Comm. Korean Math. Soc. 11 (1996), No. 2, pp. 315–321

ON LEFT, RIGHT WEAKLY PRIME IDEALS ON po–SEMIGROUPS

SANG KEUN LEE AND YOUNG IN KWON

ABSTRACT. Recently, N. Kehayopulu [4] introduced the concepts of weakly prime ideals of ordered semigroups. In this paper, we define the concepts of left(right) weakly prime and left(right) semiregular. Also we investigate the properties of them.

I. Introduction

Recently, N. Kehayopulu [4] introduced the concepts of weakly prime ideals of ordered semigroups and gave the characterizations of prime ideals of ordered semigroups analogous to the characterizations of weakly prime ideals of rings considered by N. H. McCoy [1,2] and O. Steinfeld [8]. Kehayopulu's results are the extension of the concepts of left right ideals and weakly prime ideals of ordered semigroups to the characterizations for ring given by O. Steinfeld [8]. And authors [7] gave some characterizations of weakly prime ideals of ordered semigroups which are the improvent of results due to Kehayopulu.

In this paper, we define the new concepts in an ordered po–semigroup S and we give some interesting results.

A po–semigroup(: ordered semigroup) [3-7] is an ordered set $(S, \leq)$ at the same time a semigroup such that :

$$a \leq b \quad \Longrightarrow \quad ca \leq cb \quad \text{and} \quad ac \leq bc \text{ for all } a, b, c \in S.$$

For $A, B \subseteq S$, let $AB = \{ab : a \in A, b \in B\}$.

Received November 1, 1995. Revised March 4, 1996.

1991 AMS Subject Classification: 06F05.

Key words and phrases: po–semigroup, left(right) weakly prime, weakly prime, left(right) semiregular element, left(right) semiregular, Archimedian, left(right) ideal, ideal.

DEFINITION 1[3-5, 7]. Let S be a $po-$semigroup and A is a nonempty subset of S. A is called a left(resp.right) ideal of S if

1) $SA \subseteq A$(resp. $AS \subseteq A$).
2) $a \in A, b \leq a$ for $b \in S \implies b \in A$.

A is called an ideal of S if it is a left and right ideal of S.

Now we give new concepts in $po-$semigroups which are left(resp. right) semiregular, semiregular, left(resp. right) weakly prime and left (resp. right) Archimedian.

DEFINITION 2. An element a of a $po-$semigroup S is a left(resp.right) semiregular if

$$a \leq xaya \quad (\text{resp. } a \leq ax'ay')$$

for some $x, y, x', y' \in S$. An element a of a $po-$semigroup S is a semiregular element[6] if

$$a \leq xaya \quad \text{or} \quad a \leq ax'ay'$$

for some $x, y, x', y' \in S$.

A subsemigroup T of S is semiregular(resp. left semiregular, right semiregular) if all elements of T are semiregular(resp. left semiregular, right semiregular).

DEFINITION 3. Let S be a $po-$semigroup and $T \subseteq S$. T is called left(resp.right) weakly prime if for all left(resp. right) ideals A, B of S such that

$$AB \subseteq T \implies A \subseteq T \quad \text{or} \quad B \subseteq T.$$

T is called a left(resp.right) weakly prime ideal of S if T is a left(resp. right) ideal which is weakly prime.

DEFINITION 4. A subsemigroup T of a $po-$semigroup S is called left(resp.right) Archimedian if for every $a, b \in T$ there exist $x, y \in T$ such that

$$a^n \leq xb(\text{resp. } a^n \leq by)$$

for some positive integer n.

A subsemigroup T of a $po-$semigroup S is called Archimedian[5] if for every $a, b \in T$ there exist $x, y \in T$ such that

$$a^n \leq xby$$

for some positive integer n.

NOTATION. For $H \subseteq S$,

$$(H] = \{t \in S : t \leq h \text{ for some } h \in H\}.$$

We denote by $L(a)$(resp. $R(a)$) the left(resp. right) ideal of S generated by a. One can easily prove that :

$$L(a) = (a \cup Sa], \quad R(a) = (a \cup aS].$$

We note the following lmama(see [4]).

LEMMA 1. *Let S be a po-semigroup. Then we have*

1) $A \subseteq (A]$ *for all* $A \subseteq S$.
2) $(A] \subseteq (B]$ *for* $A \subseteq B \subseteq S$.
3) $(A](B] \subseteq (AB]$ *for all* $A, B \subseteq S$.
4) $((A]] \subseteq (A]$ *for all* $A \subseteq S$.
5) *For every left(resp. right) ideal T of S,* $(T] = T$.
6) *If A and B are left(resp. right) ideals of S, then $(AB]$ and $A \cup B$ are left (resp. right) ideal of S.*
7) $(Sa]$*(resp.* $(aS]$*) is a left(resp. right) ideal of S.*

LEMMA 2. *Let T be a po-semigroup of S. Then we have the following :*

1) *T is left(resp. right) semiregular if and only if for any $a \in T$,*

$$a \in (TaTa](\text{resp. } a \in (aTaT]).$$

2) *T is semiregular if and only if for any $a \in T$,*

$$a \in (TaTa] \quad \text{or} \quad a \in (aTaT]$$

3) *T is Archimedian(resp. left Archimedian, right Archimedian) if and only if for any $a, b \in T$,*

$$a^n \in (TbT](\text{resp. } a^n \in (Tb], \ a^n \in (bT])$$

2. Main Theors

Now we give some characterizations of left(right) weakly prime.

THEOREM 1. *Let S be a po−semigroup and T a left ideal of S. Then the following are equivalent :*

1) *T is left weakly prime.*
2) *If $(aSb] \subseteq T$ for some a and $b \in S$, then $a \in T$ or $b \in T$.*
3) *If $L(a)L(b) \subseteq T$ for some a and $b \in S$, then $\in T$ or $b \in T$.*
4) *If A is any subset of S and B is a left ideal of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

PROOF. 1) $\Longrightarrow$ 2): Assume that $(aSb] \subseteq T$ for some $a, b \in S$. Then by Lemma 1, we have

$$(Sa](Sb] \subseteq (SaSb] \subseteq (S(aSb]] \subseteq (ST] \subseteq (T] \subseteq T.$$

Since $(Sa]$ and $(Sb]$ are left ideals of S and T is left weakly prime, we get

$$(Sa] \subseteq T \quad \text{or} \quad (Sb] \subseteq T.$$

Let $(Sa] \subseteq T$. Then, by Lemma 1

$$\begin{aligned}(L(a))^2 &= (a \cup Sa](a \cup Sa] \\ &\subseteq (a^2 \cup aSa \cup Sa^2 \cup SaSa] \\ &\subseteq (Sa] \subseteq T.\end{aligned}$$

Since T is left weakly prime and $L(a)$ is a left ideal of S, we have $a \in L(a) \subseteq T$.

If $(Sb] \subseteq T, b \in T$ by similar method.

2) $\Longrightarrow$ 3) : Let $a, b \in S$, $a \notin T$ and $L(a)L(b) \subseteq T$. Then, by Lma 1

$$\begin{aligned}(aSb] &\subseteq (ab \cup aSb \cup Sab \cup SaSb] \\ &= ((a \cup Sa)(b \cup Sb)] \\ &\subseteq ((a \cup Sa](b \cup Sb]] \\ &\subseteq (L(a)L(b)] \\ &\subseteq (T] \subset T.\end{aligned}$$

Thus we have $b \in T$ by 2), since $a \notin T$.

3) $\Longrightarrow$ 4) : Assume that $AB \subseteq T, A \nsubseteq T$. Let $a \in A \backslash T$ and $b \in B$. Then we have

$$\begin{aligned} L(a)L(b) &= (a \cup Sa](b \cup Sb] \\ &\subseteq (A \cup SA](B \cup SB] \\ &\subseteq (AB \cup ASB \cup SAB \cup SASB] \\ &\subseteq (AB \cup SAB] \\ &\subseteq (T \cup ST] \\ &\subseteq (T] = T. \end{aligned}$$

Hence we have $a \in T$ or $b \in T$ by 3).

4) $\Longrightarrow$ 1) : It is obvious.

We can get the following theorem 2 by the similar method to Theorem 1.

THEOREM 2. *Let S be a po$-$semigroup and T a right ideal of S. The following are equivalent :*

1) *T is a right weakly prime.*
2) *If $(aSb] \subseteq T$ for some a and $b \in S$, then $a \in T$ or $b \in T$.*
3) *If $R(a)R(b) \subseteq T$ for some a and $b \in S$, then $a \in T$ or $b \in T$.*
4) *If A is a right ideal of S and B is any subset of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

If T is an ideal of S, then we get the following theorem by Theorem 1, Theorem 2 and Theorem due to Kehayopulu[4].

COROLLARY 3. *Let S be a po$-$semigroup and T an ideal of S. The followings are equivalent :*

1) *T is left weakly prime.*
2) *T is right weakly prime.*
3) *T is weakly prime.*
4) *If $(aSb] \subseteq T$ for some a and $b \in S$, then $a \in T$ or $b \in T$.*
5) *If $L(a)L(b) \subseteq T$ for some a and $b \in S$, then $a \in T$ or $b \in T$.*
6) *If $R(a)R(b) \subseteq T$ for some a and $b \in S$, then $a \in T$ or $b \in T$.*

7) *If A is a right ideal of S and B is any subset of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*
8) *If A is any subset of S and B is a left ideal of S such that $AB \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.*

PROOF. We note that an ideal is a left and right ideal. Hence we can prove that 1) $\Longrightarrow$ 4) $\Longrightarrow$ 5) $\Longrightarrow$ 8) $\Longrightarrow$ 1) by Theorem 1, 2) $\Longrightarrow$ 4) $\Longrightarrow$ 6) $\Longrightarrow$ 7) $\Longrightarrow$ 2) by Theorem 2 and 3) $\Longleftrightarrow$ 4) by Theorem due to Kehayopulu[4].

THEOREM 4. *Let a be a left semiregular elent of a po$-$semigroup S. If L is a left ideal not containing a, then there exists a left weakly prime ideal P not containing a.*

PROOF. Let $\mathcal{L}$ be the class of left ideals not containing a. Applying Zorn's Lemma to $\mathcal{L}$ (with inclusion), there exists the maximal left ideal M not containing a. Let L_1 and L_2 be left ideals such that $L_1L_2 \subset M$, $L_1 \nsubseteq M$ and $L_2 \nsubseteq M$. Since $L_1 \cup M$ and $L_2 \cup M$ are left ideals, by the maximality of M we have $a \in L_1 \cap L_2$. Since a is a left semiregular element, by Lemma 2 we have

$$a \in (SaSa] \subseteq (SL_1SL_2] \subseteq (L_1L_2] \subseteq M.$$

Hence $a \in M$, which is a contradiction. Therefore M is a left weakly prime ideal not containing a.

We can get easily the following corollary 5 from Theorem 3.

COROLLARY 5. *Let a be a left semiregular element of a po$-$semigroup S. If $a^n \notin L$ for some left ideal L and for all positive integers n, then there exists a left weakly prime ideal P of S such that $a^n \notin P$ for all positive integers n.*

COROLLARY 6. *If $a \in P_l^*$, the intersection of all left weakly prime ideals of a po$-$semigroup S, and L is any proper left ideal of a po$-$semigroup S, then $a^n \in L$ for some positive integer n.*

PROOF. Assume that $a^n \notin L$ for all positive integers n. By Corollary 5, there exists a left weakly prime ideal P such that $a^n \notin P$ for all positive integers n. Hence $a \notin P$ and so $a \notin P_l^*$, which is a contradiction.

THEOREM 7. *P^*, the intersection of all weakly prime ideals of a po−semigroup S, is left Archimedian and right Archimedian.*

PROOF. It is obvious that P^* is a subsemigroup.

Let $a, b \in P^*$. Since P^* is also a left ideal of S, by Corollary 6 we get

$$a^n \in (b \cup Sb]$$

for some positive integer n. Since P^* is also a right ideal of S,

$$\begin{aligned} a^{n+1} &\in (a](b \cup Sb] \subseteq (ab \cup aSb] \\ &\subseteq (P^*b \cup P^*Sb] \subseteq (P^*b] \end{aligned}$$

Hence P^* is a left Archimedean. By the similar method,

$$a^{n+1} \in (bP^*]$$

for any $a, b \in P^*$ and for some positive integer n. Hence P^* is a right Archimedian.

REMARK. Wobtain the dualities for right semiregular, right Archimedian, right weakly prime as Theorem 4, Corollary 5, Corollary 6.

References

1. N. H. McCoy, *Prime ideals in general rings*, Amer. J. Math. **71** (1949), 823-833.
2. ______, *The theory of rings*, MacMillan.
3. N. Kehayopulu, *On completely regular poe−sigroups*, Math. Japon. **37** (1992), 123-130.
4. ______, *On weakly prime ideals of ordered semigroups*, Math. Japon. **35** (1990), 1051-1056.
5. ______, *On weakly commutative poe−sigroups*, Math. Japon. **36** (1991), 427-432.
6. S. K. Lee and Y. I. Kwon, *On completely regular and semiregular po−semigroups*, to appear.
7. ______, *On weakly prime ideals of po−semigroups*, to appear.
8. O. Steinfeld, *Rarks on a paper of N. H. McCoy*, Publ Math. Debrecen **3** (1953-54), 171-173.

Department of Mathematics
Gyeongsang National University
Chinju 660-701, Korea