Comm. Korean Math. Soc. **10** (1995), No. 1, pp. 195–205

PROBABILITY INEQUALITIES FOR PRODUCT OF INDEPENDENT POISSON PROCESSES

JOONG SUNG KWON

1. Introduction

Let $(\mathbf{S}, \mathcal{S}, P)$ be a finite measure space. A process
$Y = \{Y(f) \mid f : \text{nonnegative measurable function on } \mathbf{S}\}$
is said to be a *Poisson process with parameter* λ, where $\lambda = P(\mathbf{S})$ if it has independent increments, in the sense that $Y(A_1)$, $Y(A_2)$, ..., $Y(A_k)$ are independent whenever A_1, A_2, ... , A_k are disjoint subsets of $\mathbf{S}$, and the marginal distributions are Poisson with parameters $P(A_i)$. We can represent such a Poisson process as follows : Let $\{U_i\}$ be a sequence of independent identically distributed $\mathbf{S}$-valued random variables and $N = Y(\mathbf{S})$. Then, for a function f on $\mathbf{S}$, we can write

$$Y(f) = \sum_{i=1}^{N} f(U_i). \tag{1.1}$$

Note that Y is a well defined independently scattered atomic random measure, that is, it is a countably additive set function on a measurable space $(\mathbf{S}, \mathcal{S})$ with values in $L^0(\Omega, \mathbf{P})$ which is independently scattered where $(\Omega, \mathbf{P})$ denotes underlying probability space. Following the classical measure theory, since the process Y mentioned above is an atomic random measure the existence of product random measures is evident due to Fubini theorem. But resulting product measure is no longer independently scattered.

Received April 20, 1994.
AMS 1980 Subject Classifications: Primary 60E15; Secondary 60G17.
Key words and phrases: Probability inequality, Poisson Process, Product Poisson Process. Empirical Process.

Let Y_1 and Y_2 denote two Poisson processes on $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively. Let $\{U_i\}$ and $\{V_i\}$ be sequences of independent identically distributed $\mathbf{S}_1$- and $\mathbf{S}_2$- valued random variables. For a function f on $\mathbf{S}_1 \times \mathbf{S}_2$,

$$Y_1 \times Y_2(f) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} f(U_i, V_j),$$

where $N_i = P_i(\mathbf{S}_i)$, $i = 1, 2$. This product process naturally arises as building blocks of product of infinitely divisible processes when we use series representation as does (1.1) in the infinitely divisible processes(Adler and Feigin(1984), Bass and Pyke(1984)). However they are not studied enough to be understood. Especially their probability bounds are not known yet as far as we know, which is the motivation of our study. In section 2 we devote to deriving of exponential probability bounds for product of independent Poisson processes. As a by-product, we obtain probability inequalities for product of independent empirical measures. In section 3 we will calculate variances of various products of Poisson's appeared throughout the paper.

2. Probability inequalities

In this section we will obtain probability bounds for product of independent Poisson processes. For this we need the following elementary lemma whose proof appeared in Bass and Pyke(1984)

LEMMA 2.1. (Bass and Pyke(1984)) *Let N be a Poisson random variable with parameter λ. Then*

$$\begin{aligned} P(N > \eta) &< \exp(-\lambda)\exp(-\eta[\ln(\eta/\lambda) - 1]) && \text{if } \eta \geq \lambda \\ &< \exp(-\lambda)\exp(-\eta) && \text{if } \eta \geq e^2\lambda. \end{aligned}$$

PROPOSITION 2.2. *Let N_1 and N_2 be independent Poisson random variables with parameters λ and μ respectively. Then*

$$\begin{aligned} P(N_1 N_2 > \eta) &< \exp\left\{-(\lambda+\mu) + \sqrt{2\eta}(1 - \log(\sqrt{2\eta}/(\lambda+\mu))\right\} \\ &\qquad \text{if} \quad \eta > (\lambda+\mu)^2/2, \\ &< \exp(-(\lambda+\mu) - \sqrt{2\eta}) \qquad \text{if} \quad \eta > e^4(\lambda+\mu)^2/2. \end{aligned}$$

Proof. Note that

$$N_1 N_2 \leq (N_1^2 + N_2^2)/2 \leq (N_1 + N_2)^2/2. \tag{2.1}$$

By using Chebyshev's inequality and the convexity of e^x, we obtain

$$\begin{aligned} P(N_1 N_2 > \eta) &= P\left(\exp(s\sqrt{N_1 N_2}) > \exp(s\sqrt{\eta})\right) \\ &\leq \inf_{s>0} E(\exp(s(\sqrt{N_1 N_2} - \sqrt{\eta}))). \end{aligned}$$

Since N_1 and N_2 are independent, (2.1) implies

$$E\left(\exp(s\sqrt{N_1 N_2})\right) \leq \exp(-(\lambda + \mu)) \exp\left((\lambda + \mu)\exp(s/\sqrt{2})\right),$$

where we used $E(\exp(cN)) = \exp(\nu(\exp c - 1))$, for N Poisson with parameter ν.
Let $\varphi(s) = (\lambda + \mu)e^{s/\sqrt{2}} - s\sqrt{\eta}$. Then from $\varphi'(s) = 0$ we have $s_1 = \sqrt{2}\log(\sqrt{2\eta} - (\lambda + \mu))$ and $\varphi''(s_1) > 0$. Hence,

$$P(N_1 N_2 > \eta) \leq \exp(-(\lambda + \mu)) \exp\left(\sqrt{2\eta}\left(1 - \log(\sqrt{2\eta} - (\lambda + \mu))\right)\right).$$

In particular, if $\eta > e^4(\lambda + \mu)^2/2$, then $P(N_1 N_2 > \eta) \leq \exp(-(\lambda + \mu)) \exp(-\sqrt{2\eta})$.

PROPOSITION 2.3. *Under the same assumptions as in Proposition 2.2,*

$$\begin{aligned} P(N_1 N_2 > \eta) &< \exp(-\lambda)\exp(-\sqrt{\eta}/a) + \exp(-\mu)\exp(-a\sqrt{\eta}) \\ &\qquad \text{if} \quad a > 0 \quad \text{and} \quad \eta > \max\{a^2 e^4 \lambda^2, e^4 \mu^2/a^2\} \\ &< 2\exp(-\lambda)\exp(-\sqrt{\eta}/a) \\ &\qquad \text{if} \quad a = \left\{(\lambda - \mu) + \sqrt{(\lambda - \mu)^2 + 4\eta}\right\}/2\sqrt{\eta} \\ &\qquad\qquad \text{and} \quad \eta > \max\left\{a^2 e^4 \lambda^2, e^4 \mu^2/a^2\right\}. \end{aligned}$$

Proof. Notice that $P(N_1 N_2 > \eta) \leq P(N_1 > \sqrt{\eta}/a) + P(N_2 > a\sqrt{\eta})$. By lemma 2.1, we have the first inequality, and from $\lambda + \sqrt{\eta}/a = \mu + a\sqrt{\eta}$ we have the second.

PROPOSITION 2.4. *Under the same assumptions as in Proposition 2.2*

$$P(N_1N_2 > \eta) < \exp(-\eta)\exp(-x) + \exp(-\mu)\exp(-y)$$

if $x + y = \sqrt{2\eta}$ and $\eta > e^4(\lambda+\mu)^2/2$.

$$P(N_1, N_2 > \eta) < 2\exp(-\lambda)\exp(-\{(\mu-\lambda) + \sqrt{2\eta}/2\})$$

if $\eta > \max\{[(2e^2+1)\lambda - \mu]^2/2, [(2e^2+1)\mu - \lambda]^2/2, e^4(\lambda+\mu)^2/2\}$.

Proof. Let x and y be positive real numbers such that $x + y = \sqrt{2\eta}$. Then by (2.1) we can write $P(N_1N_2 > \eta) \le P(N_1 + N_2 > \sqrt{2\eta}) \le P(N_1 > x) + P(N_2 > y)$. Now use lemma 2.1 again.

For notational simplicity we denote $Z = Y_1 \times Y_2$.

THEOREM 2.5. *Let Y_1 and Y_2 be Poisson processes on $\mathbf{S}_1$ and $\mathbf{S}_2$ with parameters λ and μ respectively. Let $N_1 = Y_1(\mathbf{S}_1)$ and $N_2 = Y_2(\mathbf{S}_2)$. Then for f a nonnegative bounded $\sigma(S_1 \times S_2)$-measurable function with $\delta = \int f(dP_1 \times P_2)$ and $\delta_f^2 = \int f^2 d(P_1 \times P_2) - \delta^2$, we have*

$$\begin{aligned} &P(Z(f) > \eta) \\ &\le \exp\left(-\mu\frac{\delta^2(\eta-\tau)^2}{4\tau(\tau\delta_f^2 + \delta(\eta-\tau/3))}\right) + \exp\left(-\lambda\frac{\delta^2(\eta-\tau)^2}{4\tau(\tau\delta_f^2 + \delta(\eta-\tau/3))}\right) \\ &\quad + P(N_1N_2 > \tau/\delta), \end{aligned}$$

provided that

$$\frac{\delta^2(\eta-\tau)^2}{2\tau(\tau\delta_f^2 + \delta(\eta-\tau/3))} < 1.$$

Note that $E\{(Y_1 \times Y_2(f))/N_1N_2\} = \delta$. When $N_1 = 0$ or $N_2 = 0$ we interpret $(Y_1 \times Y_2(f))/N_1N_2 = 1$. Using this expression, we can proceed to the proof of theorem 2.5.

Proof of Theorem 2.5. Observe that for any $\tau > 0$,

$$\begin{aligned} &P(Z(f) > \eta) \\ &= P(Z(f) > \eta, N_1N_2\delta > \tau) + P(Z(f) > \eta, N_1N_2\delta \le \tau) \\ &\le P(N_1N_2 > \tau/\delta) + P\left(\frac{Z(f)}{N_1N_2\delta} > \eta/\tau\right). \end{aligned} \tag{2.2}$$

The first term in (2.2) can be bounded by proposition 2.2. Hence it remains to handle the second. For the second term note that

$$P\left[\frac{Z(f)}{N_1 N_2 \delta} > \eta/\tau | N_1 = m, N_2 = n\right] = P\left[T_{mn}(f) > \eta\delta/\tau\right]$$

and

$$T_{mn} := \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f(U_i, V_j) := P_m \times Q_n(f)$$

where P_m (resp. Q_n) is the empirical measure based on $U_1, U_2, \cdots, U_m$ (resp. $V_1, V_2, \cdots, V_n$) with $\mathcal{L}(U) = P_1(\cdot)/P_1(\mathbf{S}_1)$ (resp. $\mathcal{L}(V)$ $= P_2(\cdot)/P_2(\mathbf{S}_2)$). Here we will call $P_m \times Q_n$ the product of empirical measures P_m and Q_n. Since the bound for the product of empirical measures is of independent interest we state it separately as:

THEOREM 2.6. *For* $\gamma > 0$, $\delta = \int f d(P_1 \times P_2)$ *and* $\sigma_f{}^2 = Var(f(U_i, V_j))$

$$P(T_{mn} > \delta + \gamma) \leq \exp\left\{-\frac{\min(m,n)\gamma^2}{2(\sigma_f{}^2 + \gamma/3)}\right\},$$

Proof. Assume that $\min(m,n) = m$ and write $T_{mn} = n^{-1}\sum_{k=1}^{n} T_{mn}^{(k)}$, where

$$T_{mn}^{(k)} = m^{-1}\left\{\sum_{i=1}^{n-k-1} f(U_i, V_{i+k+=1}) + \sum_{i=n-k+2}^{m} f(U_i, V_{i+k-n-1})\right\}.$$

That is, each $T_{mn}^{(k)}$ is a sum of m-independent identically distributed random variables and each $T_{mn}^{(k)}$ has the same distribution. Hence by Hoeffding's inequality(1963)

$$\exp(sT_{mn}) \leq n^{-1}\sum_{k=1}^{n} \exp(sT_{mn}^{(k)}), \quad \text{and so}$$

$$E(\exp(sT_{mn})) \leq E(\exp(sT_{mn}^{(1)})).$$

Thus $E(\exp(sT_{mn} - s\gamma - s\delta)) \le E(\exp(sT^{(1)}_{mn} - s\gamma - s\delta))$.
By Bernstein's inequality

$$P(T_{mn} - \delta > \gamma) \le \exp\left\{ -\frac{m\gamma^2}{2(\sigma_f^2 + \gamma/3)} \right\}. \tag{2.3}$$

If $\min(m, n) = n$, then by the same argument,

$$P(T_{mn} - \delta > \gamma) \le \exp\left\{ -\frac{n\gamma^2}{2(\sigma_f^2 + \gamma/3)} \right\}. \tag{2.4}$$

Comnine (2.3) and (2.4), then we have

$$P(T_{mn} > \delta + \gamma) \le \exp\left\{ -\frac{\min(m, n)\gamma^2}{2(\sigma_f^2 + \gamma/3)} \right\}$$

which completes the proof of theorem 2.6.

Back to the proof of Theorem 2.5. To complete the proof of theorem 2.5, we use theorem 2.6.

$$\begin{aligned} P(T_{mn}(f) > \eta\delta/\tau) &= P(T_{mn}(f) > \delta + \delta(\eta/\tau - 1)) \\ &\le \exp\left(-\frac{\min(m, n)\delta^2(\eta - \tau)^2}{\tau(\tau\delta_f^2 + \delta(\eta - \tau/3))} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} &P\Big(\frac{Z(f)}{N_1 N_2 \delta} > \eta/\tau\Big) \\ \le &E\Big(\exp\Big(-\frac{N_1\delta^2(\eta-\tau)^2}{2\tau(\tau\delta_f^2 + \delta(\eta - \tau/3))}\Big)\Big) + E\Big(\exp\Big(-\frac{N_2\delta^2(\eta-\tau)^2}{2\tau(\tau\delta_f^2 + \delta(\eta - \tau/3))}\Big)\Big) \\ \le &\exp\Big(\lambda\Big(\exp\Big(-\frac{\delta^2(\eta-\tau)^2}{2\tau(\tau\delta_f^2 + \delta(\eta - \tau/3))}\Big) - 1\Big)\Big) \\ &+ \exp\Big(\mu\Big(\exp\Big(-\frac{\delta^2(\eta-\tau)^2}{2\tau(\tau\delta_f^2 + \delta(\eta - \tau/3))}\Big) - 1\Big)\Big). \end{aligned}$$

Since $1 - e^{-x} = x - x^2/2 + (x^3/3! - x^4/4!) + \cdots \geq x - x^2/2 > x/2$, for $0 < x < 1$, we have, if $\frac{\delta^2(\eta-\tau)^2}{2\tau(\tau\delta_f^2+\delta(\eta-\tau/3))} < 1$, then

$$P\left(\frac{Z(f)}{N_1 N_2 \delta} > \eta/\tau\right) \leq \exp\left(-\lambda\frac{\delta^2(\eta-\tau)^2}{4\tau(\tau\delta_f^2+\delta(\eta-\tau/3))}\right) + \exp\left(-\mu\frac{\delta^2(\eta-\tau)^2}{4\tau(\tau\delta_f^2+\delta(\eta-\tau/3))}\right),$$

which completes the proof of our main theorem.

3. Remarks

In this section, for the completeness of the paper, we include the variances of the products which appeared in many places throughout the paper.

PROPOSITION 3.1. *Let $|A| = \delta$. Let Y_1 and Y_2 be Poission processes on $\mathbf{I}^{d_1}$ and $\mathbf{I}^{d_2}$ with parameters λ and μ respectively. Let $N_1 = Y_1(\mathbf{I}^{d_1})$ and $N_2 = Y_2(\mathbf{I}^{d_2})$. Then*

(1) $Var(N_1 N_2|A|) = |A|^2(\lambda\mu + \lambda^2\mu + \lambda\mu^2)$.
(2) $Var(Y_1 \times Y_2(A)) = \lambda\mu|A| + \lambda\mu^2 E|A_{2U}|^2 + \lambda^2\mu E|A_{2U_1} \cap A_{2U_2}|$. *or, by symmetry,* $Var(Y_1 \times Y_2(A)) = \lambda\mu|A| + \lambda^2\mu E|A_{1V}|^2 + \lambda\mu^2 E|A_{1V_1} \cap A_{1V_2}|$.
(3) $Var(T_{mn}) = (mn)^{-1}\{|A| + (1-m-n)|A|^2 + (n-1)E|A_{1V_1} \cap A_{1V_2}| + (m-1)E|A_{2U_1} \cap A_{2U_2}|\}$, *where* $T_{mn} = (mn)^{-1}\sum_{i=1}^m \sum_{j=1}^n \delta_{(U_i,V_j)}(A)$.
(4) $Var[(Y_1 \times Y_2(A))/N_1 N_2|A|] = C|A|^{-1}$, *where C is a constant depending on N_1, N_2 and $|A|$.*

Proof. (1). $Var(N_1 N_2|A|) = E(N_1 N_2|A|)^2 - \{E(N_1 N_2|A|)\}^2 = |A|^2 (\lambda\mu + \lambda^2\mu + \lambda\mu^2)$.

For (2) recall $Y_1 \times Y_2(A) = \int_{\mathbf{I}^{d_1}} Y_2(A_{2x})dY_1(x) = \sum_{i=1}^{N_1} Y_2(A_{2x_i})$, where $N_1 = Y_1(\mathbf{I}^{d_1})$ and $N_2 = Y_2(\mathbf{I}^{d_2})$.

Then $E(Y_1 \times Y_2(A)) = EN_1\mu|A_{2U}| = \lambda\mu E|A_{2U}| = \lambda\mu|A|$ where U is a uniformly distributed random variable on $\mathbf{I}^{d_1}$ and $E|A_{2U}| = \int_{\mathbf{I}^{d_1}} |A_{2U}|dU = |A|$.

$$\begin{aligned}
&E[Y_1 \times Y_2(A)]^2 \\
=&E\sum_{i=1}^{N_1}\sum_{j=1}^{N_1} Y_2(A_{2x_i})Y_2(A_{2x_j}) \\
=&E\{N_1E[Y_2^2(A_{2U}|U)] + N_1(N_1-1)E[Y_2(A_{2U_1})Y_2(A_{2U_2})]\} \\
:=&\ \mathrm{I}_1 + E(N_1(N_1-1))E(\ \mathrm{I}_2).
\end{aligned}$$

Clearly $\mathrm{I}_1 = \lambda[E\mu|A_{2U}| + \mu^2|A_{2U}|^2] = \lambda\mu|A| + \lambda\mu^2E(|A_{2U}|^2)$. And $EN_1(N_1-1) = \lambda^2$. Finally, notice that A_{2U_1} and A_{2U_2} are independent identically distributed. Hence

$$\begin{aligned}
\mathrm{I}_2 =& E[Y_2(A_{2U_1})Y_2(A_{2U_2})] \\
=&E\left[Y_2(A_{2U_1}\setminus A_{2U_2}) + Y_2(A_{2U_1}\cap A_{2U_2})\right]\left[Y_2(A_{2U_2}\setminus A_{2U_1})\right. \\
&\left.+Y_2(A_{2U_1}\cap A_{2U_2})\right] \\
=&E[Y_2(A_{2U_1}\setminus A_{2U_2})Y_2(A_{2U_2}\setminus A_{2U_1}) \\
&+ Y_2(A_{2U_1}\setminus A_{2U_2})Y_2(A_{2U_1}\cap A_{2U_2}) \\
&+ Y_2(A_{2U_1}\cap A_{2U_2})Y_2(A_{2U_2})Y_2(A_{2U_2}\setminus A_{2U_1}) + Y_2^2(A_{2U_2}\cap A_{2U_1})] \\
=&\mu^2[|A_{2U_1}\setminus A_{2U_2}||A_{2U_2}\setminus A_{2U_1}| + |A_{2U_1}\setminus A_{2U_2}||A_{2U_1}\cap A_{2U_2}| \\
&+ |A_{2U_1}\cap A_{2U_2}||A_{2U_2}\setminus A_{2U_1}| + |A_{2U_2}\cap A_{2U_1}|^2]\} + \mu|A_{2U_2}\cap A_{2U_1}|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E\ \mathrm{I}_2 =&\mu^2E\{(|A_{2U_1}\setminus A_{2U_2}| + |A_{2U_1}\cap A_{2U_2}|)(|A_{2U_2}\setminus A_{2U_1}| \\
&+ |A_{2U_1}\cap A_{2U_2}|)\} + \mu E\ |A_{2U_1}\cap A_{2U_2}| \\
=&\mu^2E(|A_{2U_1}||A_{2U_2}|) + \mu E\ |A_{2U_1}\cap A_{2U_2}| \\
=&\mu^2|A|^2 + \mu E\ |A_{2U_1}\cap A_{2U_2}|,
\end{aligned}$$

where we used the fact that A_{2U_1} and A_{2U_2} are independent and

$$E(|A_{2U_1}\setminus A_{2U_2}||A_{2U_1}\cap A_{2U_2}|) = E(|A_{2U_2}\setminus A_{2U_1}||A_{2U_1}\cap A_{2U_2}|).$$

Therefore

$$Var(Y_1 \times Y_2(A)) = \lambda\mu|A| + \lambda\mu^2 E|A_{2U}|^2 + \lambda^2\mu E|A_{2U_1} \cap A_{2U_2}|.$$

Similiary, when we condition on Y_2 we have, by symmetry,

$$Var(Y_1 \times Y_2(A)) = \lambda\mu|A| + \lambda^2\mu E|A_{1V}|^2 + \lambda\mu^2 E|A_{1V_1} \cap A_{1V_2}|.$$

For (3), clearly $E[T_{mn}] = |A|$. Now

$$\begin{aligned} E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\delta_{(U_i,V_j)}(A)\right]^2 &= E\left[\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\sum_{j=1}^{n}\sum_{l=1}^{n}\delta_{(U_i,V_j)}(A)\delta_{(U_k,V_l)}(A)\right)\right] \\ &= E\left[\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\sum_{l=1}^{n}\delta_{(U_i,V_j)}(A)\delta_{(U_i,V_l)}(A)\right)\right] \\ &\quad + E\left[\sum_{i\neq k}^{m}\left(\sum_{j=1}^{n}\sum_{l=1}^{n}\delta_{(U_i,V_j)}(A)\delta_{(U_k,V_l)}(A)\right)\right] \\ &:= \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

On the one hand,

$$\begin{aligned} \mathrm{I}_1 &= \sum_{i=1}^{m}\sum_{j=1}^{n} E[\delta_{(U_i,V_j)}(A)] + \sum_{i=1}^{m}\sum_{j\neq k}^{n} E[\delta_{(U_i,V_j)}(A)\delta_{(U_i,V_l)}(A)] \\ &= mn|A| + m(n^2-n)E|A_{1V_1} \cap A_{1V_2}|. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathrm{I}_2 &= \sum_{j=1}^{n}\sum_{i\neq k}^{m} E(\delta_{(U_i,V_j)}(A)\delta_{(U_k,V_j)}(A)) + \sum_{i\neq k}^{m}\sum_{j\neq l}^{n} E(\delta_{(U_i,V_j)}(A)\delta_{(U_k,V_l)}(A)) \\ &= n(m^2-m)E|A_{2U_1} \cap A_{2U_2}| + (n^2-n)(m^2-m)|A|^2, \end{aligned}$$

where $\delta_{(U_i,V_j)}(A)$ and $\delta_{(U_k,V_l)}(A)$, $i \neq k$ and $j \neq l$, are independent identically distributed. Therefore

$$\begin{aligned} Var(T_{mn}) &= (mn)^{-2}(\mathrm{I}_1 + \mathrm{I}_2) - |A|^2 \\ &= (mn)^{-1}\{|A| + (1-m-n)|A|^2 + (n-1)E|A_{1V_1} \cap A_{1V_2}| \\ &\quad + (m-1)E|A_{2U_1} \cap A_{2U_2}|\}. \end{aligned}$$

For (4) we assume, by convention, that $(Y_1 \times Y_2(A))/N_1N_2|A| = 1$ on the event $\{N_1N_2 = 0\}$. Clearly

$$\begin{aligned}
&E\left[\frac{Y_1 \times Y_2(A)}{N_1N_2|A|}\mathbf{1}_{[N_1N_2\geq 1]} + \mathbf{1}_{[N_1N_2\geq 1]^c}\right] \\
=&E\left[E\left(\frac{Y_1 \times Y_2(A)}{N_1N_2|A|}\mathbf{1}_{[N_1N_2\geq 1]} \;\middle|\; N_1, N_2\right)\right] \\
&+ P(N_1 = 0 \text{ or } N_2 = 0) = 1.
\end{aligned}$$

Let us calculate the variance of $(Y_1 \times Y_2(A))/N_1N_2|A|$.

$$\begin{aligned}
&Var[(Y_1 \times Y_2(A))/N_1N_2|A|] \\
=&E\left[\left(\frac{Y_1 \times Y_2(A)}{N_1N_2|A|}\mathbf{1}_{[N_1N_2>0]}\right)^2\right] + \exp(-(\lambda+\mu)) - 1 \\
=&E\left[E\left(\left(\frac{Y_1 \times Y_2(A)}{N_1N_2|A|}\mathbf{1}_{[N_1N_2>0]}\right)^2 \;\middle|\; N_1,\ N_2\right)\right] + \exp(-(\lambda+\mu)) - 1 \\
=&|A|^{-1}E[(N_1N_2)^{-1}\mathbf{1}_{[N_1N_2>0]} + (N_1-1)(N_2-1)(N_1N_2)^{-1}\mathbf{1}_{[N_1N_2>0]} \\
&+ |A|^{-1}(N_1-1)(N_1N_2)^{-1}E|A_{1V_1} \cap A_{1V_2}|\mathbf{1}_{[N_1N_2>0]} \\
&+ |A|^{-1}(N_2-1)(N_1N_2)^{-1}E|A_{2U_1} \cap A_{2U_2}|\mathbf{1}_{[N_1N_2>0]}] \\
&+ \exp(-(\lambda+\mu)) - 1 \\
=&C|A|^{-1},
\end{aligned}$$

which completes the proof of (4)(note that $C < 1$).

References

1. R. J. Adler and P. D. Feigin, *On the cadlaguity of random measures*, Ann. Prob. **12** (1984), 615-630.
2. R. F. Bass and R. Pyke, *The existence of set-indexed Lévy processes*, Z. Wahrsch. verw. Gebiete **66** (1984), 157-172.
3. R. F. Bass and R. Pyke, *A central limit theorem for $\mathcal{D}(\mathcal{A})-$valued processes*, Stochastic Processes and their Applications **24** (1987), 109-131.
4. G. Bennett, *Probability inequalities for the sums of bounded random variables*, J. A. S. A. **57** (1962), 33-45.
5. D. L. Hanson and F. T. Wright, *A bound on tail probabilities for quadratic forms in independent random variables*, Ann. Math. Statist. **42** (1971), 1079-1083.

6. W. Hoeffding, *Probability inequalities for sums of bounded random variables*, J. A. S. A. **57** (1963), 33-45.
7. J. Kwon and R. Pyke, *Probability Bounds for Product Poisson processes*, summitted., 1994.
8. M. Ossiander, *Product measure and partial sums of Bernoulli random variables.*, (1991) In preperation.
9. R. Pyke, *Probability in Mathematics and Statistics: A century's predictor of future directions*, Preprint (1991), 37.

Department of Mathematics
Sun Moon University
Asan Chungnam 337-840, Korea