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# THE ANALYTIC FEYNMAN INTEGRAL OVER PATHS IN ABSTRACT WIENER SPACE

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## 1. Introduction

In their papers [2,3], Cameron and Storvick introduced some classes $S''_n$ and $S''$ of functionals on classical Wiener space $C_0[a,b]$. For such functionals, they showed that the analytic Feynman integral exists and they gave some formulas for this integral. Moreover they obtained that the functionals of the form

$$F(x) = \exp\left\{ \int_a^b \theta(s, x(s))ds \right\} \tag{1.1}$$

are in $S''$ where they assumed that the potential $\theta : [a,b] \times R \longrightarrow C$ satisfies (i) for each $s \in [a,b]$, $\theta(s,\cdot)$ is the Fourier–Stieltjes transform of $\sigma_s \in M(R)$, (ii) for each Borel subset $E$ of $[a,b] \times R$, $\sigma_s(E^{(s)})$ is a Borel measurable function of s on [a,b], and (iii) the total variation $\| \sigma_s \|$ of $\sigma_s$ is bounded as a function of s. Also, under some measurability assumptions, they proved in [3] that the analytic Feynman integral of functionals which are essentially of the form (1.1) gives a solution to an integral equation formally equivalent to Schroedinger equation. Further, Johnson and Skoug [6,7] extended the results of [2,3] to arbitrary dimension under somewhat less stringent conditions on $\theta$. Also Kallianpur and his coworkers introduced the analytic Feynman integral on abstract Wiener space $B$ with its related topics [8,9].

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Let $(B,\mathcal{B}(B),m)$ be an abstract Wiener space and let $C_B$ denote the set of all $B$–valued continuous functions on [0,T] into $B$ which vanish at origin. From [10] it follows that $C_B$ is a real separable Banach space with the norm

$$(1.2)\qquad \| x \|_{C_B} = \sup_{s\in[0,T]} \| x(s) \|_B$$

and the minimal $\sigma$–algebra making the mapping $x \longrightarrow x(s)$ measurable consists of the Borel subsets of $C_B$. Moreover the Brownian motion in $B$ induces a probability measure $m_B$ on $(C_B,\mathcal{B}(C_B))$ which is mean-zero Gaussian.

In [12] Ryu introduced an operator–valued function space integral on $C_B$ except analytic extension. Furthermore the concept of the analytic Feynman integral has so far been defined on classical Wiener space $C_0[a,b]$ and abstract Wiener space $B$ [2,3,6,7,8,9].

In this paper, we define the analytic Feynman integral over $C_B$ and we prove the existences of this integral for certain classes $\&_n''$ and $\&''$ of functionals on $C_B$ which correspond to $S_n''$ and $S''$ in [2,3]. Moreover we investigate the analytic Feynman integrability of functionals on $C_B$ of the form

$$(1.3)\qquad \exp\left\{\int_0^T \theta(s,x(s))ds\right\},\quad \exp\left\{\int_0^T \theta(s,x(s))ds\right\}\psi(x(T))$$

and

$$(1.4)\quad \exp\left\{\int_0^T \theta(s,x(s))d\eta(s)\right\},\quad \exp\left\{\int_0^T \theta(s,x(s))d\eta(s)\right\}\psi(x(T))$$

which are of interest in Feynman integration theory and quantum mechanics.

## 2. Definitions and preliminaries

Let $(B,\mathcal{B}(B),m)$ and $(C_B,\mathcal{B}(C_B),m_B)$ be given as in the introduction. We begin with introducing a concrete form of $m_B$ [12]. Let

$\vec{s} = (s_1, \cdots, s_n)$ be given with $0 = s_0 < s_1 < \cdots < s_n \le T$ and let $T_{\vec{s}} : B^n \longrightarrow B^n$ be defined by

$$T_{\vec{s}}(y_1, \cdots, y_n) = \Big(\sqrt{s_1 - s_0}\, y_1, \sqrt{s_1 - s_0}\, y_1 + \sqrt{s_2 - s_1}\, y_2, \cdots, \sum_{k=1}^{n} \sqrt{s_k - s_{k-1}}\, y_k\Big)$$

Then we define a Borel measure $\nu_{\vec{s}}$ on $\mathcal{B}(B^n)$ by $\nu_{\vec{s}}(E) = (X_1^n m)\ (T_{\vec{s}}^{-1}(E))$ for every $E \in \mathcal{B}(B^n)$. Let $J_{\vec{s}}^{-1}(X_1^n E_k)$ is called the I–set and then the collection $I$ of all such I–sets is an algebra. We define a set function $m_B$ on $I$ by $m_B(J_{\vec{s}}^{-1}(X_1^n E_k)) = \nu_{\vec{s}}(X_1^n E_k)$. Then $m_B$ is well- defined and countably additive on $I$ . Using the Caratheodory process, we have a Borel measure $m_B$ on $C_B$.

Now we introduce some integration formula which plays a key role throughout this paper. This formula is easily obtain by the change of variable theorem [4].

LEMMA 2.1. *Let $\vec{s} = (s_1, \cdots, s_n)$ be given with $0 = s_0 < s_1 < \cdots < s_n \le T$ and let $f : B^n \longrightarrow C$ be a Borel measurable function. Then*

$$\text{(2.1)} \qquad \int_{C_B} f(x(s_1), \cdots, x(s_n))\, dm_B(x) \overset{*}{=} \int_{B^n} f \circ T_{\vec{s}}(y_1, \cdots, y_n)\, d(X_1^n m)(y_1, \cdots, y_n)$$

*where by $\overset{*}{=}$ we mean that if either side exists then both sides exist and they are equal.*

Next we define the analytic Wiener integral and the analytic Feynman integral over $C_B$.

DEFINITION 2.2. Let $F$ be a $C$–valued measurable functional on $C_B$ such that

$$J_F(z) = \int_{C_B} F(z^{-1/2}x)\, dm_B(x)$$

exists for all real $z > 0$. If there exists an analytic function $J_F^*$ on $\Omega = \{z \in C : Re z > 0\}$ such that $J_F^*(z) = J_F(z)$ for all real $z > 0$, then we define $J_F^*$ to be the analytic Wiener integral of $F$ over $C_B$ with parameter $z$, and for $z \in \Omega$, we write

$$I_a^z(F) = J_F^*(z). \tag{2.2}$$

If the following limit (2.3) exists for non–zero real $q$, then we call it the analytic Feynman integral of $F$ over $C_B$ with parameter $q$, and we write

$$I_a^q(F) = \lim_{z \to -iq} I_a^z(F) \tag{2.3}$$

where $z$ approaches $-iq$ through $\Omega$.

Let $H$ be an infinite dimensional separable Hilbert space and let $\{e_n\}$ be a complete orthonormal (C.O.N.) set in $H$ such that the $e_n$'s are in $B^*$. For each $(h,x) \in H \times B$, let

$$(h,x)^\sim = \lim_{n \to \infty} \sum_{k=1}^{n} < h, e_k > (e_k, x)$$

if the limit exists and let $(h,x)^\sim = 0$ otherwise. Then $(\cdot,\cdot)^\sim$ is a Borel measurable functional on $H \times B$, and if both $h$ and $x$ are in $H$, then Parseval's identity gives $(h,x)^\sim = < h, x >$. In particular, $x \longrightarrow (h,x)^\sim$ is Gaussian with mean zero and variance $|h|^2$ [8,9].

We now construct some classes $\&_n''$ and $\&''$ of functionals on $C_B$ as mentioned in the introduction. Let $\Delta_n = \{(s_1, \cdots, s_n) \in [0,T]^n : 0 = s_0 < s_1 < \cdots < s_n \le T\}$. Let $M_n'' = M_n''(\Delta_n \times H^n)$ be the class of complex Borel measures on $\Delta_n \times H^n$ and let $\| \mu \| = var\mu$, the total variation of $\mu \in M_n''$.

DEFINITION 2.3. Let $\&_n'' = \&_n''(\Delta_n \times H^n)$ be the space of functionals of the form

$$F(x) = \int_{\Delta_n \times H^n} \exp\left\{ i \sum_{k=1}^{n} (h_k, x(s_k))^\sim \right\} d\mu(\vec{s}, \vec{h}) \tag{2.4}$$

for $x \in C_B$ where $\mu \in M_n''$. Here we take $\| F \|_n'' = inf\{\| \mu \|\}$ where the infimum is taken over all $\mu$'s so that $F$ and $\mu$ are related by (2.4).

Let $M'' = M''(\sum \Delta_n \times H^n)$ be the class of sequences of measures $\{\mu_n\}$ such that each $\mu_n \in M_n''$ and $\sum_{n=1}^{\infty} \| \mu_n \| < \infty$.

DEFINITION 2.4. Let $\&'' = \&''(\sum \Delta_n \times H^n)$ be the space of functionals on $C_B$ of the form

$$F(x) = \sum_{n=1}^{\infty} F_n(x) \tag{2.5}$$

where each $F_n \in \&''_n$ and $\sum_{n=1}^{\infty} \| F_n \|''_n < \infty$. The norm of $F$ is defined by $\| F \|'' = inf\{\sum_{n=1}^{\infty} \| F_n \|''_n\}$ where the infimum is taken over all representation of $F$ given by (2.5).

Note that if $n$ and $k$ are positive integers then $\&''_n \subset \&''_{n+k}$. And if $F \in \&''_n$ then $\| F \|''_n \geq \| F \|''_{n+k}$ and $|F(x)| \leq \| F \|''_n$ for all $x \in C_B$. For completeness, we define $\&''_0$ to be constant functionals and define their norms to be their absolute values. For $F \in \&''$, the series in (2.5) converges absolutely and uniformly over $C_B$. Also if $F \in \&''$ then $|F(x)| \leq \| F \|''$ for all $x \in C_B$. Moreover we can show that $\&''$ is a Banach algebra with the norm $\| \cdot \|''$ which corresponds to Theorem 4.1 in [2].

## 3. The analytic Feynman integral over $C_B$

In this section, we begin by proving the existence theorems of the analytic Feynman integral over $C_B$ for the classes $\&''_n$ and $\&''$ of functionals on $C_B$.

THEOREM 3.1. *Let $F \in \&''_n$ be such that*

$$F(x) = \int_{\Delta_n \times H^n} \exp\left\{ i \sum_{k=1}^{n} (h_k, x(s_k))^{\sim} \right\} d\mu(\vec{s}, \vec{h})$$

*for $x \in C_B$ where $\mu \in M''_n$. Then $F$ is analytic Feynman integrable and if $q$ is non-zero real*

$$I_a^q(F) = \int_{\Delta_n \times H^n} \exp\left\{ \frac{1}{2qi} \sum_{k=1}^{n} \sum_{j=1}^{k} (2 - \delta_{j,k}) < h_j, h_k > (s_j - s_0) \right\} d\mu(\vec{s}, \vec{h}). \tag{3.1}$$

*Proof.* From the Fubini theorem and Lemma 2.1, it follows that

$$J_F(z) = \int_{C_B} F(z^{-1/2}x)dm_B(x)$$

$$= \int_{\Delta_n\times H^n}\int_{B^n} \exp\Big\{iz^{-1/2}\sum_{k=1}^{n}\sum_{j=1}^{k}\sqrt{s_j-s_{j-1}}(h_k,y_j)^{\sim}\Big\}$$
$$d(X_1^n m)(y_1,\cdots,y_n)d\mu(\vec{s},\vec{h})$$

$$= \int_{\Delta_n\times H^n} \exp\Big\{-\frac{1}{2z}\sum_{k=1}^{n}(s_k-s_{k-1})\Big|\sum_{j=k}^{n}h_j\Big|^2\Big\}d\mu(\vec{s},\vec{h})$$

$$= \int_{\Delta_n\times H^n} \exp\Big\{-\frac{1}{2z}\sum_{k=1}^{n}\sum_{j=1}^{k}(2-\delta_{j,k}) < h_j,h_k >$$
$$(s_j-s_0)\Big\}d\mu(\vec{s},\vec{h})$$

for $z>0$. The integrand of the last member of the above equation is an analytic function of $z$ in $\Omega$ and is bounded by one. Using Cauchy's theorem, Fubini's theorem, and Morera's theorem, $J_F(z)$ has an analytic extension to $\Omega$. Thus we have for $z\in\Omega$

$$I_a^z(F) = \int_{\Delta_n\times H^n} \exp\Big\{-\frac{1}{2z}\sum_{k=1}^{n}(s_k-s_{k-1})\Big|\sum_{j=k}^{n}h_j\Big|^2\Big\}d\mu(\vec{s},\vec{h})$$

An application of the dominated convergence theorem enables us to pass the limit as $z\longrightarrow -iq$ and hence we obtain the formula (3.1).

THEOREM 3.2. *Let $F\in \&''$ be given by*

$$F(x) = \sum_{n=1}^{\infty}F_n(x) = \sum_{n=1}^{\infty}\int_{\Delta_n\times H^n}\exp\Big\{i\sum_{k=1}^{n}(h_k,x(s_k))^{\sim}\Big\}d\mu_n(\vec{s},\vec{h})$$

*where each $F_n\in \&_n''$ and $\mu_n\in M_n''$ with $\sum_{n=1}^{\infty}\|F_n\|_n''<\infty$. Then $F$ is analytic Feynman integrable over $C_B$ and if $q$ is non-zero real*

$$I_a^q(F) = \sum_{n=1}^{\infty}I_a^q(F_n) = \sum_{n=1}^{\infty}\int_{\Delta_n\times H^n}\exp\Big\{\frac{1}{2qi}\sum_{k=1}^{n}\sum_{j=1}^{k} \tag{3.2}$$
$$(2-\delta_{j,k}) < h_j,h_k > (s_j-s_0)\Big\}d\mu_n(\vec{s},\vec{h}).$$

*Proof.* We note that if $F_n \in \&_n''$ then $|F_n(x)| \leq \| F_n \|_n''$ for $x \in C_B$. By the dominated convergence theorem, we obtain

$$(3.3)\qquad \int_{C_B} F(z^{-1/2}x)dm_B(x) = \sum_{n=1}^{\infty} \int_{C_B} F_n(z^{-1/2}x)dm_B(x)$$
$$= \sum_{n=1}^{\infty} \int_{\Delta_n \times H^n} \exp\Big\{ -\frac{1}{2z} \sum_{k=1}^{n} \sum_{j=1}^{k} (2-\delta_{j,k}) < h_j, h_k >$$
$$(s_j - s_0)\Big\} d\mu_n(\vec{s}, \vec{h})$$

for $z \in \Omega$. Since $\sum_{n=1}^{\infty} \| F_n \|_n'' < \infty$, the series (3.3) converges uniformly and so is analytic in $\Omega$. Thus, using the same argument as in the proof of Theorem 3.1, we have the formula (3.2).

Next we prove the existence of the analytic Feynman integral for functions of the form (1.3). Let $\mathcal{G}$ be the set of all $C$-valued functions $\theta$ on $[0,T] \times B$ which have the form

$$(3.4)\qquad \theta(s,y) = \int_H \exp\{i(h,y)^{\sim}\} d\sigma_s(h)$$

where $\{\sigma_s : s \in [0,T]\}$ is the family from $M(H)$ satisfying the following conditions ; (i) for each Borel set $E$ of $H$, $\sigma_s(E)$ is a Borel measurable function of $s$ on $[0,T]$, and (ii) $\| \sigma_s \| \in L_1[0,T]$.

THEOREM 3.3. *Let $\theta \in \mathcal{G}$ be given by (3.4). Then the functions $F_n(x) = [\int_0^T \theta(s, x(s))ds]^n$, and $F(x) = \exp\{\int_0^T \theta(s,x(s))ds\}$ are elements of $\&''$ for $x \in C_B$. Hence they are analytic Feynman integrable and if $q$ is non-zero real*

$$(3.5)\qquad I_a^q(F) = \sum_{n=0}^{\infty} \frac{1}{n!} I_a^q(F_n) = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n \times H^n} \exp\Big\{ \frac{1}{2qi} \sum_{k=1}^{n} \sum_{j=1}^{k}$$
$$(2-\delta_{j,k}) < h_j, h_k > (s_j - s_0)\Big\} d\mu_n(\vec{s}, \vec{h})$$

*where $d\mu_n(\vec{s}, \vec{h}) = X_1^n d\sigma_{s_k}(h_k)ds_k$.*

*Proof.* By the definition of $\&_n''$, $\int_0^T \theta(s,x(s))ds \in \&_1'' \subset \&''$. Since $\&''$ is a Banach algebra, $F_n(x)$ and $F(x)$ are elements of $\&''$. From the Fubini theorem, Lemma 2.1 and unsymmetric Fubini theorem [5], it follows that for $z \in \Omega$

$$\int_{C_B} F_n(z^{-1/2}x)dm_B(x) = \int_{C_B} \Big[\prod_{k=1}^{n} \int_0^T \theta(s_k, x(s_k))ds_k\Big] dm_B(x)$$

$$= n! \int_{\Delta_n \times H^n} \int_{B_n} \exp\Big\{iz^{-1/2} \sum_{k=1}^{n}\sum_{j=1}^{k} \sqrt{s_j - s_{j-1}}(h_k, y_j)^{\sim}\Big\}$$
$$d(X_1^n m)(y_1, \cdots, y_n) X_1^n d\sigma_{s_k}(h_k)ds_k$$
$$= n! \int_{\Delta_n \times H^n} \exp\Big\{ -\frac{1}{2z}\sum_{k=1}^{n}(s_k - s_{k-1})\Big|\sum_{j=k}^{n} h_j\Big|^2\Big\} d\mu_n(\vec{s}, \vec{h})$$

where $d\mu_n(\vec{s}, \vec{h}) = X_1^n d\sigma_{s_k}(h_k)ds_k$. Moreover for $z \in \Omega$ we have

$$\sum_{n=0}^{\infty} \frac{1}{n!}|F_n(z^{-1/2}x)| \leq \sum_{n=0}^{\infty}\frac{1}{n!}\Big[\int_0^T \parallel \sigma_s \parallel ds\Big]^n < \infty.$$

Using the dominated convergence theorem and the same argument as in the proof of Theorem 3.1, we obtain the formula (3.5).

Let $\mathcal{F}(B)$ be the class of functions on $B$ of the form

$$\psi(y) = \int_H \exp\{i(h, y)^{\sim}\} d\nu(h) \tag{3.6}$$

for $y \in B$ where $\nu \in M(H)$.

THEOREM 3.4. *Let $\theta \in \mathcal{G}$ and $\psi \in \mathcal{F}(B)$ be given by (3.4) and (3.6) respectively. Then the functions $F_n(x) = [\int_0^T \theta(s, x(s))ds]^n \psi(x(T))$, and $F(x) = \exp\{\int_0^T \theta(s, x(s))ds\}\psi(x(T))$ are elements of &″ for $x \in C_B$. Hence they are analytic Feynman integrable and if $q$ is non-zero real*

$$\begin{aligned} I_a^q(F) = \sum_{n=0}^{\infty}\frac{1}{n!}I_a^q(F_n) = \sum_{n=0}^{\infty}\int_{\Delta_n \times H^{n+1}} \exp\Big\{\frac{1}{2qi}\Big[\sum_{k=1}^{n} \\ \Big((s_k - s_{k-1})\Big|\sum_{j=k}^{n+1} h_j\Big|^2\Big) + (T - s_n)|h_{n+1}|^2\Big]\Big\} d\mu_n(\vec{s}, \vec{h}) \end{aligned} \tag{3.7}$$

where $d\mu_n(\vec{s},\vec{h}) = (X_1^n d\sigma_{s_k}(h_k)ds_k)d\nu(h_{n+1})$.

*Proof.* We have already known that $[\int_0^T \theta(s,x(s))ds]^n$ and $\exp\{\int_0^T \theta(s,x(s))ds\} \in \&''$. Moreover we know that $\psi(x(T)) = \int_0^T \psi(x(s)) d\nu_T(s) = \int_0^T \int_H \exp\{i(h,x(s))^\sim\}d\mu(h)d\nu_T(s) \in \&_1'' \subset \&''$ where $\nu_T$ is the unit measure concentrated at $s=T$. Since $\&''$ is a Banach algebra, each $F_n$ and $F$ are elements of $\&''$ and hence they are analytic Feynman integrable. Now we consider the following function

$$
\begin{aligned}
F_n(x) =& \Big[\prod_{k=1}^n \int_0^T \theta(s_k,x(s_k))ds_k\Big]\psi(x(T)) \\
=& n!\int_{\Delta_n} \prod_{k=1}^n \theta(s_k,x(s_k))X_1^n ds_k \psi(x(T)) \\
=& n!\int_{\Delta_n\times H^{n+1}} \exp\Big\{i\sum_{k=1}^{n+1}(h_k,x(s_k))^\sim\Big\}(X_1^n d\sigma_{s_k}(h_k)ds_k)d\nu(h_{n+1})
\end{aligned}
$$

where $s_{n+1}=T$. From Fubini theorem, Lemma 2.1 and unsymmetric Fubini theorem, it follows that for $z\in\Omega$

$$
\begin{aligned}
&\int_{C_B} F_n(z^{-1/2}x)dm_B(x) \\
=& n!\int_{\Delta_n\times H^{n+1}}\int_{B^{n+1}} \exp\Big\{iz^{-1/2}\sum_{k=1}^{n+1}\sum_{j=1}^{k}\sqrt{s_j-s_{j-1}}(h_k,y_j)^\sim\Big\} \\
& d(X_1^{n+1}m)(y_1,\cdots,y_{n+1})d\mu_n(\vec{s},\vec{h}) \\
=& n!\int_{\Delta_n\times H^{n+1}} \exp\Big\{-\frac{1}{2z}\sum_{k=1}^{n+1}(s_k-s_{k-1})\Big|\sum_{j=k}^{n+1}h_j\Big|^2\Big\}d\mu_n(\vec{s},\vec{h}) \\
=& n!\int_{\Delta_n\times H^{n+1}} \exp\Big\{-\frac{1}{2z}\Big[\sum_{k=1}^{n}(s_k-s_{k-1})\Big|\sum_{j=k}^{n+1}h_j\Big|^2+ \\
& (T-s_n)|h_{n+1}|^2\Big]\Big\}d\mu_n(\vec{s},\vec{h})
\end{aligned}
$$

where $d\mu_n(\vec{s},\vec{h}) = (X_1^n d\sigma_{s_k}(h_k)ds_k)d\nu(h_{n+1})$. Moreover, for $z \in \Omega$,

$$\sum_{n=0}^{\infty} \frac{1}{n!}|F_n(z^{-1/2}x)| \le \sum_{n=0}^{\infty} \frac{\| \nu \|}{n!}\Big[\int_0^T \| \sigma_s \| \, ds\Big]^n < \infty.$$

Using the dominated convergence theorem and the same argument as in the proof of Theorem 3.1, we obtain the formula (3.6).

The rest of this paper, we establish the existence of the analytic Feynman integral for functions of the form (1.4). Let $\eta$ be a $C$-valued Borel measure on $(0,T)$. Then $\eta = \mu + \nu$ can be decomposed uniquely into its continuous part $\mu$ and discrete part $\nu$. Let $\delta_{\tau_p}$ denote the Dirac measure with total mass one concentrated at $\tau_p$.

Let $\mathcal{G}^*$ be the set of all $C$-valued functions $\theta$ on $[0,T]\times B$ which have the form (3.4) where $\{\sigma_s : s \in [0,T]\}$ is the family from $M(H)$ satisfying the following conditions; (i) for each Borel set $E$ of $H$, $\sigma_s(E)$ is a Borel measurable function of $s$ on $[0,T]$, and (ii) $\| \sigma_s \| \in L_1([0,T],\mathcal{B}([0,T]),|\eta|)$.

THEOREM 3.5. *Let* $\eta = \mu + \sum_{p=1}^{r} w_p\delta_{\tau_p}$ *where* $0 < \tau_1 < \cdots < \tau_r < T$ *and the* $w_p$*'s are in* $C$ *for* $p = 1,2,\cdots,r$. *Let* $\theta \in \mathcal{G}^*$ *be given by* (3.4). *Then the functions* $F_n(x) = [\int_0^T \theta(s,x(s))d\eta(s)]^n$ *and* $F(x) = \exp\{\int_0^T \theta(s,x(s))d\eta(s)\}$ *are analytic Feynman integrable and if* $q$ *is non-zero real*

$$I_a^q(F) = \sum_{n=0}^{\infty}\frac{1}{n!}I_a^q(F_n) = \sum_{n=0}^{\infty}\sum_{q_0+q_1+\cdots+q_r=n}\frac{w_1^{q_1}\cdots w_r^{q_r}}{q_1!\cdots q_r!}$$

$$W_1\Big(\sum_{i=v}^{j_u} h_{u-1,i} + \sum_{p=u}^{r}\big(\sum_{i=1}^{j_{p+1}} h_{p,i} + \sum_{i=1}^{q_p} k_{p,i}\big), \sum_{i=v}^{j_{r+1}} h_{r,i}, \sum_{p=u}^{r}\sum_{i=1}^{q_p} k_{p,i}, 0\Big) \tag{3.8}$$

*where* $W_1(A,B,C,D) = \sum_{j_1+\cdots+j_{r+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_{r+1}}} \int_{H^{q_0+q_1+\cdots+q_r}}$

$$\exp\left\{\frac{1}{2qi}\Big[\sum_{u=1}^{r}\sum_{v=1}^{j_u}\alpha_{u,v}|A|^2 + \sum_{v=1}^{j_{r+1}}\alpha_{r+1,v}|B|^2 + \sum_{u=1}^{r}\alpha_{u,j_u+1}|C|^2 + D\Big]\right\}$$

$d(X_{p=0}^{r} X_{i=1}^{j_{p+1}} \sigma_{s_{p,i}})(h_{p,i}) d(X_{p=1}^{r} X_{i=1}^{q_p} \sigma_{\tau_p})(k_{p,i}) d(X_{p=0}^{r} X_{i=1}^{j_{p+1}} \mu)(s_{p,i})$, $\alpha_{u,v} = s_{u-1,v} - s_{u-1,v-1}$, and $\Delta_{q_0;j_1,\cdots,j_{r+1}} = \{(s_{0,1},\cdots,s_{0,j_1},s_{1,1}, s_{1,2}\cdots,s_{r,j_{r+1}}) : 0 = s_{0,0} < s_{0,1} < \cdots < s_{0,j_1} < \tau_1 < s_{1,1} < s_{1,2} < \cdots < s_{r-1,j_r} < \tau_r < s_{r,1} < \cdots < s_{r,j_{r+1}} < T = \tau_{r+1}\}$.

*Proof.* From the multinomial expansion theorem, the simplex trick [12], the Fubini theorem, the relabeling ($s_{j_1+\cdots+j_p+i} = s_{p,i}$ and $\tau_p = s_{p,0} = s_{p-1,j_p+1}$), and Lemma 2.1, it follows that

$$
\begin{aligned}
&\int_{C_B} F_n(z^{-1/2}x)\,dm_B(x) \\
&= \int_{C_B} \Big[\int_0^T \theta(s, z^{-1/2}x(s))\,d\mu(s) + \sum_{p=1}^{r} \theta(\tau_p, z^{-1/2}x(\tau_p))\Big]^n dm_B(x) \\
&= \sum_{q_0+q_1+\cdots+q_r=n} \frac{n!}{q_0!q_1!\cdots q_r!} w_1^{q_1}\cdots w_r^{q_r} \int_{C_B} \Big(\int_0^T \theta(s, z^{-1/2}x(s)) \\
&\quad d\mu(s)\Big)^{q_0} \prod_{p=1}^{r} \theta(\tau_p, z^{-1/2}x(\tau_p))^{q_p}\,dm_B(x) \\
&= n! \sum_{q_0+q_1+\cdots+q_r=n} \frac{w_1^{q_1}\cdots w_r^{q_r}}{q_1!\cdots q_r!} \sum_{j_1+\cdots+j_{r+1}=q_0} \Big[\int_{\Delta_{q_0;j_1,\cdots,j_{r+1}}} \Big\{\int_{C_B} \prod_{i=1}^{q_0} \\
&\quad \theta(s_i, z^{-1/2}x(s_i)) \prod_{p=1}^{r} \theta(\tau_p, z^{-1/2}x(\tau_p))^{q_p}\,dm_B(x)\Big\} d(X_1^{q_0}\mu)(s_i)\Big] \\
&= n! \sum_{q_0+q_1+\cdots+q_r=n} \frac{w_1^{q_1}\cdots w_r^{q_r}}{q_1!\cdots q_r!} \sum_{j_1+\cdots+j_{r+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_{r+1}}} \int_{C_B} \Big\{\prod_{p=0}^{r} \\
&\quad \prod_{i=1}^{j_{p+1}} \theta(s_{p,i}, z^{-1/2}x(s_{p,i}))\Big\}\Big\{\prod_{p=1}^{r} \theta(s_{p,0}, z^{-1/2}x(s_{p,0}))^{q_0}\Big\} dm_B(x) \\
&\quad d(X_{p=0}^{r} X_{i=1}^{j_{p+1}} \mu)(s_{p,i}) \\
&= n! \sum_{q_0+q_1+\cdots+q_r=n} \frac{w_1^{q_1}\cdots w_r^{q_r}}{q_1!\cdots q_r!} \sum_{j_1+\cdots+j_{r+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_{r+1}}} \int_{B^{q_0+r}} \\
&\quad \Big\{\prod_{p=0}^{r} \prod_{i=1}^{j_{p+1}} \theta\Big(s_{p,i}, z^{-1/2}\Big(\sum_{u=1}^{p}\sum_{v=1}^{j_u} \sqrt{\alpha_{u,v}}\,y_{u-1,v} + \sum_{v=1}^{i} \sqrt{\alpha_{p+1,v}}\,y_{p,v}\Big)\Big)\Big\} \\
&\quad \Big\{\prod_{p=1}^{r} \theta\Big(s_{p,0}, z^{-1/2}\Big(\sum_{u=1}^{p}\sum_{v=1}^{j_u+1} \sqrt{\alpha_{u,v}}\,y_{u-1,v}\Big)\Big)^{q_p}\Big\} d(X_1^{q_0+r} m)
\end{aligned}
$$

$$(y_{0,1},\cdots,y_{0,j_1},y_{0,j_1+1},\cdots,y_{r-1,1},\cdots,y_{r-1,j_r},y_{r-1,j_r+1},$$
$$y_{r,1},\cdots,y_{r,j_r+1})d(X_{p=0}^{r}X_{i=1}^{j_p+1}\mu)(s_{p,i})$$
$$=n!\sum_{q_0+q_1+\cdots+q_r=n}\frac{w_1^{q_1}\cdots w_r^{q_r}}{q_1!\cdots q_r!}\sum_{j_1+\cdots+j_{r+1}=q_0}\int_{\Delta_{q_0;j_1,\cdots,j_{r+1}}}$$
$$\int_{H^{q_0+q_1+\cdots+q_r}}\exp\Big\{-\frac{1}{2z}\Big[\sum_{u=1}^{r}\sum_{v=1}^{j_u}\alpha_{u,v}\Big|\sum_{i=v}^{j_u}h_{u-1,i}+\sum_{p=u}^{r}\Big(\sum_{i=1}^{j_p+1}h_{p,i}+$$
$$\sum_{i=1}^{q_p}k_{p,i}\Big)\Big|^2+\sum_{v=1}^{j_r+1}\alpha_{r+1,v}\Big|\sum_{i=v}^{j_r+1}h_{r,i}\Big|^2+\sum_{u=1}^{r}\alpha_{u,j_u+1}\Big|\sum_{p=u}^{r}\sum_{i=1}^{q_p}k_{p,i}\Big|^2\Big]\Big\}$$
$$d(X_{p=0}^{r}X_{i=1}^{j_p+1}\sigma_{s_{p,i}})(h_{p,i})d(X_{p=1}^{r}X_{i=1}^{q_p}\sigma_{\tau_p})(k_{p,i})d(X_{p=0}^{r}X_{i=1}^{j_p+1}\mu)(s_{p,i})$$

for $z\in\Omega$. Moreover for $z\in\Omega$ we have

$$\sum_{n=0}^{\infty}\frac{1}{n!}|F_n(z^{-1/2}x)|\le\sum_{n=0}^{\infty}\frac{1}{n!}\Big[\int_0^T\|\sigma_s\|\,d|\eta|(s)\Big]^n<\infty.$$

Using the dominated convergence theorem and the same argument as in the proof of Theorem 3.1, we obtain the formula (3.9).

COROLLARY 3.6. *Let* $\eta=\sum_{p=1}^{r}w_p\delta_{\tau_p}$ *where* $0<\tau_1<\tau_2<\cdots<\tau_r<T$ *and the* $w_p$*'s are in* $C$ *for* $p=1,2,\cdots,r$. *Let* $\theta\in\mathcal{G}^*$ *be given by* (3.4). *Then the functions* $F_n$ *and* $F$ *in Theorem 3.5 are analytic Feynman integrable and if* $q$ *is non–zero real*

$$I_a^q(F)=\sum_{n=0}^{\infty}\frac{1}{n!}I_a^q(F_n)=\sum_{n=0}^{\infty}\sum_{q_1+\cdots+q_r=n}\frac{w_1^{q_1}\cdots w_r^{q_r}}{q_1!\cdots q_r!}\tag{3.9}$$
$$W_2\Big(\sum_{u=1}^{r}\sum_{k=1}^{q_p}h_{p,k},0\Big)$$

where $W_2(A,B)=\int_{H^n}\exp\Big\{\frac{1}{2qi}\sum_{u=1}^{r}(\tau_u-\tau_{u-1})|A|^2-B\Big\}d(X_{p=1}^{r}X_{k=1}^{q_p}\sigma_{\tau_p})(h_{p,k})$.

COROLLARY 3.7. *Let $\eta = \mu$ and let $\theta \in \mathcal{G}^*$ be given by (3.4). Then the functions $F_n$ and $F$ in Theorem 3.5 are analytic Feynman integrable and if $q$ is non-zero real*

$$I_a^q(F) = \sum_{n=0}^{\infty} \frac{1}{n!} I_a^q(F_n) = \sum_{n=0}^{\infty} W_3\Big(\sum_{j=k}^{n} h_j, 0\Big) \tag{3.10}$$

*where* $W_3(A,B) = \int_{\Delta_n \times H^n} \exp\left\{\frac{1}{2qi} \sum_{k=1}^{n} (s_k - s_{k-1})|A|^2 + B\right\}$ $d(X_1^n \sigma_{s_k})(h_k) d(X_1^n \mu)(s_k)$.

THEOREM 3.8. *Let $\eta$ be as in Theorem 3.5. Let $\theta \in \mathcal{G}^*$ and $\psi \in \mathcal{F}(B)$ be given by (3.4) and (3.6) respectively. Then the functions $F_n(x) = [\int_0^T \theta(s, x(s)) d\eta(s)]^n \psi(x(T))$ and $F(x) = \exp\{\int_0^T \theta(s, x(s)) d\eta(s)\} \psi(x(T))$ are analytic Feynman integrable and if $q$ is non-zero real*

$$\begin{aligned} I_a^q(F) = \sum_{n=0}^{\infty} \frac{1}{n!} I_a^q(F_n) &= \sum_{n=0}^{\infty} \sum_{q_0+q_1+\cdots+q_r=n} \frac{w_1^{q_1} \cdots w_r^{q_r}}{q_1! \cdots q_r!} \int_H \\ & W_1\Big(\sum_{i=v}^{j_u} h_{u-1,i} + \sum_{p=u}^{r} \Big(\sum_{i=1}^{j_{p+1}} h_{p,i} + \sum_{i=1}^{q_p} k_{p,i}\Big) \\ & + h_{r,j_{r+1}+1}, \sum_{i=v}^{j_{r+1}} h_{r,i} + h_{r,j_{r+1}+1}, \\ & \sum_{p=u}^{r} \sum_{i=1}^{q_p} k_{p,i} + h_{r,j_{r+1}+1}, (T - s_{r,j_{r+1}})|h_{r,j_{r+1}+1}|^2\Big) d\nu(h_{r,j_{r+1}+1}) \end{aligned} \tag{3.11}$$

*where $W_1(A,B,C,D)$ is as in Theorem 3.5 .*

COROLLARY 3.9. *Let $\eta$ be as in Corollary 3.6. Let $\theta \in \mathcal{G}^*$ and $\psi \in \mathcal{F}(B)$ be given by (3.4) and (3.6) respectively. Then the functions $F_n$ and $F$ in Theorem 3.8 are analytic Feynman integrable and if $q$ is non-zero real*

$$\begin{aligned} I_a^q(F) = \sum_{n=0}^{\infty} \frac{1}{n!} I_a^q(F_n) &= \sum_{n=0}^{\infty} \sum_{q_0+q_1+\cdots+q_r=n} \frac{w_1^{q_1} \cdots w_r^{q_r}}{q_1! \cdots q_r!} \\ & \int_H W_2\Big(\sum_{p=u}^{r} \sum_{k=1}^{q_p} h_{p,k} + h^*, (T - \tau_r)|h^*|^2\Big) d\nu(h^*) \end{aligned} \tag{3.12}$$

*where $W_2(A,B)$ is as in Corollary 3.6.*

COROLLARY 3.10. *Let $\eta$ be as in Corollary 3.7. Let $\theta \in \mathcal{G}^*$ and $\psi \in \mathcal{F}(B)$ be given by (3.4) and (3.6) respectively. Then the functions $F_n$ and $F$ in Theorem 3.8 are analytic Feynman integrable and if $q$ is non-zero real*

$$I_a^q(F) = \sum_{n=0}^{\infty} \frac{1}{n!} I_a^q(F_n) = \sum_{n=0}^{\infty} \int_H W_3\Big(\sum_{j=k}^{n+1} h_j, (T - s_n)|h_{n+1}|^2\Big) d\nu(h_{n+1}) \tag{3.13}$$

*where $W_3(A,B)$ is as in Corollary 3.7.*

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