Finite Element Analysis for Plastic Large Deformation and Anisotropic Damage

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Abstract

An improved analysis model for material nonlinearity induced by elasto-plastic deformation and damage including a large strain response was proposed. The elasto-plastic-damage constitutive model based on the continuum damage mechanics approach was adopted to overcome limitations of the conventional plastic analysis theory. It can manage the anisotropic tensorial damage evolved during the time-independent plastic deformation process of materials. Updated Lagrangian finite element formulation for elasto-plastic damage coupling problems including large deformation, large rotation and large strain problems was completed to develop a numerical model which can predict all kinds of structural nonlinearities and damage rationally. Finally a finite element analysis code for two-dimensional plane problems was developed and the applicability and validity of the numerical model was investigated through some numerical examples. Calculations showed reasonable results in both geometrical nonlinear problems due to large deformation and material nonlinearity including the damage effect.

1 Introduction

The nonlinear behavior of a structure can be separated into material and geometric nonlinear effects. Geometric nonlinearities has been set theoretically and numerical algorithms are being improved continuously. A constitutive relation which globally represent material nonlinearities due to molecular bindings, polycrystalline structures, and other micro-level characteristics of a material has been a crucial interest to modern researchers. The currently available constitutive model can not distinguish the intact and damaged zone of a deformed body. Moreover, an explicit description of the damage is almost impossible.

Continuum damage mechanics(CDM) proposed by Kachanov[1] has shown that damage, similar to strain, can be a measure of characterizing material responses, and thus can be treated as an internal state variable. Continuum damage is considered as an average density of microcracks or microvoids over a certain volume in a body. Nonlinear responses of a material can be correctly described using the damage concept. Since late 1980s, lots of researches have been successful in developing a rational CDM model and its applications

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to engineering fields. Among them is an analysis of a welded tubular joint of offshore platforms[2].

In this paper a general formulation of geometric nonlinear problems including large displacements, large rotations, and large strains is considered. The elastic-plastic-damage constitutive model proposed by Nho et al[3] is incorporated into the finite element formulation. The general two-dimensional plane problems are coded. The selected numerical examples show the validity and applicability of the proposed model.

2 Elasto-Plastic Damage Constitutive Model

Damage variable and effective stress

Kachanov[1] assumed that the damage of creeping metals was due to the nucleation and growth of micro-voids and defined a damage variable Ψ as the reduction of the effective sectional area caused by the micro-voids. Then the effective stress σ^* acting on a real material can be described as Fig.1. Since Ψ is a scalar, the directional characteristics of damage can not be expressed correctly.

Generally material damages exhibit as micro-cracks with directional characteristics, and the isotropic damage theory considering the damage as micro-void with uniform projection area to any direction is not appropriate to such an anisotropic nature due to damage evolution. Therefore Vakulenko and Kachanov[4] introduced a new concept of tensorial damage. Murakami and Ohno[5] simplified the Vakulenko-type tensorial damage and showed that the eigenvalues of the damage tensor meant the reduction ratios of the effective area in the direction of their eigenvectors. They expressed an anisotropic damage tensor $\mathbf{\Omega}$ as a following generalized form:

$$\mathbf{\Omega} = \sum_{i=1}^{3} \omega_i (\nu_i \otimes \nu_i) \tag{1}$$

where ν_i is the directional cosine vector of damage, ω_i means the ratio of effective area reduction in the ν_i direction. Symbol $\otimes$ denotes the tensor product of two vectors. The effective stress tensor σ^* is defined by the damage tensor as:

$$\sigma^* = \sigma \cdot \phi, \qquad \phi = (\mathbf{I} - \mathbf{\Omega})^{-1} \tag{2}$$

where $\cdot$ denotes a single dot product. Since, in general, σ^* is a non-symmetric tensor, it is not useful for most numerical analysis procedures. To avoid this disadvantage, a new effective stress tensor $\tilde{\sigma}$ which is the symmetric part of σ^* is introduced as[6]:

$$\tilde{\sigma} = {\sigma^*}_{\text{sym}} = \frac{1}{2}\left(\sigma^* + \sigma^{*\mathrm{T}}\right) = \frac{1}{2}\left(\phi \cdot \sigma + \sigma \cdot \phi\right) = \mathbf{\Phi} : \sigma \tag{3}$$

where : is the double dot product, and $\mathbf{\Phi}$ is the symmetric fourth-order damage tensor defining the relationship between effective stress $\tilde{\sigma}$ and stress σ.

subsection*Elasto-plastic damage constitutive equation

Assume that plastic strains, hardening parameters and damage variable as internal state variables. They express the change of microscopic structures or energy dissipation phenomena during material deformation. The evolution equations of internal variables for elasto-plastic-damage materials are derived using the thermodynamic formulation procedure of constitutive equations. Applying the energetic identification hypothesis of Sidoroff[7] and effective stress-damage tensor relations of rate form into the yield function and consistency condition, the elasto-plastic-damage constitutive equation for the time independent plastic deformation can be derived as[3] :

$$\dot{\sigma} = \tilde{\mathbf{C}}_{ep} : \dot{\varepsilon} - \mathbf{C}_d : \dot{\mathbf{\Omega}} \tag{4}$$

where

$$\dot{(\ )} = \text{material derivative}$$

$$\tilde{\mathbf{C}}_{ep} = \mathbf{\Phi}^{-1} : \mathbf{C}_{ep} : \mathbf{\Phi}^{-T} \tag{5}$$

$$\mathbf{C}_{ep} = \left[\mathbf{C}_e - \frac{(\mathbf{C}_e : \partial f / \partial \tilde{\sigma}) \otimes (\partial f / \partial \tilde{\sigma} : \mathbf{C}_e)}{H'_I + H'_K + \partial f / \partial \tilde{\sigma} : \mathbf{C}_e : \partial f / \partial \tilde{\sigma}} \right]$$

$$\mathbf{C}_d = \left[\tilde{C}_{ep} : \frac{\partial \mathbf{\Phi}}{\partial \mathbf{\Omega}} : \mathbf{C}_e^{-1} - \frac{\partial \mathbf{\Phi}^{-1}}{\partial \mathbf{\Omega}} \right] : \tilde{\sigma}.$$

Here H'_I and H'_K mean the isotropic and kinematic hardening modulus. The damage tensor can be expressed by the additive form of isotropic and deviatoric (anisotropic) components. It is assumed that the evolution of isotropic components depend on the hydrostatic components of damage Ω_m and that of deviatoric component depends on the $\nu^{(i)}$ direction component of damage $\mathbf{\Omega}_{\nu^{(i)}}$ [8]. Then the evolution equation of damage can be postulated as:

$$\dot{\mathbf{\Omega}} = \mathbf{V}_{\Omega} \dot{p} \tag{6}$$

$$\mathbf{V}_{\Omega} = C_1 \ (1 - \beta_1 \Omega_m) \langle \tilde{\sigma}_m \rangle \mathbf{I} + C_2 \sum_{i=1}^{3} (1 - \beta_2 \Omega_{\nu^{(i)}}) \left\langle \tilde{\sigma}_i^d \right\rangle (\nu^{(i)} \otimes \nu^{(i)}). \tag{7}$$

where

$$\begin{aligned}
C_1,\ C_2,\ \beta_1,\ \beta_2 \ &: \ \text{material constants} \\
\tilde{\sigma}_i \ &: \ \text{principal effective stress} \\
\nu^{(i)} \ &: \ \text{directional cosine vectors of principal direction} \\
< a > \ &= \ 1 \quad \text{when } a > \text{o} \\
&= \ 0 \quad \text{when } a \leq \ \text{o} \\
\Omega_m \ &= \ (\Omega_{11} + \Omega_{22} + \Omega_{33})/3 \\
\tilde{\sigma}_m \ &= \ (\tilde{\sigma}_{11} + \tilde{\sigma}_{22} + \tilde{\sigma}_{33})/3 \\
\tilde{\sigma}_i^d \ &= \ \tilde{\sigma}_i - \tilde{\sigma}_m \\
\Omega_{\nu^{(i)}} \ &= \ \nu^{(i)} \cdot \mathbf{\Omega} \cdot \nu^{(i)}.
\end{aligned}$$

To describe pure mechanical behaviors of materials in large rotation problems, the constitutive equation should be invariant under the changes of reference frame. Hence, the rates of variables are replaced by the objective Jaumann rates. Introducing the formulation process of McMeeking and Rice[9], Eqs. 4 and 6 can be transformed to the following forms:

$$\check{\tau} = \widetilde{C}_{ep} : \mathbf{D} - \mathbf{C}_d : \check{\boldsymbol{\Omega}} \tag{8}$$

$$\check{\boldsymbol{\Omega}} = \mathbf{V}_{\Omega}\dot{p}. \tag{9}$$

3 Finite Element Formulation

A two-dimensional plane finite element analysis for elasto-plastic-damage problems was formulated. In general, the plastic behavior of materials tend to be large deformations and shows geometrical nonlinearity as well as material nonlinearity. Therefore, a geometrical nonlinear problem is formulated based on the nonlinear kinematics[10][11][12] and also a material nonlinearity of large deformation is formulated using the elasto-plastic damage constitutive equation described in Section 2.

The motion of a body B in a stationary Cartesian coordinate system is shown in Fig.2. An updated Lagrangian(U.L.) description of a motion of the body at time $t + \Delta t$ is made based on its configuration at time t. Hence, the external and internal virtual power of a body B at time $t + \Delta t$ can be written as:

$$\dot{W}_{internal} = \int_{V^t} s_{ij}^{t+\Delta t} \delta \dot{F}_{ji} dV^t \tag{10}$$

$$\dot{W}_{external} = \int_{V^t} b_i^{t+\Delta t} \delta v_i dV^t + \int_{S^t} f_i^{t+\Delta t} \delta v_i dV^t \tag{11}$$

where δ denotes a variation, v_i is a velocity, $b_i^{t+\Delta t}$ is a body force vector, and $f_i^{t+\Delta t}$ is a surface traction vector. $s_{ij}^{t+\Delta t}$ is the first Piola-Kirchhoff stress component at time $t + \Delta t$ with respect to the reference configuration at time t and $\dot{F}_{ji}$ is the deformation gradient tensor conjugate to $s_{ij}^{t+\Delta t}$. They can be redefined as:

$$s_{ij}^{t+\Delta t} = s_{ij}^t + \Delta s_{ij} = \sigma_{ij}^t + \Delta s_{ij} \tag{12}$$

$$\dot{F}_{ji} = \frac{\partial \dot{x}_j^{t+\Delta t}}{\partial x_i^t} = \frac{\partial (\dot{x}_j^t + \dot{u}_j)}{\partial x_i^t} = \frac{\partial v_j^{t+\Delta t}}{\partial x_i^t} \tag{13}$$

where σ_{ij}^t is the Cauchy stress. By inserting Eqs. 12 and 13 into the equivalence condition of Eqs. 10 and 11, the principle of virtual power can be written as:

$$\int_{V^t} (\sigma_{ij}^t + \Delta s_{ij}) \delta (\frac{\partial v_j^{t+\Delta t}}{\partial x_i^t}) \, dV^t = R^{t+\Delta t} \tag{14}$$

where

$$R^{t+\Delta t} = \int_{V^t} b_i^{t+\Delta t} \delta v_i^{t+\Delta t} dV^t + \int_{S^t} f_i^{t+\Delta t} \delta v_i^{t+\Delta t} dS^t.$$

For convenient applications of the constitutive equation(Eq. 8), $\dot{s}_{ij}$ is converted to the following expression:

$$\begin{aligned} \dot{s}_{ij} &= \check{\tau}_{ij} - \sigma_{kj} D_{ki} - \sigma_{ik} D_{kj} + \sigma_{ik}\left(\frac{\partial v_k}{\partial x_j}\right) \\ &= \check{\tau}_{ij} - \frac{1}{2}\sigma_{kj}(v_{k,i} + v_{i,k}) - \frac{1}{2}\sigma_{ik}(v_{k,j} + v_{j,k}) + \sigma_{ik} v_{j,k} \\ &= \check{\tau}_{ij} + \Sigma_{ijkl} v_{l,k} \end{aligned} \quad (15)$$

where D_{ij} is the velocity strain and

$$\Sigma_{ijkl} = -\frac{1}{2}[\sigma_{lj}\delta_{ik} + \sigma_{kj}\delta_{il} + \sigma_{il}\delta_{jk} - \sigma_{ik}\delta_{jl}].$$

Then Eq. 14 becomes (from now on the superscript t will be abbreviated if possible.)

$$\Delta t \int_V (\check{\tau}_{ij} + \Sigma_{ijkl} v_{l,k}) \delta v_{j,i} dV = R^{t+\Delta t} - \int_V \sigma_{ij} \delta v_{j,i} dV. \quad (16)$$

Since τ_{ij} and σ_{ij} are symmetric, the principle of virtual power is described by the Eulerian variables D_{ij}, v_i, σ_{ij}, Ω_{ij} and others as:

$$\begin{aligned} &\Delta t \int_V \{\tilde{C}^{ep}_{ijkl} D_{kl} \delta D_{ij} + \Sigma_{ijkl} v_{l,k} \delta v_{j,i} dV \\ &= R^{t+\Delta t} - \int_V (\sigma_{ij} - C^d_{ijkl} \check{\Omega}_{kl} \Delta t) \delta D_{ij} dV. \end{aligned} \quad (17)$$

Finite element discretization was applied to Eq. 17 and resulted in the incremental form of stiffness equation as:

$$([K_L] + [K_{NL}])\{\Delta u\} = \{F_f\} - \{F_s\} - \{F_d\} \quad (18)$$

where $[K_L]$ is a linear stiffness matrix, $[K_{NL}]$ is a non-linear stiffness matrix attributed to the initial stress terms. $\{F_f\} - \{F_s\}$ term is the out-of-balance force due to the piecewise linearization assumption. The peculiar term $\{F_d\}$ means an additional load vector caused by the evolution of damage.

4 Numerical Calculations and Discussions

An elasto-plastic-damage finite element analysis code for two-dimensional plane stress/strain problems was developed. Four- and eight-node isoparametric elements were implemented. To solve the nonlinear incremental equilibrium equations, the arc-length method based on the Riks algorithm[13] and the modified Newton-Raphson(m.N.R.) method were adopted.

A 90° rigid-body-rotation case

To remove the effects of rigid body rotation and to consider the pure deformation of the material itself numerical calculation algorithm of integrating the Jaumann rate of Cauchy stress tensor at each time step was adopted.

The accuracy of this algorithm was checked by the example calculation of the 90deg rigid body rotation. As shown in Fig.3, a square plate initially strained by $\sigma_x = 100$ was rotated with increment $\Delta\theta$ up to 90°. Results of the analysis with a 4-node element is shown at Table 1. The smaller increment $\Delta\theta$ reduces the magnitudes of error compared with the theoretical solution $\sigma_x = 0$, $\sigma_y = 100$, $\tau_{xy} = 0$ by the coordinate transformation.

Table 1 Cauchy stress components after 90° rigid body rotation with initial stress;$\sigma_x = 100$, $\sigma_y = 0$, $\tau_{xy} = 0$.

Increment $\Delta\theta$	σ_x	σ_y	τ_{xy}
5°	-15.4	115.4	-2.30
2°	-5.75	105.8	-0.32
1°	-2.79	102.8	-0.08
0.5°	-1.37	101.4	-0.02
0.25°	-0.68	100.7	0
Exact sol.	0	100	0

Elastic large deflection of cantilever beam

Large deformation analysis of an elastic cantilever beam with a concentrated load P at end was performed by three 8-node elements. The results of the calculation was plotted in Fig.4 compared with the analytic solution and other numerical results[14]. The present solution had some deviations from the analytic solution based on the simple beam theory. However, two numerical solutions analyzed by two-dimensional plane stress approaches showed very close results.

Plastic damage in a large strain problem: tensile necking

The tensile necking phenomenon is chosen as a large strain problem to investigate the evolution process of plastic deformation and anisotropic damage. A square plate with a central hole dealt by Argyris et al[15] is analyzed. Finite element modeling and boundary conditions are described in Fig. 5.

Fig. 6 shows the calculated $P - \delta$ curves, i.e. variation of reaction force P with increasing elongation δ. To recognize the effect of damage evolution, the numerical result of the undamaged case was plotted together. At the initial stage, the reaction force P is increasing very sharply. After P shows an extreme value at the vicinity of $\delta = 3.5cm$, however, it reverses to the decreasing phase. This phenomenon explains that from that region the decreasing rate of sectional area due to elongation begins to exceed the increasing rate of stiffness due to the hardening of the material. Therefore the tangential stiffness becomes a negative value, and the unstable abrupt softening behavior with necking and plastic unloading occur. Though the tendency of the undamaged case(dashed line) is similar to that of the damaged case, reaction forces of the two cases show about a 20 % discrepancy. This exhibits the deterioration effect of the material due to damage evolution.

Fig. 7 and Fig. 8 show the distribution of principal stresses and damage contrasting the contour map. The central narrow section of the plate yields first and the neighbouring regions become to yield successively. With further gradual extension, however, deformation tends to be concentrated at the central region and necking takes place finally. At that time, most part of the plastic zone are unloaded elastically except the central stress-concentrated region. The distribution of anisotropic damage are very similar to that of stress.

The magnitude of damage does not exceed 20 % even at the maximum point. It seems to be caused by limitations of damage evolution equation (Eq. 6) which modeled only the initial stage of deformation with stable damage evolution except the region just before fracturing where damage evolves drastically. Therefore, to develop a more reasonable constitutive equation and to get the more realistic calculation results of hardening and damage evolution behavior, precise material tests including the large strain region should be recommended.

5 Conclusion

In this paper, a continuum damage mechanics approach is incorporated into the time-independent plasticity to explore material damage during large deformation process. It helps to cope with limitations of classical plasticity theories and understand more rational safety assessments of structures. Though applications to offshore structural problems are not made at present, some of the major contributions of this paper are:

(1) Based on nonlinear kinematics, a rigorous finite element formulation is completed for large displacements, large rotations, and elasto-plastic-damage materials.

(2) An anisotropic damage and an effective stress tensor are newly defined physically and mathematically. The constitutive relation of an elasto-plastic-damage material used in this paper is proved to be a reasonable assessment of structural safety.

(3) Numerical examples show good agreements with published results and more rational results for large deformation and material nonlinear problems.

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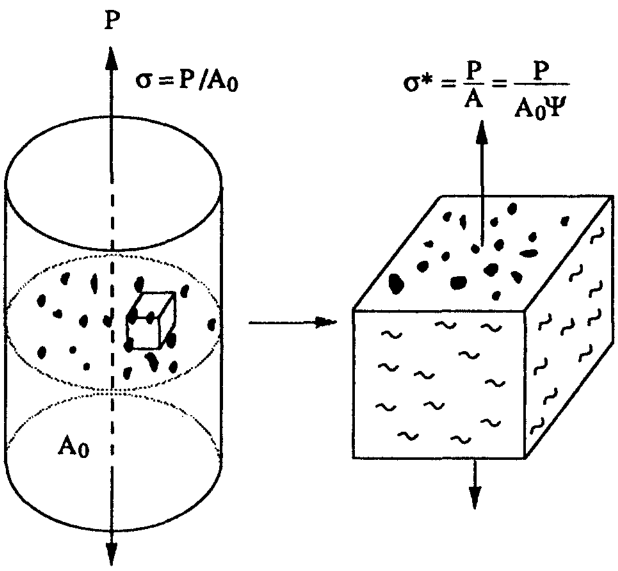


Fig.1 Damage Model of Kachanov[1].

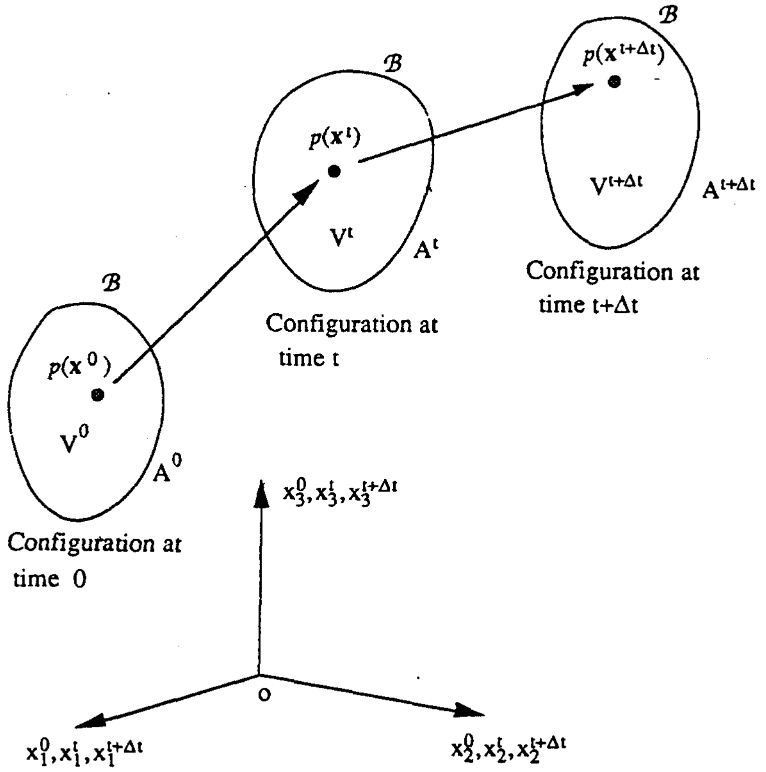


Fig.2 Motion of body in a stationary Cartesian coordinate system.

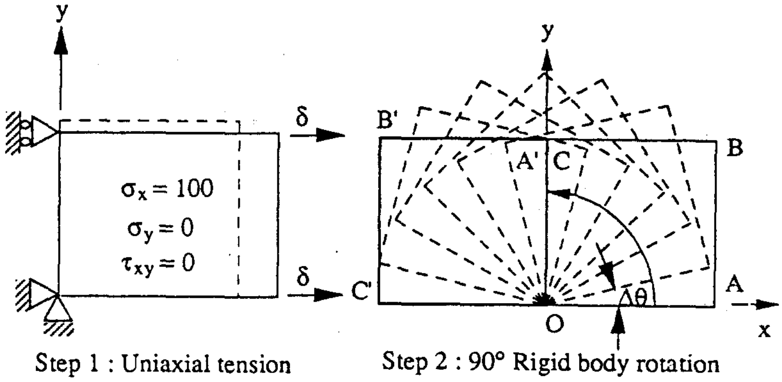
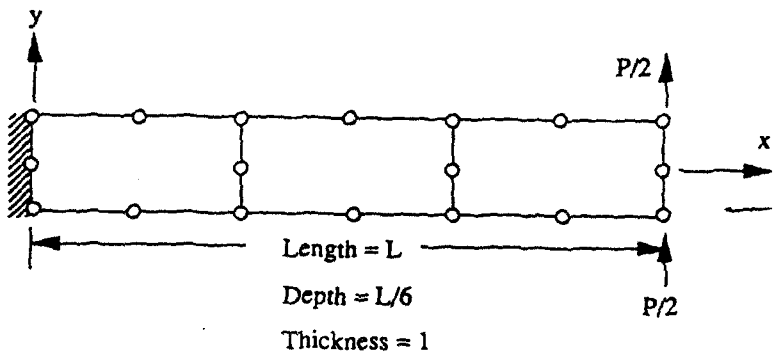


Fig.3 90° rigid body rotation with initial stress.



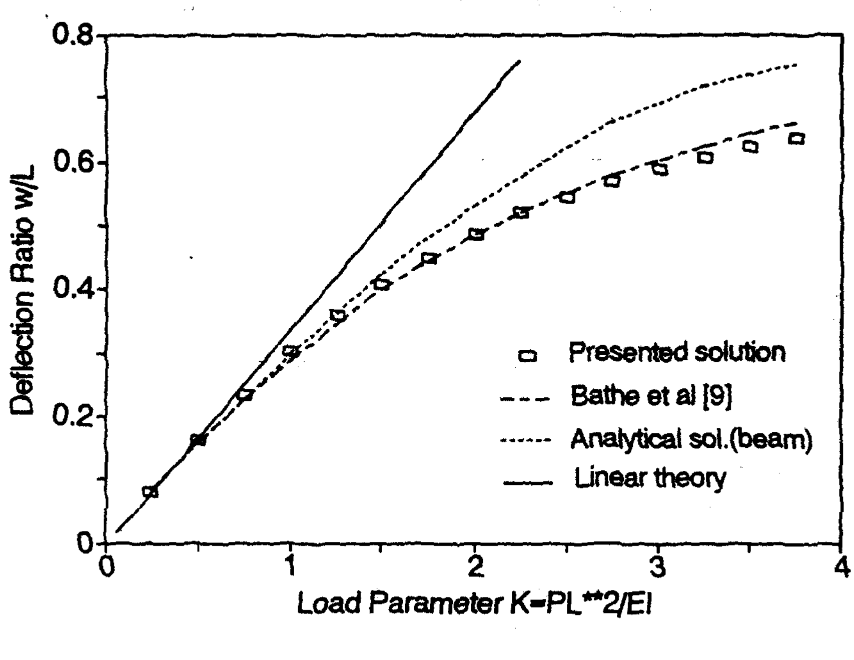


Fig.4 Finite deflection of a cantilever beam.

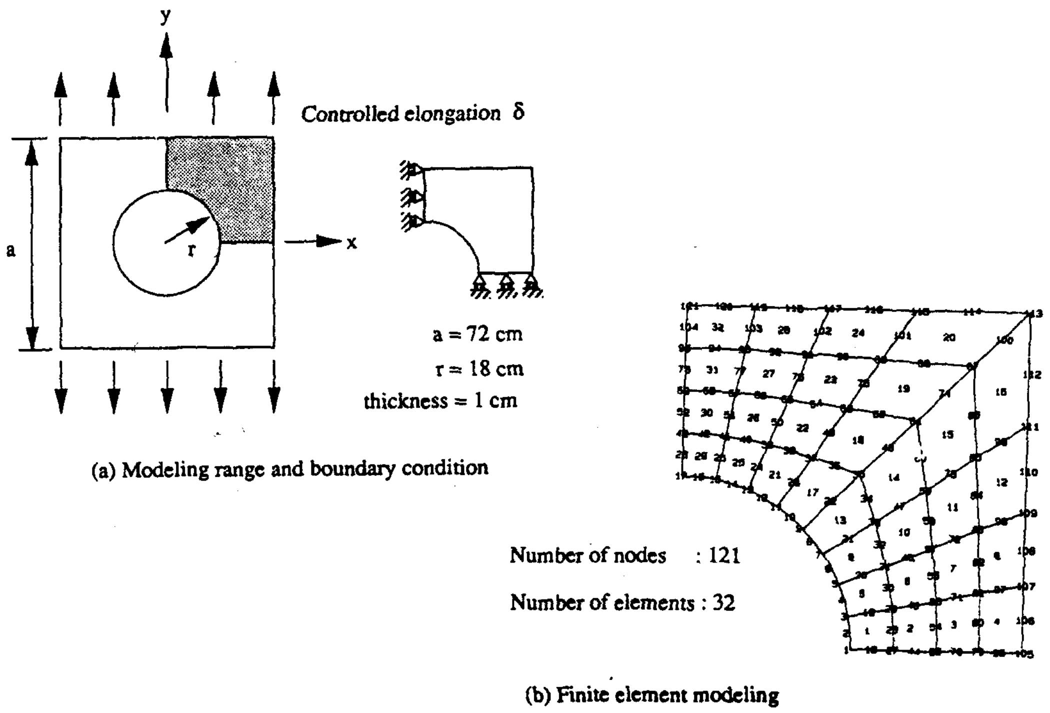


Fig.5 Extension of a square plate with a central hole.

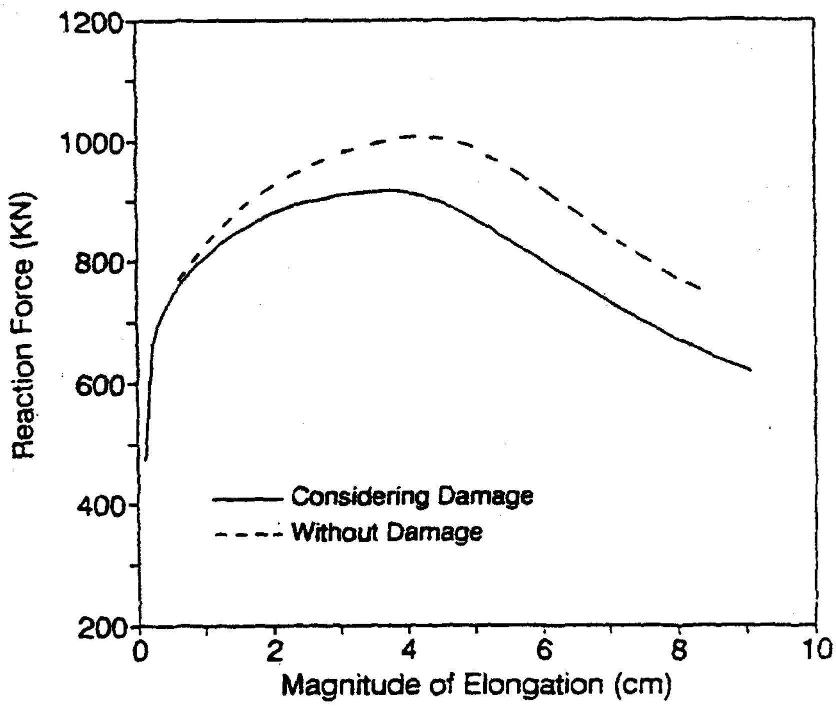


Fig.6 Calculated P-d curve of the square plate.

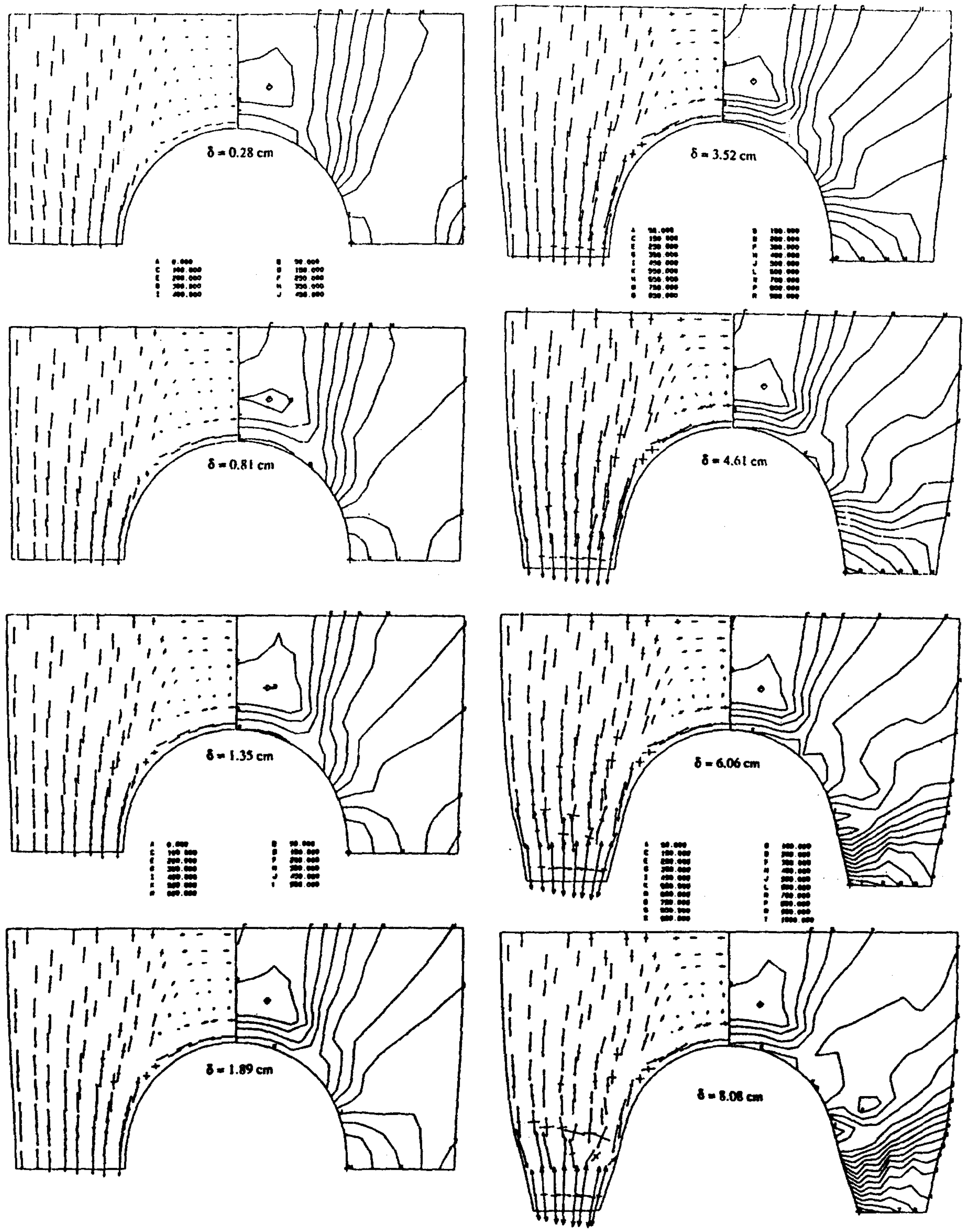


Fig.7 Principal directions and contour map of effective stresses.

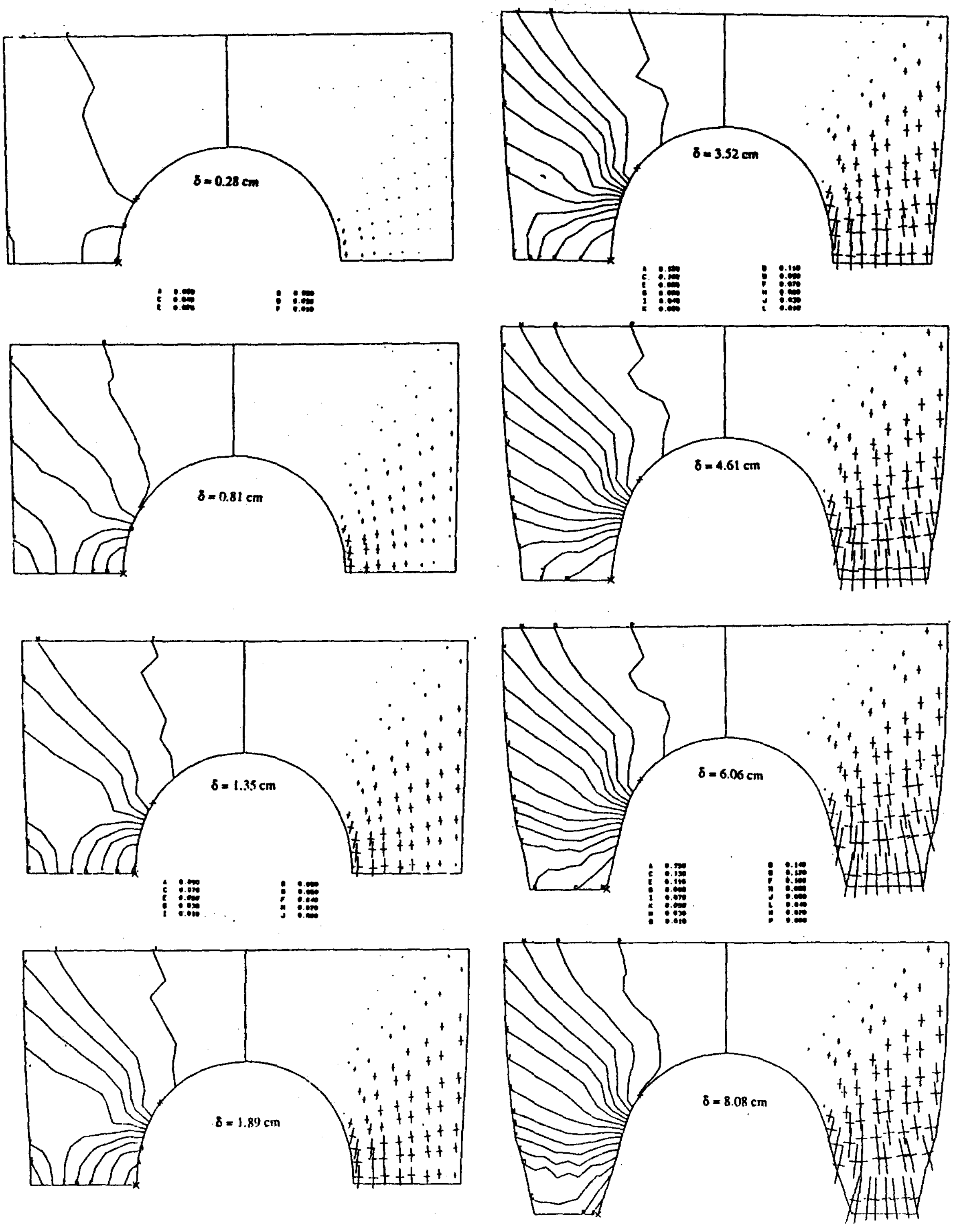


Fig.8 Evolution of anisotropic damage and its contours.