g}$ is simply complete, the result is true. If $\mathfrak{g}$ is not simply complete, there exists a nonzero minimal complete ideal A of $\mathfrak{g}$ such that $\mathfrak{g} = A \oplus \mathrm{C}_\mathfrak{g}(A)$ by Theorem 2.4. Continuing this process to

$C_{\mathfrak{g}}(A)$, we get the decomposition of $\mathfrak{g}$ into simply complete ideals since an ideal of $C_{\mathfrak{g}}(A)$ is also an ideal of $\mathfrak{g}$. 2) This follows from 1).

LEMMA 3.3. *Let $\mathfrak{g} = B \oplus C$, A be a subalgebra of $\mathfrak{g}$ and $B \subset A$. Then $A = B \oplus (C \cap A)$ and $A \triangleleft \mathfrak{g}$ if and only if $(A \cap C) \triangleleft C$. Here B and C are ideals but need not be graded.*

PROOF. It is clear.

DEFINITION 3.2. If an endomorphism (which is homogeneous of degree zero) φ of a Lie superalgebra $\mathfrak{g}$ satisfies $\varphi \mathrm{ad} x = (\mathrm{ad} x)\varphi$ for $\forall x \in \mathfrak{g}$, then φ is called a $\mathfrak{g}$-endomorphism of $\mathfrak{g}$.

EXAMPLE. Let $\mathfrak{g}$ be a Lie superalgebra with decomposition of ideals, i.e., $\mathfrak{g} = A \oplus B$, and π be the projection into A with respect to this decomposition. Then π is a $\mathfrak{g}$-endomorphism of $\mathfrak{g}$.

In fact, for any $x = x_1 + x_2, y = y_1 + y_2, x_1 \in A, y_1 \in A, x_2 \in B$ and $y_2 \in B$, we have

$$\pi \mathrm{ad} x(y) = < x_1, y_1 > = \mathrm{ad} x \pi(y).$$

LEMMA 3.4. *Let φ be a $\mathfrak{g}$-endomorphism of the Lie superalgebra $\mathfrak{g}$. Then there exists $k \in \mathbb{N}$ such that*

$$\mathfrak{g} = Ker\varphi^k \oplus Im\varphi^k.$$

Furthermore, if $\mathfrak{g}$ is indecomposable, then we have

$$\varphi^k = 0 \text{ or } \varphi \in Aut\mathfrak{g}.$$

PROOF. Let $f(\lambda) = \lambda^k x(\lambda)$ be the minimal polynomial of φ, where $(\lambda, x(\lambda)) = 1$. In fact, if $k = 0$ then $f(\lambda) = x(\lambda)$ and $x(\varphi) = 0$. Hence $\varphi^n + \alpha\varphi^{n-1} + \cdots + \beta\varphi + \gamma = 0$ and $1 = -\gamma^{-1}\varphi(\varphi^{n-1} + \cdots + \beta)$. Therefore $\mathfrak{g} \subset \mathrm{Im}\varphi$.

Now suppose $k \geq 1$, then there exist polynomials $u(\lambda)$ and $v(\lambda)$ such that $u(\lambda)x(\lambda) + v(\lambda)\lambda^k = 1$. And $u(\varphi)x(\varphi) + v(\varphi)\varphi^k$ is the identity.

If $y \in \mathfrak{g}$, then $y = u(\varphi)x(\varphi)(y) + v(\varphi)\varphi^k(y) \in \mathrm{Ker}\varphi^k + \mathrm{Im}\varphi^k$ and $\mathfrak{g} = \mathrm{Ker}\varphi^k + \mathrm{Im}\varphi^k$. If $y \in \mathrm{Ker}\varphi^k \cap \mathrm{Im}\varphi^k$, then $\varphi^k(y) = 0$ and $y = \varphi^k(z)$

for some $z \in \mathfrak{g}$. Hence $y = u(\varphi)x(\varphi)\varphi^k(z) = u(\varphi)f(\varphi)(z) = 0$ and $\mathfrak{g} = \mathrm{Ker}\varphi^k \oplus \mathrm{Im}\varphi^k$ as a vector space.

If $x_0 + x_1 \in \mathrm{Ker}\varphi^k$, where $x_0 \in \mathfrak{g}_0$ and $x_1 \in \mathfrak{g}_1$, then $\varphi^k(x_0) = \varphi^k(x_1) = 0$ since φ is homogeneous of degree 0. Hence $\mathrm{Ker}\varphi^k$ is graded and $\mathrm{Ker}\,\varphi^k \triangleleft \mathfrak{g}$.

Let $a = a_0 + a_1 \in \mathrm{Im}\varphi^k$, where $a_0 \in \mathfrak{g}_0$ and $a_1 \in \mathfrak{g}_1$. Then $a_0 + a_1 = \varphi^k(y_0 + y_1)$ for some $y_0 \in \mathfrak{g}_0$, $y_1 \in \mathfrak{g}_1$. Hence $\varphi^k(y_0) + \varphi^k(y_1) = a_0 + a_1$ and $\varphi^k(x_0) - a_0 = a_1 - \varphi^k(y_1) \in \mathfrak{g}_0 \cap \mathfrak{g}_1$. Then $a_0, a_1 \in \mathrm{Im}\varphi^k$ and $\mathrm{Im}\varphi^k$ is graded. Let $x \in \mathfrak{g}$ and $a \in \mathrm{Im}\varphi^k$. Then $a = \varphi^k(y)$ for some $y \in \mathfrak{g}$. Hence $< x, a >=< x, \varphi^k(y) >= \varphi^k < x, y > \in \mathrm{Im}\varphi^k$ since φ^k is a $\mathfrak{g}$-endomorphism. Therefore $\mathrm{Im}\varphi^k \triangleleft \mathfrak{g}$.

Since $\mathfrak{g}$ is indecomposable, $\mathrm{Ker}\,\varphi^k = \mathfrak{g}$ or $\mathrm{Im}\varphi^k = \mathfrak{g}$. If $\mathrm{Ker}\,\varphi^k = \mathfrak{g}$, then $\varphi^k = 0$. If $\mathrm{Im}\varphi^k = \mathfrak{g}$, then $\varphi^k \in \mathrm{Aut}\mathfrak{g}$.

LEMMA 3.5. *Let $\mathfrak{g}$ be an indecomposable Lie superalgebra. Let $\varphi_1, \varphi_2, \cdots, \varphi_n$ and $\sum_{k=1}^{j} \varphi_k (j = 1, 2, \cdots n)$ be $\mathfrak{g}$-endomorphisms of $\mathfrak{g}$ such that*

$$\varphi_1 + \varphi_2 + \cdots + \varphi_n = id.$$

Then there exists an index i with $\varphi_i \in Aut\mathfrak{g}$.

PROOF. By induction on n. In the case of $n = 1$, it is trivial.

For $n = 2$, since $\varphi_1 + \varphi_2 = \mathrm{id}$ and $\varphi_1(\varphi_1 + \varphi_2) = (\varphi_1 + \varphi_2)\varphi_1$, $\varphi_1\varphi_2 = \varphi_2\varphi_1$. Now suppose that $\varphi_1, \varphi_2 \notin \mathrm{Aut}\mathfrak{g}$. By Lemma 3.4, there exists $k_i (i = 1, 2)$ such that $\varphi_i^{k_i} = 0$. Choose $k > k_1 + k_2$. Then id$=(\varphi_1 + \varphi_2)^k = \sum_i^j \binom{k}{j} \varphi_1^{k-j} \varphi_2^j = 0$. It is a contradiction. Therefore $\varphi_1 \in \mathrm{Aut}\mathfrak{g}$ or $\varphi_2 \in \mathrm{Aut}\mathfrak{g}$.

For $n > 2$, set $\psi = \sum_{i=1}^{n-1} \varphi_i$. So ψ and φ_n are $\mathfrak{g}$-endomorphisms of $\mathfrak{g}$ and $\psi + \varphi_n = \mathrm{id}$. Hence $\varphi_n \in \mathrm{Aut}\mathfrak{g}$ or $\psi \in \mathrm{Aut}\mathfrak{g}$. If $\varphi_n \in \mathrm{Aut}\mathfrak{g}$, then the result is true. If $\psi \in \mathrm{Aut}\mathfrak{g}$, then $\psi^{-1}, \varphi_1\psi^{-1}, \cdots, \varphi_n\psi^{-1}$ are $\mathfrak{g}$-endomorphisms and $(\sum_{i=1}^{n-1} \varphi_i)\psi^{-1} = \psi\psi^{-1}$ =id. By inductive assumption, $\varphi_i\psi^{-1} \in \mathrm{Aut}\mathfrak{g}$ for some i. Therefore $\varphi_i \in \mathrm{Aut}\mathfrak{g}$.

THEOREM 3.6. *Let $\mathfrak{g}$ be a Lie superalgebra with trivial centre. Suppose $\mathfrak{g}$ has the decomposition of direct sum of ideals :*

$$\mathfrak{g} = H_1 \oplus H_2 \oplus \ldots \oplus H_m \tag{3.1}$$

and

$$\mathfrak{g} = T_1 \oplus T_2 \oplus \ldots \oplus T_n \tag{3.2}$$

where $H_1, \cdots, H_m$ *and* $T_1, \cdots, T_n$ *are indecomposable. Then*

$$m = n$$

and if necessary, by a permutation of the summands

$$H_i = T_i,\ i = 1, 2, \cdots, m.$$

PROOF. We prove by induction on n. For $n = 1$. $\mathfrak{g}$ is indecomposable, so $m = n = 1$ and $H_1 = T_1 = \mathfrak{g}$.

Now suppose $n > 1$, so that $m > 1$. Denote the projection of $\mathfrak{g}$ onto H_1 with respect to the decomposition (3.1) by π, the imbedding of H_1 into $\mathfrak{g}$ by σ, the projection of $\mathfrak{g}$ onto T_i with respect to the decomposition (3.2) by ρ_i and the imbedding of T_i into $\mathfrak{g}$ by τ_i. Then $\pi, \rho_1, \cdots, \rho_n$ and $\sum_{i=1}^{j} \rho_i (j = 1, 2, \cdots, n)$ are $\mathfrak{g}$-endomorphisms of $\mathfrak{g}$ and $\rho_1 + \rho_2 \cdots, \rho_n = \mathrm{id}_{\mathfrak{g}}$. Set $\pi_i^* = \pi\tau_i = \pi\ |_{T_i}$ and $\rho_i^* = \rho_i\sigma = \rho_i\ |_{H_1}$ for any $i = 1, 2, \cdots, n$. Then $\pi_i^*\rho_i^*$ is an H_1-endomorphism of H_1. The map $\sum_{i=1}^{j} \tau_i\rho_i$ defined by

$$\left[\sum_{i=1}^{j} \tau_i\rho_i\right](x) = \sum_{i=1}^{j} \tau_i\rho_i(x),\ \text{with } x \in \mathfrak{g},$$

is a $\mathfrak{g}$-endomorphism of $\mathfrak{g}$. Hence

$$\pi\left[\sum_{i=1}^{n} \tau_i\rho_i\right]\sigma = \sum_{i=1}^{j} \pi_i^*\rho_i^* = \sum_{i=1}^{j} \pi_i^*\rho_i\ |_{H_1}$$

is an H_1-endomorphism of H_1. For each $h \in H_1$, we have

$$h = \pi(h) = \pi\left[\sum_{i=1}^{n} \rho_i(h)\right] = \sum_{i=1}^{n} \pi_i^*\rho_i^*(h).$$

i.e.,

$$\sum_{i=1}^{n} \pi_i^* \rho_i^* = \mathrm{id}_{H_1}.$$

Therefore, by Lemma 3.5, there exists an index i such that $\pi_i^* \rho_i^* \in \mathrm{Aut} H_1$. If necessary, by a permutation of $T_1, T_2, \cdots, T_n$, we can assume that $i = 1$. i.e., $\pi_1^* \rho_1^* \in \mathrm{Aut} H_1$. This implies that ρ_1^* is injective.

Set $H = H_2 \oplus \cdots \oplus H_m$ and $T = T_2 \oplus \cdots \oplus T_n$. Then $\mathrm{C}(H)=\mathrm{C}(T)=0$ by Lemma 3.1 and $H = \mathrm{C}_{\mathfrak{g}}(H_1)$, $T = \mathrm{C}_{\mathfrak{g}}(T_1)$, $\mathrm{C}_{\mathfrak{g}}(H) = H_1$, $\mathrm{C}_{\mathfrak{g}}(T) = T_1$ and $T = \mathrm{Ker}\rho_1$. Therefore

$$0 = \mathrm{Ker}\rho_1^* = H_1 \cap \mathrm{Ker}\rho_1 = H_1 \cap T.$$

So $H_1 \subset \mathrm{C}_{\mathfrak{g}}(T)$. By Lemma 3.3, we have

$$T_1 = H_1 \oplus (T_1 \cap H).$$

But T_1 is indecomposable, so $T_1 = H_1$. Hence $H = T$. By inductive assumption, the theorem follows.

COROLLARY 3.7. *Let $\mathfrak{g}$ be a complete Lie superalgebra. Then*

1) $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$, where $\mathfrak{g}_i$ is a simply complete Lie superalgebra and an ideal of $\mathfrak{g}$ for each i.

2) This decomposition is unique up to the order of the ideals.

PROOF. 1) follows from Theorem 3.2.

2) Since $\mathfrak{g}$ is complete, $\mathrm{C}(\mathfrak{g}) = 0$. It follows from Theorem 3.6.

4. Complete Lie superalgebra Der$\mathfrak{g}$

DEFINITION 4.1. A subspace A of a Lie superalgebra $\mathfrak{g}$ is called a characteristic ideal of $\mathfrak{g}$, if $D(A) \subset A$ for $\forall D \in \mathrm{Der}\mathfrak{g}$.

Notice that a characteristic ideal is also an ideal.

THEOREM 4.2. *Let $\mathfrak{g}$ be an Lie superalgebra with trivial centre and ad$\mathfrak{g}$ be a characteristic ideal of Der$\mathfrak{g}$. Then Der$\mathfrak{g}$ is complete. Furthermore, if $\mathfrak{g}$ is indecomposable and $< \mathfrak{g}, \mathfrak{g} >= \mathfrak{g}$. Then Der$\mathfrak{g}$ is simply complete.*

PROOF. Since $C(\mathfrak{g})=0$, $\mathfrak{g} \simeq \mathrm{ad}\mathfrak{g}$. Set $\mathfrak{a} = \mathrm{Der}\mathfrak{g}$, so $\mathfrak{g} \lhd \mathfrak{a}$. Let $\mathfrak{b}$ be a splitting extension by $\mathfrak{a}$. i.e., $\mathfrak{a} \lhd \mathfrak{b}$. Then we have $\mathrm{ad}b \in \mathrm{Der}\mathfrak{a}$ for $\forall b \in \mathfrak{b}$. Since $\mathfrak{g}$ is a characteristic ideal of $\mathfrak{a}$, there exists an element $a \in \mathfrak{a}$ such that $\mathrm{ad}a\,|_{\mathfrak{g}} = \mathrm{ad}b\,|_{\mathfrak{g}}$. Then $\mathrm{ad}(a-b)\,|_{\mathfrak{g}} = 0$. i.e., $a - b \in C_{\mathfrak{b}}(\mathfrak{g})$. Thus we obtain $\mathfrak{b} = \mathfrak{a} + C_{\mathfrak{a}}(\mathfrak{g})$. But $\mathfrak{a} \cap C_{\mathfrak{b}}(\mathfrak{g}) = C_{\mathfrak{a}}(\mathfrak{g}) = 0$ and $\mathfrak{a} \lhd \mathfrak{b}$. Therefore $\mathfrak{b} = \mathfrak{a} \oplus C_{\mathfrak{b}}(\mathfrak{g})$. From this, we have immediately $C_{\mathfrak{b}}(\mathfrak{g}) \subset C_{\mathfrak{b}}(\mathfrak{a})$. Therefore, we have $\mathfrak{b} = \mathfrak{a} \oplus C_{\mathfrak{b}}(\mathfrak{a})$. Then $\mathfrak{a}$ is a complete Lie superalgebra by Theorem 2.6.

Now, suppose that $\mathrm{Der}\mathfrak{g}$ is not simply complete. Then there exists a simply complete ideal A. Then there exists an ideal B such that $\mathfrak{a} = A \oplus B$ by Theorem 2.4.

For $y, z \in \mathfrak{g}$, there exist $y_1, z_1 \in A, y_2, z_2 \in B$ such that $y = y_1 + y_2$ and $z = z_1 + z_2$. Hence $< y, z > = < y_1, z > + < y_2, z >$ and $< y_1, z > \in (A \cap \mathfrak{g})$ and $< y_2, z > \in B \cap \mathfrak{g}$. Then $\mathfrak{g} = < \mathfrak{g}, \mathfrak{g} > = (A \cap \mathfrak{g}) \oplus (B \cap \mathfrak{g})$. But $A \cap \mathfrak{g} = 0$ (or $B \cap \mathfrak{g} = 0$) because $\mathfrak{g}$ is indecomposable. Then $\mathfrak{g} \subset (B \cap \mathfrak{g})$ and $\mathfrak{g} \subset B$. Therefore $A \subset C_{\mathfrak{a}}(\mathfrak{g}) = 0$. Hence $\mathfrak{a}$ is indecomposable and $\mathrm{Der}\mathfrak{g}$ is simply complete by Theorem 3.2.

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