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RADIAL SYMMETRY AND SPHERICAL NODAL SET OF SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS

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1. Introduction

In this note, we will investigate the radial symmetry of some kind of solutions of nonlinear ellipitic equations

$$\begin{aligned} \Delta U &= f(U) && \text{in } B \\ U &= 0 && \text{on } \partial B \\ U &\in C^2(\overline{B}) \end{aligned} \tag{1.1}$$

Here f is C^1 and B denotes a n-dimensional unit ball in R^n. And $C^2(\overline{B})$ denotes the space of all functions which have continous second partial derivatives up to order 2. Gidas-Ni-Nirenberg proved that the positive solutons of (1.1) must be radially symmetric. Here we are interesed in the radial symmetry of solutions of (1.1) only with $U(0) > 0$ but which have spherical nodal set.

2. Radial symmetry results

The nodal set of solution U of (1.1) is the set

$$\{x \in B; U(x) = 0\}.$$

If U is rotationally symmetric, it is obvious that the nodal set is spherical. Conversely,if the nodal set is spherical, let the nodal set of solution U of (1.1) with $U(0) > 0$ be

$$\bigcup_{\lambda \in \Lambda} S(\lambda)$$

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where $\Lambda \subset [0,1]$ and $S(\lambda) = \{x \in B; |x| = \lambda\}$. From now on, let us denote

$$\overline{\lambda} = inf_{\lambda \in \Lambda}\{\lambda\}. \tag{2.1}$$

LEMMA. *If the nodal set of solution U of (1.1) with $U(0) > 0$ is spherical, then $\overline{\lambda}$ is positive and isolated.*

Proof. If Λ is finite set, it is obvious that $\overline{\lambda}$ is isolated and $\overline{\lambda} > 0$ since $0 \notin \Lambda$. Now if Λ is an infinite set, obviously $\overline{\lambda} > 0$. Otherwise, there is a sequence $\{\lambda_i\}$ with $\lambda_i \to 0$, then $U(0) = 0$.It contradicts $U(0) > 0$,which shows $\overline{\lambda} > 0$. Now suppose that $\overline{\lambda}$ is not isolated, there is a sequence $\{\lambda_i\}$ with $\lambda_i \to \overline{\lambda}(> 0)$. Then in the polar coordinates $x = r \cdot \xi$ where $\xi \in S^{n-1}$ and $r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$,

$$\begin{aligned} &U = U_r = U_{rr} = 0 \quad \text{for} \quad r = \overline{\lambda} \quad \text{and} \\ &U = D_\xi U = D_\xi^2 U = 0 \quad \text{on} \quad S(\overline{\lambda}) \end{aligned} \tag{2.2}$$

which implies

$$U = \Delta U = 0 \quad \text{on} \quad S(\overline{\lambda}). \tag{2.3}$$

So in the case $f(0) \neq 0$, it contradicts (1.1). Now in the case $f(0) = 0$, we set

$$C(x) = \int_0^1 f'(tU(x))\, dt,$$

then U solves the Cauchy problem

$$\begin{aligned} \Delta U &= C(x)U \qquad \text{in} \quad B(\overline{\lambda}) \\ U &= U_r = 0 \qquad \text{on} \quad S(\overline{\lambda}) \end{aligned} \tag{2.4}$$

By uniqueness of solutions to Cauchy problem of linear elliptic equations, U constantly equals 0 in $B(\overline{\lambda})$. It also contadicts $U(0) > 0$.This proves the lemma.

We now state and prove the main theorem.

THEOREM. *A solution U of (1.1) with $U(0) > 0$ is radially symmetric if its nodal set is spherical.*

Proof. Let us denote $\overline{\lambda}$ as in the Lemma and $B(\overline{\lambda}) = \{x \in R^n; |x| < \overline{\lambda}\}$.Now in $B(\overline{\lambda})$, U is positive since there are no nodal points in $B(\overline{\lambda})$, together with $B(\overline{\lambda})$ is simply connected. By the result of [GNN], U must be radial symmetric in $B(\overline{\lambda})$, in the polar coordinates,we obtain

$$U(r) = 0, \frac{\partial U}{\partial r}\,(r) = \text{const.} \quad \text{on} \quad S(\overline{\lambda}). \tag{2.5}$$

Let $T : R^n \to R^n$ be any rotation transform.Then $V = U(Tx)$ also solves equation (1.1) since (1.1) is invariant under the transform T .Obviously

$$V = U, V_r = U_r = \text{const.} \quad \text{on} \quad S(\overline{\lambda}). \tag{2.6}$$

Then $W = (V - U)$ is a solution to the Cauchy problem

$$\begin{aligned} \Delta W &= \left(\int_0^1 f'(tV + (1-t)U)\,dt\right) W \quad \text{in} \quad B \\ W &= W_r = 0 \quad \text{on} \quad S(\overline{\lambda}) \end{aligned} \tag{2.7}$$

By the uniqueness of the Cauchy problem, $(V - U)$ constantly equals 0,which means $U(Tx) = U(x)$ in B for any rotation transform T, which implies U is radially symmetric throughout B.

REMARK. The result of [GNN] is a special case of the problem since the nodal set of a positive solution is the sphere ∂B.

References

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