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DUAL LIMIT THEOREMS FOR THE GENERALIZED CURIE–WEISS MODEL: MULTIPLE GLOBAL MINIMA CASE

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1. Introduction

For a probability measure Q with $\Phi_Q(t) = \int_R \exp(tx)Q(dx) < \infty$ for $-h < t < h,\ h > 0$, let L_Q be the class of probability measure P such that for $|t| < k,\ k > 0$

$$\Phi_P(t) = \int_R \exp(tx)P(dx) < \infty, \tag{1.1}$$

and

$$\int_R \Phi_Q(x)P(dx) < \infty. \tag{1.2}$$

Let $\{X_j^{(n)} : j = 1, 2, \ldots, n\}$ $, n = 1, 2, \ldots$ be a triangular array of dependent and identically distributed random variables with the joint distribution given by

(1.3)

$$\mu_n^G(dx_1, \ldots, dx_n) = z_n^{-1} \exp\left[n\Psi_Q\{(x_1 + \cdots + x_n)/n\}\right] \prod_{j=1}^{n} P(dx_j),$$

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where $P \in L_Q$, $\Psi_Q(t) = \log \Phi_Q(t)$ and z_n is the normalizing constant,

$$z_n = \int_{R^n} \exp[n\Psi_Q\{(x_1 + \cdots + x_n)/n\}] \prod_{j=1}^{n} P(dx_j). \tag{1.4}$$

The model in (1.3) defines the generalized Curie - Weiss model which is a direct generalization of the classical mean field model or Curie - Weiss model in which the joint distribution is postulated as

$$\mu_n^{CW}(dx_1, \cdots, dx_n) = z_n^{-1} \exp[(x_1 + \cdots + x_n)^2/2n] \prod_{j=1}^{n} P(dx_j) \tag{1.5}$$

The asymptotic behavior of $S_n = X_1 + \cdots + X_n$ as $n \to \infty$ has been studied in great detail for the Curie - Weiss model. The asymptotic distribution of S_n for this model when P is symmetric Bernoulli was obtained by Simon and Griffiths(1973). Dunlop and Newman(1975) have extended the result to the case where the random variables are vectors. Ellis and Newman(1978 a,b) generalized the result of Simon and Griffiths to a large class of probabilities.[see also Ellis and Rosen(1980)]. The similar limit theorems for the generalized Curie - Weiss model were obtained by Jeon(1978). Chaganty and Sethuraman(1985,1987) considered the limit theorems for further extended models. Recently Choi, Kim and Jeon(1989) extended the result of Ellis and Newman(1978) for a generalized model. On the hand, interchanging the role of P and Q in the generalized Curie - Weiss model(1.3), Lee, Kim and Jeon(1993) for more details on these models.

Let the distribution function F_Q of a probability measure Q be such that

$$\begin{aligned} F_Q(x) &= 0, \qquad x < a \\ 0 < F_Q(x) &< 1, \qquad a < x < b \\ F_Q(x) &= 1, \qquad b < x \end{aligned}$$

and $D_Q \equiv (a, b)$ where $-\infty \leq a < b \leq \infty$. We define the dual of the generalized Curie - Weiss model. The dual model is defined as the joint distribution

$$\begin{aligned} &\mu_n^{GD}(dx_1, \cdots, dx_n) \\ &= d_n^{-1} \exp[n\Psi_P\{(x_1 + \cdots + x_n)/n\}] \prod_{j=1}^{n} Q(dx_j). \end{aligned} \tag{1.6}$$

Note that dual of the original model is obtained simply by exchanging the role of Q and P in (1.3). For well-definedness of the dual model, see Lee, Kim and Jeon(1993).

For a probability measure Q, define a function γ_Q by

$$\gamma_Q(t) = \sup_{s \in R}[ts - \Phi_Q(s)] \qquad t \in \mathbf{R}, \tag{1.7}$$

where $\Psi_Q(s) = \log \int_R \exp(sx)Q(dx)$. In statistics this function is called the large deviation rate of Q.

For the probability measures Q and $P \in L_Q$, define

$$G_{QP}(t) = \gamma_Q(t) - \Psi_P(t) \qquad t \in D_Q, \tag{1.8}$$

where $\Psi_P(t) = \log \int_R \exp(tx)P(dx)$ and $D_Q = (a, b) = \{Q(t) : t \in (c, d)\}$.

Note that $\Psi'_Q(t)$ is strictly increasing on (c, d) [see Daniels(1954)]. This function was studied by Ellis and Newman(1978 a,b). For the dual model, the function corresponding G_{QP} of the original model (1.3) is

$$G_{PQ}(t) = \gamma_P(t) - \Psi_Q(t) \qquad t \in D_P, \tag{1.9}$$

where $\gamma_P(t) = \sup_{s \in D_P} [ts - \log \int_R \exp(sx)P(dx)]$ and $D_P = (c, d) = \{\Psi'_P(t) : t \in (a, b)\}$.

Ellis, Newman and Rosen(1980) proved similar limit theorems for the Curie - Weiss model. In this paper we prove some dual limit theorems by conditioning technique in the case where G_{QP} (or G_{PQ}) has several global minima.

2. Limit theorems for the conditional case

For a random variable X and an event A, we write $P(X \in dw|A)$ to denote the measure $P_A(dw)$ defined by $P_A(B) = P(X \in B|A)$ for every Borel set B in $\mathbf{R}$. Given F a probability distribution on $\mathbf{R}$, we write $(X_n|A_n) \xrightarrow{d} F$ to mean $P(X_n \in dw|A)$ converges weakly to $F(dw)$.

DEFINITION 2.1. A real number m is said to be a global minimum for G_{QP} if

$$G_{QP}(u) \geq G_{QP}(m) \qquad \text{for all } \ u \in D_Q.$$

When $D_Q = (-\infty, \infty)$, it is well-known the following Lemma.

LEMMA 2.2. *Given a probability measure Q, let $P \in L_Q$. Then G_{QP} is analytic and $G_{QP}(s) \longrightarrow \infty$ as $|s| \to \infty$. Thus, G_{QP} has a finite number of minima. See Lee, Kim and Jeon(1993).*

DEFINITION 2.3. For given probability measure Q and $P \in L_Q$, a local minimum m for G_{QP} is said to be of type k if

$$G_{QP}(m+u) - G_{QP}(m) = c_{2k}\frac{u^{2k}}{(2k)!} + o(u^{2k}), \quad \text{as } \ u \to 0, \tag{2.1}$$

where $c_{2k} = G_{QP}^{(2k)}(m) > 0$.

DEFINITION 2.4. For probability measures Q and $P \in L_Q$, let $m_1, \cdots, m_h$ be the global minima of G_{QP} with types be $k_1, \cdots, k_h$, respectively. Then $k = \max\{k_1, \cdots, k_h\}$ is said to be the maximal type of G_{QP}.

DEFINITION 2.5. Let Q be a probability measure. Then $P \in L_Q$ is said to be pure with respect to Q if G_{QP} has a unique global minimum.

Let $h_n(z)$ be a function on the set $n^{1/2k}D_Q - m = \{x \in R : x = n^{1/2k}u - m, u \in D_Q\}$ satisfying

$$h_n(z) = \exp[-n\{G_{QP}(zn^{-1/2k} + m) - G_{QP}(m)\}]\sigma^{-1}(t_n)[1 + o(1)],$$

where $\sigma^2(t) = \Psi''_Q(t), \Psi'_Q(t_n) = zn^{-1/2k} + m$. Similar, define the function $h_n^D(\cdot)$ for the corresponding dual model.

DEFINITION 2.6. Let Q be a probability measure. A probability measure $P \in L_Q$ is said to be semipure with respect to Q if there is only one global minimum of G_{QP} with maximal type.

LEMMA 2.7. *Assume that $G_{QP}(\cdot)$ has an extreme of type k at m. Then there exist $\beta = \beta(m) > 0$ and $\beta^D = \beta(m^D) > 0$ such that for some $c(m) > 0$ as $n \to \infty$,*

$$(2.2) \qquad h_n(z) \le c(m) \exp[-c_{2k} z^{2k}/(2k)!] \quad \text{for } |z| \le \beta n^{1/2k},$$

$$(2.3) \qquad h_n(z) \longrightarrow \exp[-c_{2k} z^{2k}/(2k)!] \sigma_Q^{-1}(z) \quad \text{for each } z$$

and for some $c(m^D) > 0$ as $n \to \infty$,

$$(2.4) \qquad h_n^D(z) \le c(m^D) \exp[-c'_{2k} z^{2k}/(2k)!] \quad \text{for } |z| \le \beta^D n^{1/2k},$$

$$(2.5) \qquad h_n^D(z) \longrightarrow \sigma_P^{-1}(z) \exp[-c'_{2k} z^{2k}/(2k)!] \quad \text{for each } z$$

where $m^D = \Psi'_P(m)$.

Proof. We only have to prove (2.2) and (2.3) due to the symmetricity. Note that

$$G_{QP}(m+u) - G_{QP}(m) = c_{2k} u^{2k}/(2k)! + o(u^{2k}) \quad \text{as} \quad u \to 0,$$

and there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that for some $c(m) > 0$,
$|G_{QP}(m+u) - G_{QP}(m) - c_{2k} u^{2k}/(2k)!| \le c(m) u^{2k+1}$ for all $|u| < \beta_1$ and $O(u^{2k+1}) = o(u_{2k}) \le (1/2) c_{2k} u^{2k}/(2k)!$ for all $|u| < \beta_2$. Hence (2.2) and (2.3) are asserted by choosing $\beta = \min\{\beta_1, \beta_2\}$.

LEMMA 2.8. *Let $X_1^{(n)}, , \cdots, X_n^{(n)}$ be independent random variables with common distribution*

$$M_{n,z}(dx) = \exp[(m + zn^{-1/2k})x - \Psi_P(m + zn^{-1/2k})] P(dx)$$

and its dual

$$M_{n,z}^D(dx) = \exp[(m^D + zn^{-1/2k})x - \Psi_Q(m^D + zn^{-1/2k})] Q(dx).$$

Then

$$(2.6) \qquad \frac{S_n - nm^D}{n^{1-1/2k}} \xrightarrow{d} \begin{cases} \delta(s - m_1^D z) & \text{if } k \ge 2 \\ N(m_1^D z, m_1^D) & \text{if } k = 1 \end{cases}$$

and

$$(2.7) \qquad \frac{S_n^D - nm}{n^{1-1/2k}} \xrightarrow{d} \begin{cases} \delta(s - m_1 z) & \text{if } k \ge 2 \\ N(m_1 z, m_1) & \text{if } k = 1, \end{cases}$$

where $m^D = \Psi'_P(m), m_1^D = \Psi''_P(m), m = \Psi'_Q(m^D)$ and $m_1 = \Psi''_Q(m^D)$.

Proof. See Theorem 4.1 in Lee, Kim and Jeon(1993).

COROLLARY 2.9. *Under the same assumption of Lemma 2.8, we have*

$$S_n/n - m^D \xrightarrow{d} 0 \quad \text{and} \tag{2.8}$$

$$S_n^D/n - m \xrightarrow{d} 0 \tag{2.9}$$

Proof. They are trivially true due to Lemma 2.8.

A variation of Laplace's formulas will be used later to prove transfer principles. We restate it for our purpose as follows(cf. Erdélyi(1956), p.37).

LEMMA 2.10. (Laplace.) *Let g and h be functions on the interval (α, β) for which the integral (2.10) exists for each sufficiently large $x > 0$, Suppose that h is continuous at $t = \alpha$, continuously differentiable on $\alpha < t \leq \alpha + \eta$ $(\eta > 0)$ and $h' > 0$ on $\alpha < t \leq \alpha + \eta$, $h(t) \geq h(\alpha) + \varepsilon$ $(\varepsilon > 0)$, on $\alpha + \eta \leq t < \beta$. Then if $h'(t) \sim a(t-\alpha)^{\lambda-1}$ and $g(t) \sim b(t-\alpha)$ as $t \to \alpha$ for some constants $\lambda, \nu > 0$,*

$$f(x) = \int_\alpha^\beta g(t)\exp\{xh(t)\}dt \sim \frac{b}{\nu}\Gamma\Big(\frac{\lambda}{\nu}\Big)\Big(\frac{\nu}{ax}\Big)^{\lambda/\nu} e^{xh(\alpha)}, \quad \text{as } x \to \infty. \tag{2.10}$$

In the following, we prove two transfer principles that will be used as basic tool to establish Theorems 2.14 and 2.15.

LEMMA 2.11. (Transfer principle 1.) *Given probability measures Q and $P \in L_Q$, there exist $\beta > 0$ and $A > 0$ depending only on m, and $\beta^D > 0$ and $A^D > 0$ depending only m^D, such that for $a \in (0, A)$ and any bounded continuous function $h(\cdot)$, as $n \to \infty$.*

(2.11)

$$\int_R \Big[\int_{|s_n/n - m^D| \leq a} h\{n^{1/2k}(s_n/n - m^D)/m_1^D\} \prod_{j=1}^n M_{n,z}(dx_j)\Big] h_n(z)dz$$
$$= \int_{|z| \leq \beta n^{1/2k}} \Big[\int_{R^n} h\{n^{1/2k}(s_n/n - m^D)/m_1^D\} \prod_{j=1}^n M_{n,z}(dx_j)\Big] h_n(z)dz + o(1)$$

and for $a \in (0, A^D)$ any bounded continuous function $h^D(x)$, as $n \to \infty$
(2.12)

$$\int_R \Big[\int_{|s_n/n-m|\le a} h^D\{n^{1/2k}(s_n/n-m)/m_1\} \prod_{j=1}^n M_{n,z}^D(dx_j)\Big] h_n^D(z)dz$$
$$= \int_{|z|\le \beta^D n^{1/2k}} \Big[\int_{R^n} h^D\{n^{1/2k}(s_n/n-m)/m_1\} \prod_{j=1}^n M_{n,z}^D(dx_j)\Big] h_n^D(z)dz + o(1)$$

where $s_n = x_1 + \cdots + x_n$.

Proof. We only prove (2.11). It suffices to prove that as $n \to \infty$,

$$\int_{|z|\le \beta n^{1/2k}} \Big[\int_{|s_n/n-m^D|>a} \prod_{j=1}^n M_{n,z}(dx_j)\Big] h_n(z)dz + o(1) \tag{2.13}$$

and

$$\int_{|z|> \beta n^{1/2k}} \Big[\int_{|s_n/n-m^D|\le a} \prod_{j=1}^n M_{n,z}(dx_j)\Big] h_n(z)dz + o(1) \tag{2.14}$$

The proof of (2.13) is relatively easy. By Corollary (2.9), we have for all $a > 0$,

$$\int_{|s_n/n-m^D|>a} \prod_{j=1}^n M_{n,z}(dx_j) \to 0, \text{ as } n \to \infty, \quad \text{for each } z. \tag{2.15}$$

Then (2.13) follows from the dominated convergence theorem by choosing $\beta > 0$ small enough that Lemma 2.1 applies. The proof of (2.14) is a little difficult. Defining ν_n as the distribution of $V_n = S_n/n - m^D$ on $(R^n, \prod_{j=1}^n P_n(dx_j))$, where $P_n(dx_j) = \exp(mx_j)/\Psi_P(m)$, we rewrite the left hand side of (2.14) as follows:

$$
\begin{aligned}
& n^{1/2k} \int_{|z|>\beta} \exp\left[-n\{G_{QP}(m+z) - G_{PQ}(m) + \Psi_P(m) - m^D z\}\right] \\
& \quad \times \sigma_Q^{-1}(m+z)[1+o(1)] \left[\int_{|v|\le a} \exp[nvz] \, \nu_n(dv)\right] dz \\
\le\ & n^{1/2k} \int_{|z|>\beta} \exp\left[-n\{\gamma_Q(m+z) - \gamma_Q(m) - m^D z\}\right] \\
& \quad \times \sigma_Q^{-1}(m+z)[1+o(1)] \left[\int_{|v|\le a} \exp[nvz]\nu_n(dv)\right] dz \\
\le\ & \tfrac{3}{2} n^{1/2k} \left[\int_{z>\beta} \exp[-(n-n_0)\{\gamma_Q(m+z) - \gamma_Q(m) - m^D z - az\}] \right. \\
& \quad \left. \times \exp[-n_0\{G_{QP}(m+z) - G_{QP}(m)\}] \sigma_Q^{-1}(m+z)[1+o(1)] dz\right] \\
& \quad + \left[\int_{z<-\beta} \exp[-(n-n_0)\{\gamma_Q(m+z) - \gamma_Q(m) - m^D z - az\}] \right. \\
& \quad \left. \times \exp[-n_0\{G_{QP}(m+z) - G_{QP}(m)\}] \sigma_Q^{-1}(m+z)[1+o(1)] dz\right] \\
\le\ & \tfrac{3}{2} n^{1/2k} \left[\int_{z>\beta} \exp[-(n-n_0)h_1(z)] g_1(z) dz \right. \\
& \quad \left. + \int_{w>\beta} \exp[-(n-n_0)h_2(w)] g_2(w) dw\right] \\
=\ & \tfrac{3}{2} n^{1/2k} [c_1 \exp\{-nh_1(\beta)\}][1+o(1)] + c_2 \exp\{-nh_2(\beta)\}[1+o(1)]] \\
=\ & O(e^{-ns}), \quad n \to \infty,
\end{aligned}
$$

where

$$
\begin{aligned}
h_1(z) =& \gamma_Q(m+z) - \gamma_Q(m) - \gamma_Q'(m)z - Az \\
g_1(z) =& \exp[-n_o\{G_{QP}(m+z) - G_{QP}(m)\}]\gamma_Q^{-1}(m+z), \\
h_2(w) =& \gamma_Q(m-w) - \gamma_Q(m) - \gamma_Q'(m)w - Aw, \\
g_2(w) =& g_1(-w) \text{ and } s = (1/2)\min\{h_1(\beta), h_2(\beta)\}.
\end{aligned}
$$

By choosing $A > 0$ so that $h_i(\beta) > 0$ and $h_i'(\beta) > 0, i = 1, 2$, we establish the lemma.

LEMMA 2.12. (Transfer principle 2.) *Under the same assumptions for P and Q as in Lemma 2.11, there exist $\beta > 0$ and $A > 0$ depending only on m and $\beta^D > 0$ and $A^D > 0$ depending only on $m^D = \Psi'_P(m)$ such that for $a \in (0, A)$ and for bounded continuous function $h(\cdot)$, as $n \to \infty$,*

(2.16)

$$\int_R \left[\int_{s_n/n \in [m^D, m^D+a]} h\{n^{1/2k}(s_n/n - m^D)/nm_1^D\} \prod_{j=1}^n M_{n}, z(dx_j) \right] h_n(z)dz$$
$$= \int_{zn^{-1/2k} \in [0,\beta]} \left[\int_{R^n} h\{n^{1/2k}(s_n/n - m^D)/nm_1^D\} \prod_{j=1}^n M_n, z(dx_j) \right]$$
$$\times h_n(z)dz + o(1)$$

and for $a \in (0, A^D)$ and any bounded continuous function $h^D(\cdot)$

(2.17)

$$\int_R \left[\int_{s_n/n \in [m, m+a]} h^D\{n^{1/2k}(s_n/n - m)/nm_1\} \prod_{j=1}^n M_{n,z}^D(dx_j) \right] h_n^D(z)dz$$
$$= \int_{zn^{-1/2k} \in [0,\beta^D]} \left[\int_{R^n} h^D\{n^{1/2k}(s_n/n - m)/nm_1\} \prod_{j=1}^n M_{n,z}^D(dx_j) \right]$$
$$\times h_n^D(z)dz + o(1),$$

where $m_1^D = \Psi''_P(m), m_1 = \Psi''_Q(m^D), m = \Psi'_Q(m^D)$ and $m^D = \Psi'_P(m)$.

Proof. It suffices to prove the following results: as $n \to 0$,

$$\text{(2.18)} \quad \int_{zn^{-1/2k} \in [0,\beta]} \left[\int_{s_n/n \in [m^D, m^D+a]} \prod_{j=1}^n M_n, z(dx_j) \right] h_n(z)dz = o(1),$$

and

$$\text{(2.19)} \quad \int_{zn^{-1/2k} \in [0,\beta]^c} \left[\int_{s_n/n \in [m^D, m^D+a]} \prod_{j=1}^n M_n, z(dx_j) \right] h_n(z)dz = o(1),$$

By Corollary 2.9 we have

$$\text{(2.20)} \quad \int_{s_n/n \in [m^D, m^D+a]^c} \prod_{j=1}^n M_n, z(dx_j) \to 0, \ n \to \infty \ \text{ for each } z > 0.$$

Hence the result (2.18) follows from the dominated convergence theorem. The proof of (2.19) is similar to that of (2.14). We first rewrite left hand side of (2.19) as follows:

$$\begin{aligned}
&n^{1/2k}\int_{z\in[0,\beta]^c}\exp\left[-(n-n_0)\{\gamma_Q(m+z)-\gamma_Q(m)-\gamma_Q'(m)z\}\right]\\
&\times\int_{v\in[0,a]}\exp[nvz]\nu_n(dv)\cdot\exp\left[-n_0\{G_{QP}(m+z)-G_{QP}(m)\}\right]\\
&\times\sigma_Q^{-1}(m+z)[1+o(1)]\exp[-n_0\{\Psi_P(m+z)-\Psi_P(m)-m^D z\}]dz\\
&\le \tfrac{3}{2}n^{1/2k}\left[\int_{z>\beta}\exp[-(n-n_0)h_1(z)]g_1(z)dz\right.\\
&\quad\left.+\int_{w\ge 0}\exp[-(n-n_0)h_3(w)]g_2(w)dw\right]\\
&\le \tfrac{3}{2}n^{1/2k}[c_1n^{-1}\exp\{-nh_1(\beta)\}[1+o(1)]+c_2n^{-1/2}[1+o(1)]]\\
&=o(1),
\end{aligned}$$

where $h_3(w)=\gamma_Q(m-w)-\gamma_Q(m)+\gamma_Q'(m)w$ and h_1, g_1, g_2 are the same functions as in Lemma 2.11. To complete the proof, we choose $A>0$ so that $h_1(\beta)>0$, and $h_1'(\beta)>0$. Similarly, we can get (2.17) by symmetry.

THEOREM 2.13. *Let P and Q be probability measures which satisfy the uniform local limit theorems of Daniels(1954). Then there exist $A=A(m)>0$ and $A^D=A(m^D)>0$ such that the following holds. If m is a local or a global minimum of $G_{QP}(\cdot)$, then for any $a\in(0,A)$,*

$$\left(\frac{S_n}{n}\,\middle|\,\frac{S_n}{n}\in[m^D-a,\ m^D+a]\right)\ \xrightarrow{d}\ \delta(u-m^D) \tag{2.21}$$

and for $a\in(0,\ A^D)$

$$\left(\frac{S_n^D}{n}\,\middle|\,\frac{S_n^D}{n}\in[m-a,\ m+a]\right)\ \xrightarrow{d}\ \delta(u-m) \tag{2.22}$$

If m is a point of inflection of $G_{QP}^{(2k)}(\cdot)$ and $c_{2k}(m)=G_{QP}(m)>0$ then for $a\in(0,\ A)$,

$$\left(\frac{S_n}{n}\,\middle|\,\frac{S_n}{n}\in[m^D,\ m^D+a]\right)\ \xrightarrow{d}\ \delta(u-m^D) \tag{2.23}$$

and for $a \in (0,\ A^D)$

$$\left(\frac{S_n^D}{n}\,\middle|\,\frac{S_n^D}{n} \in [m,\ m+a]\right) \xrightarrow{d} \delta(u-m). \tag{2.24}$$

If $c_{2k}(m) < 0$*, conditioning on* $S_n/n \in [m^D - a, m^D]$ *and* $S_n^D \in [m-a,\ m]$ *respectively, we have analogous results hold.*

Proof. The theorem follows from Theorem 2.14 and 2.15 below.

THEOREM 2.14. *Let* P *and* Q *as in Theorem 2.13. If* m *is a global or a local minimum of type* k *for* G_{QP}*, then for* $a \in (0,\ A)$*,* A *as in Theorem 2.13,*

$$\left(\frac{S_n - nm^D}{m_1^D n^{1-1/2k}}\,\middle|\,\frac{S_n}{n} \in [m^D - a,\ m^D + a]\right) \xrightarrow{d} F_{k,c_{2k}} \tag{2.25}$$

and for $a \in (0,\ A^D)$*,* A^D *as in Theorem 2.13,*

$$\left(\frac{S_n^D - nm}{m_1 n^{1-1/2k}}\,\middle|\,\frac{S_n^D}{n} \in [m-a,\ m+a]\right) \xrightarrow{d} F_{k,c'_{2k}}, \tag{2.26}$$

where $c'_{2k} = c_{2k}(\Psi''_P(m))^{-2k}$, $m_1^D = \Psi''_P(m)$, $m_1 = \Psi''_Q(m^D)$ and

$$dF_{k,c_{2k}} = \begin{cases} N(0,\ 1/m_1^D + 1/c_2) & \text{if } k=1 \\ \frac{\exp[-c_{2k}z^{2k}/(2k)!]}{\int_R \exp[-c_{2k}t^{2k}/(2k)!]dt} & \text{if } k \geq 2, \end{cases}$$

$$dF_{k,c'_{2k}} = \begin{cases} N(0,\ 1/m_1^D + 1/c'_2) & \text{if } k=1 \\ \frac{\exp[-c'_{2k}z^{2k}/(2k)!]}{\int_R \exp[-c'_{2k}t^{2k}/(2k)!]dt} & \text{if } k \geq 2. \end{cases}$$

Proof. By Lemma 2.8, we have

$$\frac{S_n - nm^D}{m_1^D n^{1-1/2k}} \xrightarrow{d} \begin{cases} \delta(w-z) & \text{if } k \geq 2 \\ N(z,\ 1/m^D) & \text{if } k=1 \end{cases}$$

under $M_{n,z}$. Also by Lemma 2.7 and the dominated convergence theorem, we have, for each z,

(2.27)

$$\frac{h_n(z)\, I_{(|z|\le \beta n^{1/2k})}(z)}{\int_{|t|\le \beta n^{1/2k}} h_n(t)\, dt} \longrightarrow \frac{\exp[-c_{2k} z^{2k}/(2k)!]}{\int_R \exp[-c_{2k} t^{2k}/(2k)!] dt} \quad \text{as } n \to \infty.$$

Hence by theorem of Sethurman (1961), we obtain

$$\frac{(S_n - nm^D)}{m_1^D n^{1-1/2k}} \xrightarrow{d} F_{k,c_{2k}}$$

under

$$\mu_n^{(1)}(dx_1, \cdots, dx_n) = \frac{\int_{|z|\le \beta n^{1/2k}} \prod_{j=1}^n M_{n,z}(dx_j) h_n(z)\, dz}{\int_{|z|\le \beta n^{1/2k}} h_n(z)\, dz}.$$

Hence using the transfer principle 1 we have the desired result

$$\text{(2.28)} \qquad \frac{(S_n - nm^D)}{m^D n^{1-1/2k}} \xrightarrow{d} F_{k,c}$$

under

$$\mu_n^{(2)}(dx_1, \cdots, dx_n) = K_n^{-1} \int_{s_n/n \in [m^D - a,\, m^D + a]} \prod_{j=1}^n M_{n,z}(dx_j) h_n(z)\, dz,$$

where K_n is a normalizing constant. Since $m^D = \Psi'_P(m)$ is a global or a local minimum of type k for G_{PQ}, we also get (2.26) in the same manner. This completes the Theorem 2.14.

THEOREM 2.15. *Let P and Q be as defined in Theorem 2.14. If m is a point of inflection of type k for G_{QP} with $c_{2k} = c_{2k}(m) > 0$. Then for $a \in (0, A)$, A as in Theorem 2.13,*

$$\text{(2.29)} \qquad \left(\frac{S_n - nm^D}{m_1^D n^{1-1/2k}} \,\middle|\, \frac{S_n}{n} \in [m^D, m^D + a] \right) \xrightarrow{d} F_{k,c_{2k}} \Big| [0, \infty)$$

and for $a \in (0, A^D)$, A^D as in Theorem 2.13,

$$(2.30)\qquad \left(\frac{S_n^D - nm}{m_1 n^{1-1/2k}} \,\middle|\, \frac{S_n^D}{n} \in [m, m+a]\right) \xrightarrow{d} F_{k,c'_{2k}}\Big|[0, \infty),$$

where for $k \geq \frac{3}{2}$, $F_{k,c_{2k}}\,||[0, \infty) = I_{(z>0)}(z)\, dF_{k,c_{2k}}$ and $F_{k,c'_{2k}}\,||[0, \infty) = I_{(z>0)}(z)\, dF_{k,c'_{2k}}$.

If $c_{2k} = c_{2k}(m) < 0$, we condition on $S_n/n \in [m^D - a, m^D]$ and $S_n^D/n \in [m - a, m]$ and then analogous results hold with the limit distribution supported on $(-\infty, 0)$.

Proof. By Lemma 2.8, we have

$$\frac{S_n - nm^D}{m_1^D n^{1-1/2k}} \xrightarrow{d} \begin{cases} \delta(w - z) & \text{if } k \geq 2 \\ N(z, 1/m_1^D) & \text{if } k = 1 \end{cases}$$

under $M_{n,z}$. Also we have, for each z, as $n \to \infty$,

$$(2.31)\qquad \frac{h_n(z)\, I_{(z\in[0,\beta n^{1/2k}))}(z)}{\int_{t\in[0,\beta n^{1/2k})} h_n(t)\, dt} \longrightarrow \frac{\exp[-c_{2k} z^{2k}/(2k)!]}{\int_0^\infty \exp[-c_{2k} t^{2k}/(2k)!]dt}.$$

by Lemma 2.7 and the dominated convergence theorem. Hence by theorem of Sethuraman (1961), we obtain

$$\frac{(S_n - nm^D)}{m_1^D n^{1-1/2k}} \xrightarrow{d} F_{k,c_{2k}}\Big|[0, \infty)$$

under

$$\mu_n^{(3)}(dx_1, \cdots, dx_n) = \frac{\int_{|z|\leq\beta n^{1/2k}} \prod_{j=1}^n M_{n,z}(dx_j) h_n(z)\, dz}{\int_0^\infty h_n(z)\, dz}$$

. By the transfer principles we have

$$(2.32)\qquad \frac{(S_n - nm^D)}{m_1^D n^{1-1/2k}} \xrightarrow{d} F_{k,c_{2k}}\Big|[0, \infty)$$

under

$$\mu_n^{(4)}(dx_1, \cdots, dx_n) = K_n^{-1} \int_{s_n/n\in[m^D, m^D+a]} \prod_{j=1}^n M_{n,z}(dx_j) h_n(z)\, dz,$$

where K_n is a normalizing constant. Similarly we get (2.30) by symmetry.

REMARK 2.16. We can immediately extend the definition of $c_{2k} = G_{QP}^{(2k)}(m)$ to the case that k is an half integer greater than or equal to $3/2$ and c_{2k} is any nonzero real number.

THEOREM 2.17. (J.Sethuramen) *Let λ_n be a sequence of probability measure on $T \times W$, where T and W are topological spaces. Let μ_n be the marginal probability measure of λ_n on T and $\nu_n(t,\cdot)$ be a conditional probability measure on W. Suppose that μ_n converges to a probability measure μ for every measurable set in T and for almost all t with respect to μ, $\nu_n(t,\cdot)$ converges weakly to $\nu(t,\cdot)$. Then λ_n converges weakly to ν, where $\lambda(A \times B) = \int \nu(t,B)\mu(dt)$ for every measurable rectangular set $A \times B$.*

3. Limit theorems for the maximal case

In this section we prove the analogous dual limit theorems in section 2 when G_{QP} has a local minimum of maximal type.

DEFINITION 3.1. Let $L'_Q \subset L_Q$ be the class of probability measures P such that

$$\inf_{t \in D_Q} G_{QP}(t) < \min\left\{ \liminf_{t \to a} G_{QP}(t),\ \liminf_{t \to b} G_{QP}(t) \right\},$$

where $D_Q = (a,\, b)$. Similarly, we define $L'_P \subset L_P$ by exchanging the role of Q and P.

REMARK 3.2. By Theorem 3.2 of Lee, Kim and Jeon(1993), if $P \in L'_Q$, then $Q \in L'_P$ and vice versa.

LEMMA 3.3. *For a probability measure Q and $P \in L'_Q$, define*

$$G^*_{QP} = \inf_{s \in D_Q} G_{QP}(s)$$

and let V be any closed (possible unbounded) subset of D_Q which contain no global minimum of G_{QP}. Then there exists $\varepsilon > 0$ so that as $n \to \infty$,

$$\exp(nG^*_{QP}) \int_V \exp[-nG_{QP}(u)]\sigma_Q^{-1}({\Psi'_Q}^{-1}(u))\,du = O(e^{-n\varepsilon}).$$

Proof. By assumption, there exists $\varepsilon > 0$ so that

$$\inf_{s\in V} G_{QP}(s) \geq \inf_{s\in D_Q} G_{QP}(s) + \varepsilon = G^*_{QP} + \varepsilon.$$

Hence,

$$\begin{aligned} &\exp(nG^*_{QP}) \int_V \exp[-nG_{QP}(u)]\sigma_Q^{-1}({\Psi'_Q}^{-1}(u))\,du \\ &\leq \exp[-(n-n_0)]\exp(G_{QP}) \int_R \exp[-n_0 G_{QP}(u)]\,du \leq Ke^{-n\varepsilon} \\ &= O(e^{-n\varepsilon}). \end{aligned}$$

Note that for $Q \in L'_P$, we can obtain the similar result.

THEOREM 3.4. *Let Q be a probability measure Q and $P \in L'_Q$. Let $\{m_1, \cdots, m_s\}$ be the set of global minima of G_{QP} of type $\{k_1, \cdots, k_s\}$ respectively, and let $k^* = \max\{k_1, \cdots, k_s\}$. Then $\{m_1^D, \cdots, m_s^D\}$ is the set of global minima of G_{PQ} of type $\{k_1, \cdots, k_s\}$, where $m_1^D = \Psi'_P(m_i), i = 1, 2, \cdots, s$. Further*

$$\frac{S_n}{n} \xrightarrow{d} \frac{\sum_{i=1}^s b(m_i)\delta(u - m_i^D)}{\sum_{i=1}^s b(m_i)} \tag{3.1}$$

and

$$\frac{S_n^D}{n} \xrightarrow{d} \frac{\sum_{i=1}^s b'(m_i^D)\delta(u - m_i^D)}{\sum_{i=1}^s b'(m_i^D)} \tag{3.2}$$

where

$$b(m_i) = \begin{cases} [c_{2k}(m_i)]^{-1/2k_i} & \text{if } k_i = k^* \\ 0 & \text{otherwise} \end{cases}$$

$$b'(m_i^D) = \begin{cases} [c'_{2k}(m_i^D)]^{-1/2k_i} & \text{if } k_i = k^* \\ 0 & \text{otherwise} \end{cases}$$

and $c_{2k}(m_i) = G_{QP}^{(2k)}(m_i)$, $c'_{2k}(m_i^D) = G_{PQ}^{(2k_i)}(m_i^D)$ $= c_{2k}(m_i)(\Psi''_P(m_i))^{-2k_i}$.

Proof. It suffices to prove that for any bounded function $h(\cdot)$, as $n \to \infty$

$$\int_{R^n} h(s_n/n)\mu_n^G(dx_1, \cdots, dx_n) \longrightarrow \frac{\sum_{i=1}^n h(m_i^D)b(m_i)}{\sum_{i=1}^n b(m_i)}. \tag{3.3}$$

Defining $P_z(dx_1, \cdots, dx_n) = \prod_{j=1}^n \exp[x_j z - \Psi_P(z)] P(dx_j)$, we express the left hand side of (3.3) as

$$\begin{aligned}(3.4)\qquad &\int_{R^n} h(s_n/n)\mu_n^G(dx_1, \cdots, dx_n) \\ &= K_n^{-1} \int_R \exp[-nG_{QP}(z)]\sigma_Q^{-1}(z)[1+o(1)] \\ &\quad \cdot \left[\int_{R^n} h(s_n/n)P_z(dx_1, \cdots, dx_n)\right] dz,\end{aligned}$$

where $K_n = \int_R \exp[-nG_{QP}(z)]\sigma_Q^{-1}(z)[1+o(1)]\, dz$ is a normalizing constant. Let $\xi > 0$ be a number such that $0 < \xi < \min\{|m_j - m_i| : 1 \le i < j \le s\}$ and let $V = R - \cup_{i=1}^s (m_i^D - \xi, m_i^D + \xi)$. By Lemma 3.3 , there exists $\varepsilon > 0$ such that

$$(3.5)\quad \exp(nG_{QP}^*) \int_V \exp[-nG_{QP}(z)]\sigma_Q^{-1}(z)\, dz = O(e^{-n\varepsilon}), \text{ as } n \to \infty.$$

For each $i = 1, \cdots, s$, let $k = k(m_i)$, $c_{2k} = c_{2k}(m_i)$. Then

$$\begin{aligned} & n^{-1/2k} \exp(nG_{QP}^*) \int_{m_i-\xi}^{m_i+\xi} \exp[-nG_{QP}(z)]\sigma_Q^{-1}(z)\,[1+o(1)] \\ & \times \left[\int_{R^n} h(s_n/n)P_z(dx_1, \ldots, dx_n)\right] dz \\ (3.6)\qquad & = n^{1/2k} \int_{-\xi}^{\xi} \exp[-n(G_{QP}(m_i+z) - G_{QP}(m_i))]\sigma_Q^{-1}(m_i+z)[1+o(1)] \\ & \times \left[\int_{R^n} h(s_n/n)P_z(dx_1, \ldots, dx_n)\right] dz \\ & = \int_{|z|\le \xi n^{1/2k}} \left[\int_{R^n} h(s_n/n)\prod_{j=1}^n M_{n,z}(dx_j)\right] h_n(z)\, dz \\ & = h_n(m_i^D) c_{2k}^{-1/2k} \int_R \exp[-z^{2k}/(2k)!]\, dz + o(1), \quad n \to \infty. \end{aligned}$$

In deriving the last equation, we used Lemma 2.7, Corollary 2.9 and the dominated convergence theorem. Now (3.3) follows from (3.5) and (3.6) separately applying to numerator and denominate of (3.3). (3.2) can be proved easily by symmetry.

THEOREM 3.5. *Let P and Q be as defined in Theorem 3.4. Let m be one of the global minima of maximal type k^* of G_{QP}. Then $m^D = \Psi'_P(m)$ is a global minimum type k^* of G_{PQ} and*

$$\frac{(S_n - nm^D)}{m_1^D n^{1-1/2k^*}} \xrightarrow{d} \bar{b}(m) F_{k^*, c_{2k^*}} \tag{3.7}$$

and

$$\frac{(S_n^D - nm)}{m_1 n^{1-1/2k^*}} \xrightarrow{d} \bar{b}(m) F_{k^*, c'_{2k^*}}, \tag{3.8}$$

where

$$dF_{k^*, c_{2k^*}} = \begin{cases} N(0,\, 1/m_1^D + 1/c_2) & \text{if } k^* = 1 \\ \frac{\exp[-c_{2k^*} z^{2k^*}/(2k^*)!]}{\int_R \exp[-c_{2k^*} t^{2k^*}/(2k^*)!]\, dt,} & \text{if } k^* \geq 2, \end{cases}$$

$$\bar{b}(m) = \frac{b(m)}{\sum_{i=1}^s b(m_i)}, \quad \bar{b}(m^D) = \frac{b(m^D)}{\sum_{i=1}^s b(m_i^D)}$$

and

$$dF_{k^*, c'_{2k^*}} = \begin{cases} N(0,\, 1/m_1 + 1/c'_2) & \text{if } k^* = 1 \\ \frac{\exp[-c'_{2k^*} z^{2k^*}/(2k^*)!]}{\int_R \exp[-c'_{2k^*} t^{2k^*}/(2k^*)!]\, dt} & \text{if } k^* \geq 2. \end{cases}$$

If m is not a global minimum of G_{QP} of maximal type then m^D is also not a global minimum of G_{PQ} of maximal type and

$$\frac{(S_n - nm^D)}{n^{1/c}} \xrightarrow{d} 0 \qquad \text{for } c > 1 \tag{3.9}$$

and

$$\frac{(S_n^D - nm)}{n^{1/c}} \xrightarrow{d} 0 \qquad \text{for } c > 1 \tag{3.10}$$

Proof. We define $Y_n = (S_n - nm^D)/m^\alpha$ for some positive number α and denoted by $N_n = N_n(a)$ the event $\{S_n/n \in [m^D - a,\, m^D + a]\}$. Then by conditioning we have for any Borel set B,

(3.11)

$$Pr\{Y_n \in B\} = Pr\{Y_n \in B | N_n\} P\{N_n\} + Pr\{Y_n \in B | N_n^c\} Pr\{N_n^c\}.$$

If B is bounded and $\alpha < 1$, there exists a positive number $c > 0$ so that $\{Y_n \in B\} \subset \{|S_n/n - m^D| < cn^{\alpha-1}\}$: the latter set is disjoint from N_n for all sufficiently large n, so that $Pr\{Y_n \in B | N_n^c\} \to 0$. In the case where m is not a global minimum of maximal type, we let $\alpha = 1/c$ and choose a sufficiently small so that $[m^D - a,\ m^D + a] = 0$, where $\tau(\cdot)$ is the discrete distribution given by the right side of (3.1). Then by Theorem 3.4 we have $Pr\{N_n\} \to 0$. Hence from (3.11), $Pr\{Y_n \in B\} \to 0$ for any bounded Borel set B and thus

$$\frac{(S_n - nm^D)}{n^{1/c}} \xrightarrow{d} 0$$

as desired. In the case where m is a global minimum of maximal type k, we let $\alpha = 1 - 1/2k$ and choose a in accordance with Theorems 2.13 and 2.14. It follows from Theorem 3.4 that $Pr\{N_n\} \to b(m)$. Thus by Theorem 2.14 and (3.11) we have for any bounded Borel set B that

$$Pr\{Y_n \in B\} \to b(m) \int_B dF_{k^*, c_{2k^*}}$$

as desired. By symmetry we get (3.10). This completes the proof.

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