Comm. Korean Math. Soc. **10** (1995), No. 1, pp. 123–131

# IDEMPOTENT ELEMENTS IN THE LOTKA–VOLTERRA ALGEBRA

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## 1. Introduction

The notion of our non-associative algebra is obtained from the Lotka-Volterra system of differential equation describing competition between animals or vegetals species and also in the kinetic theory of gasses.

For the struture of an algebra, the existence of idempotents is of particular interest.

But also from the biological aspect the existence of such elements is of interest because the equilibria of a population which can be described by an algebra correspond to idempotents of this algebra.

Thus we present some properties of the general natures for a Lotka-Volterra algebra associated to a weight function and idempotents elements.

## 2. Preliminary results

The Lotka-Volerra system with binary interaction treated by Kimura [4] and Mather [7] is represented by $\frac{d}{dt}x_i(t) = x_i(t)\sum_{j=1}^{n} a_{ij}\ x_j(t)$, $(i = 1, 2, \cdots, n)$ where $x_i$ are differentiable functions of time variable $t$ and for $t > t_0$ with $x_i(t_0) > 0 \quad (i = 1, 2, \cdots, n)$, $\sum_{i=1}^{n} x_i(t_0) = 1$ and the real constants $a_{ij}$ satisfies $a_{ij} + a_{ji} = 0$ for any $i$ and $j$.

It is known [2] that the structure of constants permit us to associate this system of differential equation to a commutative, nonassociative $n$-dimensional algebra over $R$.

Hereafter, we shall consider such algebras over a commutative field of characteristic not 2. Let $K$ be a commutative field of characteristic not

Received August 11, 1994.

The present studies were supported (in part) by the Basic Science Research Institute Program. Ministry of Education 1994, Project No. 1423.

2, and $(a_{ij})$ an $n \times n$ antisymmetric matrix with coefficients in $K$ where $n \geq 2$ is an integer and $A$ a vector space over $K$ of dimension $n$ with a basis $\{e_1, e_2, \cdots, e_n\}$. On $A$, consider an algebraic structure defined by the multiplication $e_i e_j = \left(\frac{1}{2} + a_{ij}\right) e_i + \left(\frac{1}{2} - a_{ij}\right) e_j \quad (i, j = 1, 2, \cdots, n)$ where $a_{ij} = -a_{ji}$ and $-\frac{1}{2} \leq a_{ij} \leq \frac{1}{2}$. From definition, we easily see that $e_i e_j = e_j e_i$, $e_i^2 = e_i$ and $e_i(e_j e_k) \neq (e_i e_j) e_k$ for $i, j, k = 1, 2, \cdots, n$. We call this commutative, nonassociative algebra as the Lotka-Volterra algebra associated to the antisymmetric matrix $(a_{ij})$.

LEMMA 1. *Let $K$ be a field of characteristic not 2 and $A$ the Lotka-Volterra $K$-algebra associated to antisymmetric matrix $(a_{ij})$. Then $A$ is a baric algebra.* [cf. 13]

*Proof.* Let $w : A \to K$ be a $K$-linear application defined by $w(e_{ij}) = 1 \quad (i = 1, 2, \cdots, n)$, it is easy to show that $w$ is a $K$-algebra morphism, that is, $w(xy) = w(x)w(y)$ for all $x$ and $y$ in $A$.

For any $x = \sum_{i=1}^{n} \lambda_i e_i$ and $y = \sum_{i=1}^{n} \mu_i e_i$ with $K$ we have the multiplication $xy = \frac{1}{2}(w(x)y + w(y)x) + \sum_{i=1}^{n}(\lambda_i w_i(y) + \mu_i w_i(x)) e_i$, where $w_i : A \to K, \quad e_j \mapsto a_{ij} \quad (i, j = 1, \cdots, n)$ are $K$-linear applications.

In particular, if all $a_{ij}$ are zero, we have the multiplication $xy = \frac{1}{2}(w(x)y + w(y)x)$, that is the multiplication in gametic algebra $G(n, 2)$, where $w$ is the weight function.

REMARK. Gametic algebra $G(n, 2)$ is Gonshor genetic (cf [13]). This means that there exists a base $(u_1, u_2, \cdots, u_n)$ of $A$ over $K$ such that the multiplication table of $A$ related to this base is $u_i u_j = \sum_{k=1}^{n} \gamma_{ijk} u_k \quad (i, j = 1, \cdots, n)$, where $\gamma_{ijk}$ satisfy the conditions $\gamma_{111} = 0$, $\gamma_{1jk} = 0$ if $k < j$ and $\gamma_{ijk} = 0$ if $k \leq \max(i, j)$ with $i \geq 2$ and $j \geq 2$.

PROPOSITION 2. *Let $A$ be a Lotka-Volterra $K$-algebra associated to a $n \times n$ antisymmetric matrix $(a_{ij})$.*

*For any extension $K \to L$ of $K$, the $L$-algebra $A \otimes_K L$ is also Lotka-Volterra algebra associated to the same matrix $(a_{ij})$ that admits a weight function. i.e., $A \otimes_K L$ is also a baric algebra*

*Proof.* If $\{e_1, e_2, \cdots, e_n\}$ is a base of $A$ with which we define the multiplication table $e_i e_j = \left(\frac{1}{2} + a_{ij}\right) e_i + \left(\frac{1}{2} - a_{ij}\right) e_j$ in $A$, then $\{e_1 \otimes 1, \ e_2 \otimes 1, \cdots, \ e_n \otimes 1\}$ is a base of $A \otimes_K L$ as an $L$-algebra and the constants of the multiplication structure of $A \otimes_K L$ related to this base

$\{e_1 \otimes 1,\ e_2 \otimes 1, \cdots,\ e_n \otimes 1\}$ are the same as that of $A$ related to the base $\{e_1, e_2, \cdots, e_n\}$. We consider $w' = w \otimes id_L$ of $A \otimes_K L$ such that $w'(e_i \otimes 1) = w(e_i)1 = 1$ as a weight function of Lotka-Volterra $L$-algebra $A \otimes_K L$.

REMARK. Since the Lotka-Volterra $K$-algebra $(A, w)$ is a gametic algebra $G(n,2)$ when all $a_{ij}$ are 0 and the weight function of $G(n,2)$ is unique (cf. [13]), there exists unique weight function $w$ in a Lokta-Volterra algebra $A$ when all entries are 0 in the associated matrix $(a_{ij})$.

In general, it is interesting to consider the condition for the uniqueness of weight function of the Lotka-Volterra algebra. We have the following results.

PROPOSITION 3. *Let $A$ be a Lotka-Volterra algebra with weight function $w$. If ker $w$ is nil, then $w$ is uniquely determined.*

*Proof.* Let $\alpha : K \to L$ be any non-trivial homomorphism and $x \in$ ker $w$. Then there is an $n \in N$ with $x^n = 0$.

Since $\alpha$ is a homomorphism from $A$ into $K$, it follows $\alpha(x) = 0$. Let $x \in A/\ker w$, then $w(x) \neq 0$. So we have $w\left(\frac{x^2}{w(x)} - x\right) = 0$, and $\frac{x^2}{w(x)} - x \in$ ker $w$. It implies that $\alpha\left(\frac{x^2}{w(x)} - x\right) = 0$, so we have $\alpha\left(\frac{\alpha x}{w(x)} - 1\right) = 0$.

Therefore, we have either $\alpha(x) = 0$ or $\alpha(x) = w(x)$. Since we have assume that $\alpha$ is nontrivial, we must have $\alpha = w$.

If $a_{ij} \neq -\frac{1}{2}$ for all $i$ and $j$ $(i, j = 1, 2, \cdots, n)$, then $w$ is uniquely determined in the Lotka-Volterra $K$-algebra (cf. [9]). In fact $e_i^2 = e_i$ implies that $w'(e_i)^2 = w'(e_i)$, and so $w'(e_i) = 0$ or $w'(e_i) = 1$.

Let $I = \{i | w'(e_i) = 1\}$ and $J = \{i | w'(e_i) = 0\}$. Since $w'$ is nontrivial, $I \neq \emptyset$. For any $i \in I$ and $j \in J$, by applying $w'$ to $e_i e_j = \frac{1}{2}(1 + 2a_{ij})e_i + \frac{1}{2}(1 + 2a_{ji})e_j$ we have $1 + 2a_{ij} = 0$, that is impossible by hypothesis, so $J = \emptyset$ and $w'(e_i) = 1$. Consequently, $w = w'$.

We have an example of the Lotka-Volterra algebra which does not have the unique weight function for some $a_{ij}$ that are $\pm\frac{1}{2}$.

EXAMPLE. Consider the 3×3 anti-symmetric matrix $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} & 0 \end{pmatrix}$.

By the multiplication table of the Lotka-Volterra algebra associated with this matrix, we have $e_1^2 = e_1$, $e_1e_2 = e_1$, $e_1e_3 = e_1$, $e_2^2 = e_2$, $e_2e_3 = \frac{1}{2}(1+\frac{1}{2})e_2 + \frac{1}{2}(1-\frac{1}{2})e_3$, $e_3^2 = e_3$ and $K$-algebra application $w' : K \to L$ defined by $w'(e_1) = 0$, $w'(e_2) = 1$ and $w'(e_3) = 1$. That is different from $w(e_i) = 1$ for $i = 1, 2, 3$.

The Proposition 3 and Remark permit us to say the following property.

PROPOSITION 4. *If either $a_{ij} \neq -\frac{1}{2}$ for all $i$ and $j$ or ker $w$ is nil, then there exist unique weight funtion $w'$ in the Lotka-Volterra $L$-algebra $A \otimes_K L$ for any extension $K \to L$ of $K$.*

*Proof.* Since $A$ and $A \otimes_K L$ have the same related matrix and the weight function $w'$ of $A \otimes_K L$ which is defined by $w$ in the Lotka-Volterra $K$-algebra $A$, we have the unique weight function in $A \otimes_K L$ as an $L$-algebra.

## 3. Idempotent elements of the Lotka-Volterra algebra

PROPOSITION 5. *Let $(A, \omega)$ be a Lotka-Volterra baric algebra and if ker $\omega$ is nil. Then $\omega(e) = 1$ for every idempotent element $e$ of $A$.*

*Proof.* For every idempotent $e$ of a Lokta-Volterra algebra $A$, we have either $\omega(e) = 0$ or $\omega(e) = 1$. In a Lokta-Volterra algebra $A$ all elements of weight 0 are nilpotent by hypothesis and an idempotent can not be nilpotent. Thus we have $\omega(e) = 1$.

Let $\mathfrak{N} =$ ker $\omega$, then $A/\mathfrak{N} \cong K$. Thus $A/\mathfrak{N}$ is a 1-dimensional associative algebra with unity and not nilpotent. Therefore there exist a Levi decomposition of $A$ with respect to $\mathfrak{N}$, i.e. $A$ is the direct sum of the vector spaces $B$ and $\mathfrak{N}$ where $B$ is a subalgebra of $A$ isomorphic to $A/\mathfrak{N}$.

It follows immediately that $A =$ Ke $\oplus$ Ker $\omega$ for any nontrivial idempotent element $e$ of $A$, and hence for any $x$ of $A$ written by $x = \omega(x)e + y$ with $\omega(y) = 0$, a nontrivial idempotent element can be written by $x = e + y$ with $w(y) = 0$ by the condition $\omega(x)^2 = \omega(x)$.

Now, generally we study the idempotent elements of the (genetic) Lotka-Volterra algebra $A$. At first we restrict ourselves for the case of $n = 2$. Let $x = x_1e_1 + x_2e_2$ and $y = y_1e_1 + y_2e_2$ be element in $A$, where

$x_1, x_2, y_1$ and $y_2$ are in $K$. If we consider the identity $xy = x$ for all $x, y \in A$, we have the (right) identity element

$$y = \frac{1}{\left(\frac{1}{2} - a_{12}\right)x_1 + \left(\frac{1}{2} - a_{21}\right)x_2}\left[\left(\frac{1}{2} - a_{12}\right)x_1 e_1 + \left(\frac{1}{2} - a_{21}\right)x_2 e_2\right]$$

in $A$. Furthermore, if we replace $y$ by $x$, i.e. $x^2 = x$, for $x \in A$, then we have $x = \frac{1}{2a_{12}}[-e_1 + e_2]$ as a nonzero idempotent element in $A$. However, if we consider the homogeneous case of the last equation, we can easily see that there is no nonzero nilpotent elements in $A$. So, we can say that there is no nonzero nilpotent elements, but there does exist nonzero idempotent elements in $A$ for the case of $\dim A = 2$. Furthermore, using $a_{ij} = -a_{ji}$, we can see that there are 4 idempotent elements in $A$. i.e.,

$$x = \frac{1}{2a_{12}}(-e_1 + e_2),\ \frac{1}{2a_{12}}(e_1 - e_2),\ \frac{1}{a_{12}}(-e_1 + e_2),\ \frac{1}{a_{12}}(e_1 - e_2)$$

For the general case, let $\{e_1, e_2, \cdots, e_n\}$ be a canonical basis of $A$. Then from the multiplication

$$e_i e_j = \left(\frac{1}{2} + a_{ij}\right)e_i + \frac{1}{2}(1 - a_{ji})e_j, \quad a_{ij} + a_{ji} = 0,$$
$$-\frac{1}{2} \le a_{ij} \le \frac{1}{2}, \qquad i, j = 1, 2, \cdots, n,$$

we have

$$\begin{aligned}\left(\sum_{i=1}^n x_i e_i\right)^2 &= \sum_{i=1}^n x_i^2 e_i^2 + 2\sum_{1\le i\le j\le n} x_i x_j e_i e_j \\ &= \sum_{i=1}^n x_i^2 e_i + 2\sum_{1\le i\le j\le n} x_i x_j\left[\left(\frac{1}{2} + a_{ij}\right)e_i + \left(\frac{1}{2} - a_{ij}\right)e_j\right] \\ &= \sum x_k e_k\end{aligned}$$

where

$$\begin{aligned}x_k &= x_k^2 + 2\sum_{j\ne k}^n x_k x_j\left(\frac{1}{2} + a_{kj}\right) + 2\sum_{j\ne k} x_i x_k\left(\frac{1}{2} - a_{ik}\right) \\ &= x_k^2 + 2\sum_{j\ne k}^n x_k x_j\left(\frac{1}{2} + a_{kj}\right) + 2\sum_{j\ne k} x_i x_k\left(\frac{1}{2} + a_{ki}\right)\end{aligned}$$

$$= x_k^2 + 2\sum_{j\neq k}^{n} x_k x_j(1+2a_{kj}).$$

To consider the idempotent elements in the Lotka-Volterra algebra $A$, it suffices to consider only for $x_k(x_k + 2\sum_{j\neq k}^{n} x_j(1+2a_{kj}) - 1) = 0$.

Since the left side of the above equation must be in $K$, $x_k = 0$ or $x_k + 2\sum_{j\neq k}^{n} x_j(1+2a_{kj}) - 1 = 0$.

Let $P_k : x_k = 0$ and $Q_k : x_k + 2\sum_{j\neq k}(1+2a_{kj})x_j - 1 = 0$. If we consider $\{i_1, \cdots, i_p\}$ and $\{j_1, \cdots, j_{n-p}\}$, then we see that

$$\cup_{(i_1,\cdots,i_p)\in P(1,2,\cdots,n)}(P_{i_1}\cap\cdots\cap P_{i_p}\cap Q_{j_1}\cap\cdots\cap Q_{j_n-p})$$

becomes an ideal of $A$ ([1]). So we can write $P_1\cap\cdots\cap P_p\cap Q_{p+1}\cap\cdots\cap Q_n$ as $x_1 = 0,\ x_2 = 0, \cdots, x_p = 0,\ ((x_{p+1} + 2\sum_{J>p+1}(1+2a_{p+1j})x_j = 1))$, and it is associated to an antisymmetric matrix

$$A = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & \vdots & & & & \\ \vdots & & \vdots & (( & & & )) \\ 0 & \cdots & 0 & & & & \end{bmatrix} \begin{matrix} \\ \\ \\ p+1 \\ \vdots \\ n \end{matrix}$$

$$p+1 \quad \cdots\cdots \quad n$$

Therefore, we can consider only for $\sum_{j=p+1}^{n}(1+2a_{p+1,j})x_j = 1$ which is reduced to a matrix form as $(E+2A)x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, where $E = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$.

So, for the general case, consider a matrix form $tE + A$, where $t$ is an indetermined variable. Then since $\det(tE + A) = P(t) = \alpha t + \beta$ and rank $(E) = 1$ for the case which $n$ is even and $t = 0$, there exists a basis $\{e_1, e_2, \cdots, e_{2m}\}$ and we can reduce $A$ to

$$A' = \begin{bmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{bmatrix}$$

such that $A = R^T AR$ for some nonsingular matrix $R \in M_{2m\times 2m}$.

Let $\alpha = \frac{d}{dt}[\det(tE + A)]_{t=0}$.

To differentiate the matrix

$$\begin{bmatrix} t & t-a_{12} & \cdots & t-a_{1n} \\ t+a_{12} & t & \cdots & t-a_{2n} \\ \vdots & & \vdots & \\ t+a_{1n} & t+a_{2n} & \cdots & t \end{bmatrix},$$

consider $\frac{d}{dt}\det(v_1, \cdots, v_n) = \sum_k \det(v_1, \cdots, \frac{d}{dt}v_k, \cdots, v_n)$ where $\frac{d}{dt}v_k$ $= v_0 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and $v_i = tv_0 + c_i$, where $c_i$ is the $i$th column vector.

Since

$$\det(v_1, \cdots, \frac{d}{dt}v_k, \cdots, v_n) = \det(tv_0 + c_1, \cdots, tv_0 + c_{k-1}, v_0, \cdots, tv_0 + c_n)$$

and $tv_0 = 0$, we have

$$\frac{d}{dt}P(t) = \frac{d}{dt}\det(v_1, \cdots, v_n) = \sum_k \det(c_1, \cdots, c_{k-1}, v_0, c_{k+1}, \cdots, c_n).$$

Hence, if $n$ is even, $\{c_1, c_2, \cdots, c_n\}$ is a linearly independent set. So, $v_0 = \sum_{i-1}^n \alpha_i c_i$ and $\frac{d}{dt}P(t) = \sum_k \alpha_k(\det A) = 0$.

Since $v_0 = A\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ implies $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = A^{-1}v_0 = A^{-1}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and $A^{-1} =$ $(a'_{ij})$, we have $a_1 + \cdots + a_n = \sum_{i,j} a'_{ij} = 0$.

So we have $1 = \det A = (\det A)^2 \det A$ and $\beta = (PfA)^2 = \det A$ where $PfA$ is the genetic Phaffian of size $n$ [5]. i.e., if $n$ is even, then $\det(tE+A) = (PfA)^2$ which is a Cramer's form and then we can say that there always exists the idempotent elements in an (even) $n$-dimensional Lotka-Volterra algebra $A$.

On the other hand, if $n$ is odd and $t = 0$, then we have $\beta = \det A = 0$. So $\det(tE + A) = \left(\sum_{i=1}^n (-1)^i PfA_i\right)^2$, which is a nonzero polynomial.

Since

$$\det \begin{bmatrix} & 1 & \\ c_1 \cdots c_{k-1} & \vdots & c_{k+1} \cdots c_n \\ & 1 & \end{bmatrix} = (-1)^{k+1} \det A_{1k},$$

$$\frac{d}{dt}P(t) = \sum_k \det(c_1, \cdots, c_{k-1}, v_0, c_{k+1}, \cdots, c_n) = \sum_{ij}(-1)^{i+j} \det A_{ij}.$$

By letting $\det A_{ii} = \det A_i$, we can see that $\det A_i = (PfA_i)^2$.

Hence $\det(tE + A) = \left(\sum_{i=1}^n (-1)^i PfA_i\right)^2$ if $n$ is odd.

Therefore, we have the following result ;

THEOREM 6. *In a (genetic) Lotka-Volterra algebra $A$, the idempotent elements in $A$ are as following :*

$$i) \quad x_i = \frac{(-1)^i PfA_i}{\left[\sum_{k=1}^n (-1)^k PfA_k\right]^2} \quad \text{if } n \text{ is odd,}$$

$$ii) \quad x_i = \frac{\sum (-1)^{i+j} PfA_{ij}}{PfA} \quad \text{if } n \text{ is even.}$$

COROLLARY 7. *In a (genetic) Lotka-Volterra algebra $A$, there exists exactly $2^n$ idempotent elements.*

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