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HARMONIC DOUBLING CONDITION AND JOHN DISKS

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1. Introduction

A Jordan domain D in $\mathbb{C}$ is said to be a *c-quasidisk* if there exists a constant $c \geq 1$ such that each two points z_1 and z_2 in D can be joined by an arc γ in D such that

$$\ell(\gamma) \leq c|z_1 - z_2|$$

and

$$\min(\ell(\gamma_1), \ell(\gamma_2)) \leq c\, d(z, \partial D) \tag{1.1}$$

for all $z \in \gamma$, where γ_1 and γ_2 are the components of $\gamma \setminus \{z\}$. Quasidisks have been extensively studied and can be characterized in many different ways [1], [2], [3].

A bounded domain D in $\mathbb{C}$ is said to be a *c-John domain* if there exist a point $z_0 \in D$ and a constant $c \geq 1$ such that each point $z_1 \in D$ can be joined to z_0 by an arc γ in D satisfying

$$\ell(\gamma(z_1, z)) \leq c\, d(z, \partial D)$$

for each $z \in \gamma$. We call z_0 a *John center*, c a *John constant* and γ a *c-John arc.*

There are several equivalent definitions for John domains. For example, a domain D in $\mathbb{C}$ is a c-John domain if and only if each two points $z_1, z_2 \in D$ can be joined by an arc γ which satisfies (1.1), [9]. This

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definition can be used to define the unbounded John domains D in $\mathbb{C}$ as well [9, 2.26].

John domains were introduced by F. John [5] in connection with his work in elasticity; the term John domain is due to Martio and Sarvas [7]. John domains arise also naturally in distortion problems of conformal and quasiconformal mappings. From the definition we can see that a domain is a John domain if it is possible to move from one point to another without passing too close to the boundary.

We say that a domain D in $\mathbb{C}$ is a *c-John disk* if it is a simply connected c-John domain. Thus the class of quasidisks is properly contained in the class of John disks. The converse is not true since a John disk need not even be a Jordan domain. For example, the unit disk minus the segment $[0,1]$ is a John disk.

There are various characterizations of John disks, for example, see [4], [7], [9], [10]. The main purpose of this paper is to give a conformally invariant characterization of John disks in terms of harmonic measure.

A bounded Jordan domain D in $\mathbb{C}$ is said to satisfy a *harmonic doubling condition* if there exist a point $z_0 \in D$ and a constant $c_0 > 0$ such that

$$\omega(z_0, \alpha; D) \leq c_0 \omega(z_0, \beta; D) \tag{1.2}$$

for each pair of consecutive arcs α, β on ∂D with

$$\mathrm{dia}(\alpha) \leq 2\mathrm{dia}(\beta),$$

where ω is the harmonic measure in D.

REMARK 1.3. If D satisfies (1.2) for some $z_0 \in D$, then it satisfies (1.2) for every $z_1 \in D$ with $c_1 = c_1(c_0, z_0, z_1)$.

Proof of Remark 1.3. Fix $z_1 \in D$ and fix consecutive arcs $\alpha, \beta \subset \partial D$ with

$$\mathrm{dia}(\alpha) \leq 2\mathrm{dia}(\beta).$$

Since ω is nonnegative and harmonic, by Harnack's Theorem [8, p. 115]

$$\frac{\omega(z_1, \alpha; D)}{\omega(z_0, \alpha; D)} \leq k \qquad \text{and} \qquad \frac{\omega(z_0, \beta; D)}{\omega(z_1, \beta; D)} \leq k$$

where k is a constant depending only on z_0, z_1, $0 < k > 1$. Thus by hypothesis we have

$$\frac{\omega(z_1, \alpha; D)}{\omega(z_1, \beta; D)} \le \frac{\omega(z_0, \alpha; D)}{\omega(z_0, \beta; D)} k^2 \le c_0 k^2 = c_1$$

and hence (1.2) holds for every $z_1 \in D$ with $c_1 = c_1(c_0, z_0, z_1)$. □

In [6], Jerison and Kenig showed that a bounded Jordan domain D in $\mathbb{C}$ is a quasidisk if and only if D and $D^* = \overline{\mathbb{C}} \setminus \overline{D}$ satisfy a harmonic doubling condition. Since a John disk may be viewed as a one-sided quasidisk, it is natural to ask whether a bounded Jordan domain D in $\mathbb{C}$ satisfies a harmonic doubling condition if and only if D is a John disk. The answer is yes.

MAIN THEOREM. *A bounded Jordan domain D in $\mathbb{C}$ is a c-John disk if and only if it satisfies a harmonic doubling condition.*

2. Proof of main theorem

Let f map the unit disk $\mathbb{B}$ conformally onto the bounded Jordan domain D in $\mathbb{C}$. Then by the Caratheodory extension theorem $f : \mathbb{B} \to D$ admits an extension to a homeomorphism $f : \overline{\mathbb{B}} \to \overline{D}$. The following Lemma 2.1 describes John disks in terms of the conformal mapping $f : \mathbb{B} \to D$.

LEMMA 2.1. *Suppose that D is a bounded Jordan domain in $\mathbb{C}$, that $w_0 \in D$ and that f is as above with $w_0 = f(0)$. Then the following conditions are equivalent, where the constants c and $\delta > 0$ need not be the same in every condition:*

(1) *D is a c-John disk.*

(2)

$$\frac{\mathrm{dia} f(\beta_1)}{\mathrm{dia} f(\beta)} \le c\Big(\frac{\ell(\beta_1)}{\ell(\beta)}\Big)^{\delta}$$

for all arcs $\beta_1 \subset \beta \subset \partial\mathbb{B}$.

(3)

$$\frac{\mathrm{dia}(\alpha_1)}{\mathrm{dia}(\alpha)} \le c\Big(\frac{\omega(w_0, \alpha_1; D)}{\omega(w_0, \alpha; D)}\Big)^{\delta}$$

for all arcs $\alpha_1 \subset \alpha \subset \partial D$.

If $\text{dia}(D) \le q\, d(w_0, \partial D)$, *then the various constants* c *and* δ *depend only on* q *and on each other.*

Proof. The equivalence of (1) and (2) is proved in [10]. Next the condition (3) is a reinterpretation of the condition (2): for suppose that (2) holds, fix arcs $\alpha_1 \subset \alpha \subset \partial D$ and let $\beta_1 = f^{-1}(\alpha_1)$, $\beta = f^{-1}(\alpha)$. Since harmonic measure ω of α at 0 with respect to $\mathbb{B}$ is

$$\omega(0, \alpha; \mathbb{B}) = \frac{\theta}{2\pi} = \frac{\ell(\alpha)}{2\pi}$$

for α an arc of the circle with central angle θ, by conformal invariance of harmonic measure we have

$$\begin{aligned}\frac{\text{dia}(\alpha_1)}{\text{dia}(\alpha)} = \frac{\text{dia} f(\beta_1)}{\text{dia} f(\beta)} \le c\Big(\frac{\ell(\beta_1)}{\ell(\beta)}\Big)^{\delta} &= c\Big(\frac{\omega(0, \beta_1; \mathbb{B})}{\omega(0, \beta; \mathbb{B})}\Big)^{\delta} \\ &= c\Big(\frac{\omega(w_0, \alpha_1; D)}{\omega(w_0, \alpha; D)}\Big)^{\delta}.\end{aligned}$$

By the same reasoning as above, (3) also implies (2). □

LEMMA 2.2. *Suppose that* D *is a bounded Jordan domain in* $\mathbb{C}$ *and that* $z_0 \in D$. *Then the following conditions are equivalent:*

(1) *There exist constants* c *and* $\delta > 0$ *such that*

$$\frac{\text{dia}(\alpha_1)}{\text{dia}(\alpha)} \le c\Big(\frac{\omega(z_0, \alpha_1; D)}{\omega(z_0, \alpha; D)}\Big)^{\delta}$$

for all arcs $\alpha_1 \subset \alpha \subset \partial D$.

(2) *There exists a constant* $c > 1$ *such that*

$$\omega(z_0, \alpha; D) \le c\,\omega(z_0, \alpha_1; D) \tag{2.3}$$

for all arcs $\alpha_1 \subset \alpha \subset \partial D$ *with*

$$\text{dia}(\alpha) \le 2\text{dia}(\alpha_1).$$

Proof. First we assume that (1) holds and let $\alpha_1 \subset \alpha$ be arcs on ∂D with

$$\text{dia}(\alpha) \leq 2\text{dia}(\alpha_1).$$

Then by (1)

$$\frac{\omega(z_0, \alpha_1; D)}{\omega(z_0, \alpha; D)} \geq c^{-\frac{1}{\delta}} \left(\frac{\text{dia}(\alpha_1)}{\text{dia}(\alpha)}\right)^{\frac{1}{\delta}} \geq (2c)^{-\frac{1}{\delta}},$$

and hence we have

$$\omega(z_0, \alpha; D) \leq c' \omega(z_0, \alpha_1; D),$$

where $c' = (2c)^{\frac{1}{\delta}}$.

Next suppose that (2) holds. We show first that

$$\omega(z_0, \alpha; D) \leq c^n \omega(z_0, \alpha_1; D) \tag{2.4}$$

for all arcs $\alpha_1 \subset \alpha \subset \partial D$ with

$$\text{dia}(\alpha) \leq 2^n \text{dia}(\alpha_1).$$

By (2.3), inequality (2.4) is true for $n = 1$. Now we assume that it is true for $n = k \geq 1$ and suppose that we have arcs $\alpha_1 \subset \alpha \subset \partial D$ such that

$$\text{dia}(\alpha) \leq 2^{k+1} \text{dia}(\alpha_1).$$

Then since (2.4) is true for $n = k$, we may assume that

$$2^k \text{dia}(\alpha_1) \leq \text{dia}(\alpha) \leq 2^{k+1} \text{dia}(\alpha_1),$$

since otherwise we would have

$$\text{dia}(\alpha) < 2^k \text{dia}(\alpha_1)$$

whence

$$\omega(z_0, \alpha; D) \leq c^k \omega(z_0, \alpha_1; D) < c^{k+1} \omega(z_0, \alpha_1; D).$$

If γ is an arc with $\alpha_1 \subset \gamma \subset \alpha \subset \partial D$, then dia($\gamma$) increases continuously as the end points of γ tend to the end points of α. Hence there exists an arc γ_1 such that

$$\alpha_1 \subset \gamma_1 \subset \alpha \subset \partial D$$

and

$$\mathrm{dia}(\alpha) = 2\mathrm{dia}(\gamma_1).$$

Then

$$\mathrm{dia}(\gamma_1) \leq 2^k \mathrm{dia}(\alpha_1)$$

and by (2.4) with $n = k$ and $n = 1$,

$$\frac{\omega(z_0, \alpha; D)}{\omega(z_0, \alpha_1; D)} = \frac{\omega(z_0, \alpha; D)}{\omega(z_0, \gamma_1; D)} \frac{\omega(z_0, \gamma_1; D)}{\omega(z_0, \alpha_1; D)} \leq cc^{k+1} = c^{k+1}.$$

Thus

$$\omega(z_0, \alpha; D) \leq c^{k+1} \omega(z_0, \alpha_1; D)$$

and this establishes (2.4).

Next given any arcs $\alpha_1 \subset \alpha \subset \partial D$, there exists an integer $n > 0$ such that

$$2^{n-1}\mathrm{dia}(\alpha_1) \leq \mathrm{dia}(\alpha) \leq 2^n \mathrm{dia}(\alpha_1). \tag{2.5}$$

Then by (2.4) we have

$$\omega(z_0, \alpha; D) \leq c^n \omega(z_0, \alpha_1; D). \tag{2.6}$$

Let $\delta = \frac{log 2}{log c}$. Then by (2.5) and (2.6) we obtain

$$\begin{aligned} \frac{\omega(z_0, \alpha; D)}{\omega(z_0, \alpha_1; D)} &\leq c^n = c(2^{\frac{1}{\delta}})^{n-1} = c(2^{n-1})^{\frac{1}{\delta}} \\ &\leq c\Big(\frac{\mathrm{dia}(\alpha)}{\mathrm{dia}(\alpha_1)}\Big)^{\frac{1}{\delta}}. \end{aligned}$$

Hence we get

$$\frac{\mathrm{dia}(\alpha_1)}{\mathrm{dia}(\alpha)} \leq c'\Big(\frac{\omega(z_0, \alpha_1; D)}{\omega(z_0, \alpha; D)}\Big)^{\delta},$$

where $c' = c^\delta$. Hence (2) implies (1). $\square$

Proof of Main Theorem. Suppose first that a bounded Jordan domain D in $\mathbb{C}$ is a c-John disk with a John center z_0. We want to show that there exists a constant $c_0 > 0$ such that

$$\omega(z_0, \alpha; D) \le c_0 \omega(z_0, \beta; D)$$

for each pair of consecutive arcs α, β on ∂D with

$$\operatorname{dia}(\alpha) \le 2\operatorname{dia}(\beta).$$

Suppose not. Then for $j = 1, 2, \ldots$ there are consecutive arcs α_j, β_j on ∂D such that

$$\text{(2.7)} \qquad \operatorname{dia}(\alpha_j) \le 2\operatorname{dia}(\beta_j) \qquad \text{and} \qquad \omega(z_0, \alpha_j; D) \ge 3^j \omega(z_0, \beta_j; D).$$

Thus

$$\operatorname{dia}(\alpha_j \cup \beta_j) \le 3\operatorname{dia}(\beta_j)$$

and hence by (3) of Lemma 2.1 with $w_0 = z_0$ and (2.7)

$$\begin{aligned} \frac{1}{3} \le \frac{\operatorname{dia}(\beta_j)}{\operatorname{dia}(\alpha_j \cup \beta_j)} &\le c\Big(\frac{\omega(z_0, \beta_j; D)}{\omega(z_0, \alpha_j \cup \beta_j; D)}\Big)^\delta \\ &\le c\Big(\frac{\omega(z_0, \beta_j; D)}{\omega(z_0, \alpha_j; D)}\Big)^\delta \le c(3^{-j})^\delta \end{aligned},$$

which yields a contradiction as $j \to \infty$.

Suppose next that a bounded Jordan domain D in $\mathbb{C}$ satisfies a harmonic doubling condition (1.2). To show that D is a c-John disk, it suffices to show that D satisfies (2.3) by Lemma 2.1 with $w_0 = z_0$ and Lemma 2.2.

Let $\alpha_1 \subset \alpha$ be arcs of ∂D with

$$\operatorname{dia}(\alpha) \le 2\operatorname{dia}(\alpha_1)$$

and let $c_1 = 2(c_0 + 1)$.

Suppose first that α_1, α have a common end point. Then

$$\mathrm{dia}(\alpha \setminus \alpha_1) \leq 2\mathrm{dia}(\alpha_1)$$

and hence by (1.2) we have

$$\omega(z_0, \alpha \setminus \alpha_1; D) \leq c_0 \omega(z_0, \alpha_1; D)$$

for some $z_0 \in D$. Thus

$$\omega(z_0, \alpha; D) \leq (c_0 + 1)\omega(z_0, \alpha_1; D)$$

Next suppose that $\alpha \setminus \alpha_1$ consists of two disjoint subarcs $\alpha_2, \alpha_3 \subset \alpha$. Then

$$\mathrm{dia}(\alpha_1 \cup \alpha_2) \leq 2\mathrm{dia}(\alpha_1)$$

and hence

$$\omega(z_0, \alpha_1 \cup \alpha_2; D) \leq (c_0 + 1)\omega(z_0, \alpha_1; D)$$

by what was proved above. The same argument also gives

$$\omega(z_0, \alpha_1 \cup \alpha_3; D) \leq (c_0 + 1)\omega(z_0, \alpha_1; D)$$

and hence

$$\omega(z_0, \alpha; D) \leq 2(c_0 + 1)\omega(z_0, \alpha_1; D).$$

This completes the proof of Main Theorem. □

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