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PSEUDO VALUATION DOMAINS

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ABSTRACT. In this paper we characterize strongly prime ideals and prove a theorem: an integral domain R is a PVD if and only if every maximal ideal M of R is strongly prime.

1. Introduction

Let R be an integral domain with quotient field K. A prime ideal P of R is said to be *strongly prime ideal* if $x, y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$. A domain R is called a *Pseudo-Valuation Domain* (PVD) if each prime deal of R is strongly prime ideal. This concept of strongly prime ideals was introduced by Hedstorm and Houstone ([2]) in their study of Pseudo Valuation Domains. In this short paper, we generalize a theorem ([2],Theorem 1.4) that a quasi local domain is a PVD if and only if its maximal ideal is strongly prime.

2. Strongly Prime Ideals

PROPOSITION 2.1. ([4]) *Let P be a prime ideal of a domain R with quotient field K. Then P is strongly prime if $x^{-1}P \subset P$ whenever $x \in K \setminus R$.*

PROPOSITION 2.2. *The following statements are equivalent for a nonzero ideal I of R.*

(1) *I^{-1} is a ring.*
(2) *$I \subset xR$ and $I \subset yR$ for $x, y \in K$, then $I \subset xyR$.*

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PROOF. Let $I \subset xR$ and $I \subset yR$ for $x, y \in K$. Then $x \neq 0, y \neq 0$, $Ix^{-1} \subset R$ and $Iy^{-1} \subset R$. Thus $x^{-1} \in I^{-1}$ and $y^{-1} \in I^{-1}$. Since I^{-1} is a ring, $x^{-1}y^{-1} \in I^{-1}$ and $x^{-1}y^{-1}I \subset R$. i.e., $I \subset xyR$. Conversely, let $x \in I^{-1}$, $y \in I^{-1}, x \neq 0$ and $y \neq 0$. Then $I \subset x^{-1}R$ and $I \subset y^{-1}R$, so $I \subset x^{-1}y^{-1}R$ and $xyI \subset R$ i.e., $xy \subset I^{-1}$. □

PROPOSITION 2.3. *Let I be a nonzero ideal of R. If I is a radical ideal, then I^{-1} is a ring if and only if $I^{-1} = (I : I)$.*

PROOF. Suppose I^{-1} is a ring. If $x \in I^{-1}$, then $xI \subset R$. Since I^{-1} is a ring, we have $x^2 I \subset R$. Thus $(xI)^2 = (x^2 I)I \subset I$, so $xI \subset I$ since I is a radical ideal of R. Hence $x \in (I : I)$ and $I^{-1} = (I : I)$. Conversely, suppose $I^{-1} = (I : I)$ and $x, y \in I^{-1}$. Then $xI \subset I$ and $yI \subset I$, so $xyI \subset I$. i.e., $xy \in (I : I) = I^{-1}$. Thus I^{-1} is a ring. □

PROPOSITION 2.4. ([3]) *If I is a proper invertible ideal of R, then I^{-1} is not a subring of K.*

PROPOSITION 2.5. *For a prime ideal P of R, the followings are equivalent.*

(1) *P is strongly prime.*
(2) *For each fractional ideal I and J of R, if $IJ \subset P$ then either $I \subset P$ or $J \subset P$.*
(3) *P is comparable to each principal fractional ideal of R.*
(4) *P is comparable to each fractional ideal of R.*

PROOF. (1)⇒(2). Let $IJ \subset P$ for each fractional ideal I and J of R. If $I \not\subset P$, then there exists $x \in I \setminus P$. For any $y \in J, xy \in IJ \subset P$, so $y \in P$. i.e., $J \subset P$.

(2)⇒(1). Let $x, y \in K$ and $xy \in P$. If we take fractional ideals Rx and Ry of R, then $RxRy \subset P$. Hence $Rx \subset P$ or $Ry \subset P$.

(1)⇒(3). Suppose $xR \not\subset P$. Then $(x^{-1}P)x \subset P$ implies that $x^{-1}P \subset P$. In fact, if $x^{-1}P \not\subset P$ then there exists $y \in x^{-1}P \setminus P$. Hence $yx \in (x^{-1}P)x \subset P$. Since P is strongly prime, $x \in P$. It is a contradiction. Thus $P \subset xP \subset xR$.

(3)⇒(1). Let $x, y \in K$ and $xy \in P$. If both $x \notin P$ and $y \notin P$ then $P \subset Rx$ and $P \subset Ry$. Hence x and y are in R. This contradicts the fact that P is a prime ideal of R.

(3)⇒(4). Suppose I is any fractional ideal of R and let $I \not\subset P$. Then there exists $x \in I \setminus P$. Since $x \notin P, P \subset Rx$ and $P \subset Rx \subset RI \subset I$.

(4)⇒(3). It is trivial. □

PROPOSITION 2.6. *Let P be a nonzero strongly prime ideal of R.*

(1) *If P is not principal then $P^{-1} = (P : P)$ is a valuation domain.*
(2) *If P is principal then $(P : P) = R$ is a valuation domain.*

PROOF. (1) We first show that P^{-1} is a ring. Suppose that $P \subset xR$ and $P \subset yR$. If $P \not\subset xyR$, then $xyR \subset P$ by Proposition 2.5. Then either $xR \subset P$ or $yR \subset P$, so either $P = xR$ or $P = yR$. It is a contradiction. Hence $P \subset xyR$ and P^{-1} is a ring by Proposition 2.2. Since P is a radical ideal and P^{-1} is a ring, $P^{-1} = (P : P)$ by Proposition 2.3. If $x \in R$ then $xP \subset P$, so $x \in (P : P)$. If $x \in K \setminus R$, then $(x^{-1}P)x \subset P$ implies $x^{-1}P \subset P$. Hence x or $x^{-1} \in (P : P)$, so $(P : P)$ is a valuation domain.

(2) If P is principal then $P = xR, x \neq 0$ and $x \in R$. $y \in (P : P)$ implies $yP \in P$. i.e., $yxR \subset xR$. Hence $yx = xs$ for some $s \in R$. Thus $y = s \in R$ and $(P : P) \subset R$. Above proof shows that R is a valuation domain. □

PROPOSITION 2.7. *The following statements are equivalent for a nonzero ideal I of R.*

(1) *I is non principal strongly prime ideal.*
(2) *I^{-1} is a ring and I is comparable to each principal fractional ideal of R.*

PROOF. (1)⇒(2). It is trivial by Proposition 2.5 and Proposition 2.6.
(2)⇒(1). Let I be principal and $I = Ra, a \neq 0 \in R$. Then $I^{-1} = Ra^{-1}$. I^{-1} is not a subring of K by Proposition 2.4. Hence I is nonprincipal. Suppose $xy \in I$ for some $x, y \in K$. If both $x \notin I$ and $y \notin I$ then $I \subset xR$ and $I \subset yR$. But I^{-1} is a ring, so $I \subset xyR$ by Proposition 2.2. Hence $I = xyR$ is principal. It is a contradiction. Thus $x \in I$ or $y \in I$. □

3. Main results

THEOREM 3.1. *An integral domain R is a PVD if and only if every maximal ideal M of R is strongly prime.*

PROOF. We show that any prime ideal P which is contained in a maximal ideal M of R is strongly prime. Since each maximal ideal M is strongly prime, $x^{-1}M \subset M$ for $x \in K \backslash R$. Hence $x^{-1}P \subset x^{-1}M \subset M$. Also, $x^{-1}Px^{-1} \subset x^{-1}M \subset M$.

$(x^{-1}P)^2 = x^{-1}Px^{-1}P \subset MP \subset P$.But P is a prime,so $x^{-1}P \subset P$. Thus P is a strongly prime. Since any prime ideal of R is contained in a maximal ideal, we know that R is a PVD.

The converse is clear by definition of PVD. □

Now we have Theorem 1.4([2]) as the following corollary

COROLLARY 3.2. *A quasi local domain (R, M) is a PVD if and only if M is strongly prime.*

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