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BLOCK TENSOR PRODUCT

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1. Introduction

For a Hilbert space $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded operators on $\mathcal{H}$. For an $n \in \mathbb{N}$, it is well known that any element $T \in \mathcal{L}(\overset{n}{\oplus}\mathcal{H})$ is expressed as an $n \times n$ matrix each of whose entries lies in $\mathcal{L}(\mathcal{H})$ so that T is written as

$$T = (T_{ij}), \quad i,j = 1,2,\ldots,n, \quad T_{ij} \in \mathcal{L}(\mathcal{H}), \tag{1}$$

where $\overset{n}{\oplus}\mathcal{H}$ is the direct sum Hilbert space of n copies of $\mathcal{H}$.

Let $S = (S_{ij}) \in \mathcal{L}(\overset{n}{\oplus}\mathcal{H})$ be another such element, then the *block Schur product* $TⓢS$ is defined as an element of $\mathcal{L}(\overset{n}{\oplus}\mathcal{H})$ whose i-j entry is $T_{ij}S_{ij}$ $(1 \le i,j \le n)$.

More generally, let $p \in \mathbb{N}$ and $T^{(1)}, T^{(2)}, \ldots, T^{(p)}$ be p elements of $\mathcal{L}(\overset{n}{\oplus}\mathcal{H})$ such that each $T^{(q)} (1 \le q \le p)$ has the operator matrix expression

$$T^{(q)} = \left(T_{ij}^{(q)}\right) (i,j = 1,2,\ldots,n, T_{ij}^{(q)} \in \mathcal{L}(H)), \tag{2}$$

then the *block Schur product* $T^{(1)}ⓢT^{(2)}ⓢ\cdotsⓢT^{(p)}$ is the element of $\mathcal{L}(\overset{n}{\oplus}\mathcal{H})$ defined by

$$T^{(1)}ⓢT^{(2)}ⓢ\cdotsⓢT^{(p)} = \left(T_{ij}^{(1)}T_{ij}^{(2)}\cdots T_{ij}^{(p)}\right). \tag{3}$$

Thus,

$$\begin{aligned} &T^{(1)}ⓢT^{(2)}ⓢ\cdotsⓢT^{(p)} \\ =&(\cdots((T^{(1)}ⓢT^{(2)})ⓢT^{(3)})\cdots)ⓢT^{(p)} \\ =&\textit{the product by any other ways of taking parentheses.} \end{aligned} \tag{4}$$

The main purpose of this note is to prove the following.

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THEOREM 1. *If* $T^{(q)}_{ij}T^{(r)}_{k\ell} = T^{(r)}_{k\ell}T^{(q)}_{ij}$ *for all* $q, r = 1, 2, \ldots, p$ *and all* $i, j, k, l = 1, 2, \ldots, n$, *and if all* $T^{(q)} (1 \leq q \leq p)$ *are positive operators then the block Schur product* $T^{(1)} \circledS T^{(2)} \circledS \cdots \circledS T^{(p)}$ *is also a positive operator.*

2. Block tensor product

Keeping the notation in the introduction, we first note that $\overset{n^2}{\oplus}\mathcal{H}$ is identified with the Hilbert space tensor product $\mathcal{H} \otimes \ell^2(n) \otimes \ell^2(n)$ where $\ell^2(n)$ is the n-dimensional Hilbert space $\mathbb{C}^n$ in the following way.

Let $\{\varepsilon_i : i = 1, 2, \ldots, n\}$ be the standard basis of $\ell^2(n)$. Give the lexicographic order to the set $\{\varepsilon_i \otimes \varepsilon_j : i, j = 1, 2, \ldots, n\}$ with respect to i's and j's. A vector of $\overset{n^2}{\oplus}\mathcal{H}$ whose components are all zero except the t-th component, say $\zeta_t (1 \leq t \leq n^2)$, corresponds to $\zeta_t \otimes \varepsilon_i \otimes \varepsilon_j$, where $\varepsilon_i \otimes \varepsilon_j$ is located at the t-th position in the lexicographic order of $\{\varepsilon_i \otimes \varepsilon_j : i, j = 1, 2, \ldots, n\}$.

Then the *block tensor product* $T \odot S$ of T and S as an element of $\mathcal{L}(\overset{n^2}{\oplus}\mathcal{H}) = \mathcal{L}(\mathcal{H} \otimes \ell^2(n) \otimes \ell^2(n))$ is defined by

$$T \odot S = \sum_{1 \leq i,j,k,\ell \leq n} (T_{ij}S_{k\ell}) \otimes e_{ij} \otimes e_{k\ell}, \tag{5}$$

where $e_{ij} \in \mathcal{L}(\ell^2(n))$, whose matrix with respect to the standard basis $\{\varepsilon_i : i = 1, 2, \ldots, n\}$ is the $n \times n$ matrix of which entries are all zero, but 1 at i-j position.

It is important to note that

(6) *the* $\varepsilon_i \otimes \varepsilon_j - \varepsilon_k \otimes \varepsilon_\ell$ *block in* $T \odot S$ *is* $T_{ik}S_{j\ell} (i, j, k, \ell = 1, 2, \ldots, n)$.

This enables us to verify the following lemma, whose routine computational proof will be omitted.

LEMMA 1. *Let* S, T, U *and* V *be bounded operators on* $\overset{n}{\oplus}\mathcal{H}$.

(i) *If* $T_{ij}S_{k\ell} = S_{k\ell}T_{ij}$ *for all* $i, j, k, \ell = 1, 2, \ldots, n$, *then*

$$(T \odot S)^* = T^* \odot S^* \tag{7}$$

(ii) *If* $S_{ij}U_{k\ell} = U_{k\ell}S_{ij}$, *for all* $i, j, k, \ell = 1, 2, \ldots, n$, *then*

$$(T \odot S)(U \odot V) = (TU) \odot (SV) \tag{8}$$

For $n \geq 2$, we define two mappings π, ρ from $\mathcal{L}(\overset{n}{\oplus}\mathcal{H})$ into $\mathcal{L}(\overset{n^2}{\oplus}\mathcal{H})$ by

$$\pi(T) = T \odot I, \quad T \in \mathcal{L}(\overset{n}{\oplus}\mathcal{H}) \tag{9}$$

$$\rho(S) = I \odot S, \quad S \in \mathcal{L}(\overset{n}{\oplus}\mathcal{H}) \tag{10}$$

where I is the identity operator on $\overset{n}{\oplus}\mathcal{H}$. The following lemma is immediately proven from the previous lemma, so its proof is also omitted.

LEMMA 2. *The above defined* π, ρ *are *-representations of* $\mathcal{L}(\overset{n}{\oplus}\mathcal{H})$ *on* $\overset{n^2}{\oplus}\mathcal{H}$, *sending the identity operator on* $\overset{n}{\oplus}\mathcal{H}$ *to the identity operator on* $\overset{n^2}{\oplus}\mathcal{H}$.

COROLLARY 1. *If* $T, S \in \mathcal{L}(\overset{n}{\oplus}\mathcal{H})$ *are positive operators, and if* $T_{ij}S_{k\ell} = S_{k\ell}T_{ij}$ *for all* $i, j, k, \ell = 1, 2, \ldots, n$, *then the block tensor product* $T \odot S$ *is also a positive operator acting an* $\overset{n^2}{\oplus}\mathcal{H}$.

Proof. When $n = 1$, the assertion is well-known to be true. So, let $n \geq 2$. Applying Lemma 1, we get that

$$\begin{aligned} \pi(T)\rho(S) &= (T \odot I)(I \odot S) = T \odot S \\ &= (I \odot S)(T \odot I) \\ &= \rho(S)\rho(T). \end{aligned}$$

By Lemma 2, $\pi(T), \rho(S)$ are also positive operators.

Consequently, $T \odot S = \pi(T)\rho(S)$ is a positive operator.

COROLLARY 2. *If $U, V \in \mathcal{L}(\overset{n}{\oplus}\mathcal{H}$ are unitary operators, then $U \odot V$ is also a unitary operator acting on $\overset{n^2}{\oplus}\mathcal{H}$. ($U, V$ may not commute.).*

Proof. By Lemma 1,

$$\begin{aligned} U \odot V &= (U \odot I)(I \odot V) \\ &= \pi(U)\rho(V), \end{aligned}$$

which is a unitary operator by Lemma 2.

Proof of Theorem 1. Let Y be the bounded linear operator from $\overset{n^2}{\oplus}\mathcal{H}$ into $\overset{n}{\oplus}\mathcal{H}$ whose operator matrix with respect to the orthonormal bases of $\overset{n^2}{\oplus}\mathcal{H}$ into $\overset{n}{\oplus}\mathcal{H}$ as described previously is given as follows. Let I be the identity operator on $\mathcal{H}$. The matrix of Y is the $n \times n^2$ matrix with entries are filled with I's located at $(1,1), (2, n+2), (3, 2n+3), \ldots, (n, n^2)$ positions and the zero operator $0 \in \mathcal{L}(\mathcal{H})$ at the remaining positions. For example, when $n = 3$, the matrix of Y is 3×3^2 block operator matrix as follows.

$$Y = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

It is routine to verify that, for $T, S \in \mathcal{L}(\overset{n}{\oplus}\mathcal{H})$,

$$T \circledS S = Y(T \odot S)Y^*. \tag{11}$$

Consequently, if T and S are positive operators on $\overset{n}{\oplus}\mathcal{H}$, then $T \circledS S$ is also a positive operator because $T \odot S$ is a positive operator acting on $\overset{n^2}{\oplus}\mathcal{H}$ by Corollary 1. This proves the theorem when $p = 2$. For the case $p \geq 3$, the assertion follows from (4) and the mathematical induction.

REMARKS. To prove Theorem 1.6 ([2, p.7]), an extended version of the Stinespring dilation theorem, Lemma 1.5 ([2, p.6]) has been employed. In my opinion, our Theorem 1 will be the right one to give a complete background for the proof of Theorem 1.6 ([2, p.7]). Of course our proof of Theorem 1 is motivated by that for the case of complex matrices given in [1, p.29-30].

LEMMA 3. *For any* $T, S \in \mathcal{L}(\overset{n}{\oplus}\mathcal{H})$, *we have*

$$\|T \odot S\| \leqq \|T\|\|S\|.$$

Proof. Use Lemma 1 and Lemma 2.

COROLLARY 3. *Let* $\mathcal{A}, \mathcal{B}$ *be two* C^**-algebras and* $\alpha : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, $\beta : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ *be completely bounded maps. Then the map*

$$(x, y) \in \mathcal{A} \times \mathcal{B} \longrightarrow \alpha(x)\beta(y) \in \mathcal{L}(\mathcal{H})$$

is a completely bounded bilinear mapping.

Proof. Apply Lemma 3 and (12), noticing that $\|Y\| \leq 1$.

References

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