Comm. Korean Math. Soc. **10** (1995), No. 1, pp. 85–92

STABILITY OF TYPE CONDITION IN $\mathbb{C}^n$

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1. Introduction

In recent years, several questions about the complex function theory of a domain in $\mathbb{C}^n$ have been studied for pseudoconvex domains of finite type. In [1,2], Catlin showed that the finite type condition is a necessary and sufficient condition for the subelliptic estimates for the $\overline{\partial}$-Neumann problem. This subellipticity has been frequently the first step in studying function theories for the domains in $\mathbb{C}^n$.

Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ defined by $r < 0$, where $dr \neq 0$ on $b\Omega$, and $z_0 \in b\Omega$. We also assume that $\Delta_1(z_0)$, the type of z_0, is finite and set $T' = (\Delta_1(z_0)/2)^{n-1}$. Then the following theorem has been proved by Catlin

THEOREM C. *Let z_0 be a point in the boundary of a smoothly bounded pseudoconvex domain Ω. Assume that $\Delta_1(z_0) < \infty$. Then for any number $\epsilon > 0$, there exist a constant c_ϵ and a neighborhood U_ϵ such that if $\{M_\sigma; \sigma \in \Sigma\}$ is any family of 1-dimensional complex manifolds of diameter less than or equal to σ contained in U_ϵ, then*

$$\sup\{|r(z)|; z \in M_\sigma\} \geq c_\epsilon \sigma^{2T'+\epsilon} \tag{1}$$

REMARK 1.1. In particular, if we set $\epsilon = 1$ in above theorem, (1) says that there is a neighborhood U_1 of z_0 such that the type is stable (bounded by $2T' + 1$) on $b\Omega \cap U_1$.

In this paper, we will extend Theorem C and prove the stability of type condition under a small perturbations of the boundary of Ω near

Received August 11, 1994. Revised October 14, 1994.
1991 Mathematics Subject Classification. 32F15.
Key words and phrases: Finite 1-type, pseudoconvex domains.
Partially supported by KOSEF 941-0100-037-2 1994-1995 and by GARC-KOSEF 1994.

z_0. This theorem has been mentioned in [3,4,5]. For a function f which is smooth on $\overline{U}_1 \ni z_0$, we set

$$B_\delta^k(f, U_1) = \left\{ g \in C^\infty(\mathbb{C}^n);\ \sup_{z \in \overline{U}_1} |D^\alpha(f-g)(z)| < \delta \text{ for all } |\alpha| \le k \right\}.$$

Therefore $B_\delta^k(f, U_1)$ is a δ-neighborhood of f in C^k-norm on $\overline{U}_1$. Set $T'' = [\![T']\!]$. Here we denote $[\![x]\!]$ the smallest integer bigger than or equal to x. Then we prove

THEOREM 1. *Let $z_0 \in b\Omega$ with $\Delta_1(z_0) < \infty$. Then for any $\epsilon > 0$, there exist a constant c_ϵ and a neighborhood $U_\epsilon \subset\subset U_1$ and $\sigma_\epsilon, \delta_\epsilon > 0$ such that for any 1-dimensional manifold M_σ of diameter less than or equal to σ_ϵ contained in U_ϵ and for any $\rho \in B_{\delta_\epsilon}^{2T''+1}(r, U_1)$, we have*

$$\sup\{|\rho(z)|; z \in M_\sigma\} \ge c_\epsilon \sigma^{2T'+\epsilon} \tag{2}$$

REMARK 1.2. (2) says that the type is bounded by $2T'+\epsilon$ on $U_\epsilon \cap b\Omega_\rho$, where Ω_ρ is the domain with defining function ρ.

As an immediate corollary (take $\epsilon = 1$), we can prove

COROLLARY 2. *Let $\Delta_1(z_0) < \infty$ and $T' = (\Delta_1(z_0)/2)^{n-1}$. Then there is a neighborhood $U_0 \subset\subset U_1$ of z_0 such that the type is bounded by $2T'+1$ on $U_0 \cap b\Omega_\rho$, where Ω_ρ is an arbitrary small perturbation of Ω with defining function ρ.*

2. Notation and preliminaries

In this section, we adopt D'Angelo's notation and results in [6]. Let $\mathcal{O}_{z_0} = \mathcal{O}_{n,z_0}$ denote the local ring of germs of holomorphic functions at z_0 in $\mathbb{C}^n$.

DEFINITION 2.1. Let $\mathcal{C} = \mathcal{C}(1,0;n,z_0)$ denote the set of germs of holomorphic maps z such that $z : \mathbb{C}^1 \longrightarrow \mathbb{C}^n$ and $z(0) = z_0$.

DEFINITION 2.2. Let $\nu(g)$ denote the order of vanishing of a function g at 0. Let I be a proper ideal in $\mathcal{O}_{n,z_0}$ then define

$$\tau^*(I) = \sup_{z \in C} \inf_{g \in I} \nu(z^* g)/\nu(z) \tag{3}$$

$$D(I) = dim_{\mathbb{C}}(\mathcal{O}/I) \tag{4}$$

In [6] D'Angelo proved the following theorem which relates D(I) and τ^*.

THEOREM 2.3. (D'Angelo [6, Theorem 2.7]) *Suppose that I is a proper ideal in $\mathcal{O}_{n,z_0}$ and that I contains q independent linear forms. Then the following sharp inequalities hold*

$$\tau^*(I) \le D(I) \le (\tau^*(I))^{n-q}. \tag{5}$$

Suppose r is a local defining function for Ω. Set $M = b\Omega = \{z; r(z) = 0\}$. Then M is a smooth real hypersurface of $\mathbb{C}^n$. Let us consider hypersurfaces defined by Taylor polynomials of r. Suppose that k is an integer. We write $j_{k,p}r$ for the k-th order Taylor polynomial at the point p of the function r. Thus

$$(j_{k,p})(z, \overline{z}) = \sum_{|\alpha+\beta| \le k} \frac{(D^\alpha \overline{D}^\beta r)(p)}{\alpha!\ \beta!} (z - p)^\alpha (\overline{z} - \overline{p})^\beta.$$

Notice that $r_k = j_{k,p}r - r(p)$ is a real valued polynomial on $\mathbb{C}^n$ that vanishes at p. We write $\Delta(M_k, p)$ for the order of contact of the hypersurface defined by this polynomial r_k. Then we have the following theorem.

THEOREM 2.4. (D'Angelo [6, Theorem 4.4]) *Let M be a real hypersurface of $\mathbb{C}^n$, and let p lie on M. Then the following are equivalent:*

(i) *$\Delta_1(p)$ is finite*

(ii) *There exists an integer k_0, so that if k is larger than or equal to k_0, $\Delta(M_k, p)$ equals $\Delta(M, p)$ and is finite. Thus $\Delta(M_k, p)$ is eventually constant with final value $\Delta_1(p)$.*

(iii) *There is an integer k_0, so that for k larger than or equal to k_0, we have $\Delta(M_k, p) \le k$.*

With this theorem, we may replace r by its k-th order Taylor polynomial r_k for some k. Therefore we will study r_k from now on. D'Angelo also showed that any real valued polynomial in $\mathbb{C}^n$ can be decomposed by holomorphic functions. Let $W = W(n,k)$ denote the vector space of all polynomials on $\mathbb{C}^n$, in the variable z and $\overline{z}$, which are of degree less than or equal to k. Let $H = H(n,k)$ denote the vector space of all holomorphic polynomials in W.

PROPOSITION 2.5. (D'Angelo [6, Proposition 3.1]) *Suppose that r is a real valued element of W, and that $r(z_0) = 0$. Suppose also that the coefficients of r are continuous (smooth) functions of a parameter λ. Then there is an integer $N = N(n,k)$, independent of r, and elements $h, f_1, ..., f_N, g_1, ..., g_N$ in $H(n,k)$ so that*

(i) $r(z,\overline{z}) = 2Re(h(z)) + \sum_{j=1}^{N} |f_j(z)|^2 - \sum_{j=1}^{N} |g_j(z)|^2$
(ii) *z_0 lies in the variety defined by h and all the f_j and g_j*
(iii) *The coefficients of h,f and g_j are all continuous (smooth) functions of the parameter λ.*

DEFINITION 2.6. Let r be a real valued polynomial in $W(n,k)$. Suppose that $r(z,\overline{z}) = 2Re(h(z-z_0)) + \|f(z-z_0)\|^2 - \|g(z-z_0)\|^2$ is a holomorphic decomposition for r. Let U be a unitary martice in $\mathbb{C}^n$. Then set $I(U, z_0) = (h, f - Ug)$ equal to the ideal generated by h and the components of $f - Ug$.

Finally, we have the following relation between $\tau^*(I(U,z_0))$ and $\Delta_1(z_0)$.

THEOREM 2.7. (D'Angelo [6, Theorem 5.3]) *Suppose $b\Omega$ is pseudoconvex near $z_0 \in b\Omega$. Then $\Delta_1(z_0) = 2\sup_{U\in\mathcal{U}(N)} \tau^*(I(U,z_0))$ where $\mathcal{U}(N)$ denotes the group of $N \times N$ unitary matrices.*

3. Proof of Theorem 1

We may assume that coordinates $(z_1, ..., z_n)$ have been chosen around z_0 so that $\partial r/\partial x_n(z_0) \neq 0$, where $z_n = x_n + iy_n$. If the conclusion (2) does not hold, then there must be an $\epsilon > 0$ such that we have a sequence of manifolds $\{M_{\sigma_k}\}$ and a sequence of functions $\{r_{\delta_k}\}$, $r_{\delta_k} \in R^{2T''+1}_{\delta_k}(r, U(z_0))$ with $\{\sigma_k\}, \{\delta_k\} \to 0$ such that

(6) $$\sup\{|r_{\delta_k}(z)|; z \in M_{\sigma_k}\} \leq C\sigma_k^{2T'+\epsilon}$$

Let $g^k; B^1_{\sigma_k} \longrightarrow \mathbb{C}^n$ be a parametrization of M_{σ_k}; i.e., $g^k(0) = w^k$ and $0 < c \le |dg^k(z')| \le C$ for all $z' \in B^1_{\sigma_k}(0)$ where $B^1_{\sigma_k}$ denotes a disc around 0 with radius σ_k and c, C are independent of k. Set $r_{\delta_k} = r_k$. Since we may assume that (2) fails in every neighborhood U around z_0, we may choose the manifolds M_{σ_k} so that

$$\lim_{k\to\infty} g^k(0) = \lim_{k\to\infty} w^k = z_0.$$

Since $|r_k(g^k(0))| \le C\sigma_k^{2T'+\epsilon}$, we may certainly assume that $g^k(0) = w^k$ satisfies $r_k(g^k(0)) = 0$, for if not, we need only shift M_{σ_k} by an amount of order of magnitude $|r_k(g^k(0))|$ in a direction transverse to $b\Omega$. Using Cauchy's estimates (as in [2], Theorem 3.4), we may assume that

$$\lim_{k\to\infty} \sup\left\{|dg^k(\xi) - dg^k(0)|; \xi \in B^1_{\sigma_k}\right\} = 0$$

and that each component of g^k is a polynomial of a given degree(M) that is independent of k (Here we have to replace σ_k by $\sigma_k^b = \sigma_k'$ and ϵ by ϵ' where $b > 1$ and $\epsilon' < \epsilon$ as in [2], Theorem 3.4). By Proposition 2.5, we have a holomorphic decomposition of Taylor polynomial of r_k up to order $2T'' + 1$ for each k as follows

$$\begin{aligned} r_k(z) = 2Re(h_k) + \sum_{j=1}^{N} |F_j^k(z - w^k)|^2 - |G_j^k(z - w^k)|^2 \\ + C(|z - w^k|^{2T''+1}) \end{aligned}$$

where h_k is a polynomial of degree $2T''$ and F_j^k, G_j^k are holomorphic polynomials of degree T'' and C does not depend on k. If we write the holomorphic decomposition for r at w^k, we also have

$$\begin{aligned} r(z) = 2Re(\widetilde{h_k}) + \sum_{j=1}^{N} |\widetilde{F_j^k}(z - w^k)|^2 - |\widetilde{G_j^k}(z - w^k)|^2 \\ + C(|z - w^k|^{2T''+1}). \end{aligned}$$

Here, C also does not depend on k. Then by the definition of r_k, we have

$$
(7) \qquad
\begin{aligned}
&\sup_{\substack{|\alpha|\le 2T'' \\ z\in U(z_0)}} |D^\alpha \widetilde{h_k}(z) - D^\alpha h_k(z)| < \delta_k \\
&\sup_{\substack{|\alpha|\le T'' \\ z\in U(z_0)}} |D^\alpha \widetilde{F^k}(z) - D^\alpha F^k(z)| < \delta_k \\
&\sup_{\substack{|\alpha|\le T'' \\ z\in U(z_0)}} |D^\alpha \widetilde{G^k}(z) - D^\alpha G^k(z)| < \delta_k
\end{aligned}
$$

where $\widetilde{F^k}$, $\widetilde{G^k}$, F^k, G^k are vector valued functions. Let h_0, F_j^0, G_j^0 be the same functions (for r) constructed around the point z_0. Since w^k approaches to z_0, the construction in [6] shows that

$$\lim_{k\to\infty} h_k = h_0, \quad \lim_{k\to\infty} F_j^k = F_j^0 \quad \lim_{k\to\infty} G_j^k = G_j^0$$

From this point on, we follow Catlin's [2] proof of Theorem 3.4. Then we can show that

$$(8) \qquad |D_\xi^j(h_k \circ g^k)(0)| \le C\sigma_k^{2T''+\epsilon-j}, \quad j \le 2T''$$

and there exist $N\times N$ unitary matrices $U^k = [U_{j,l}^k]$, $1 \le j, l \le N$ such that

$$(9) \qquad |D_\xi^j((F^k - U^k G^k)\circ g^k)(0)| \le C\sigma_k^{T''-j+\epsilon/2} \quad for\ j = 1, 2, ..., T''$$

Using (8) we can assume that (through the change of coordinates as in [2], Theorem 3.4) $h_k \circ g^k$ vanishes to order at least $2T''+1$ at $\xi = 0$. Let $I_k(U^k, w^k)$ denote the ideal of germs of holomorphic functions about the point w^k generated by the components of $F^k - U^k G^k$. If f is any of these generators, (9) amounts to T'' linear relations satisfied by the derivatives of f of order at most T''. If we set $X_\alpha^k = (D_z^\alpha f)(w^k)$, where $|\alpha| \le T''$, then (9) gives T'' equations

$$(10) \qquad \sum_{0<|\alpha|\le T''} C_{j,\alpha}^k X_\alpha^k = \mathcal{O}(\sigma_k^{\epsilon/2}), \qquad j = 1, 2, .., T''$$

Since $dg^k \neq 0$ we may assume that $g_1^k(\xi) = \xi + w_1^k$. By the Lemma 3.6 in [2], we may assume that $C_{j,\alpha}^k$ are uniformly bounded, and this matrix has a $T'' \times T''$ minor whose determinant is bounded away from zero(uniformly in k). After extracting a subsequence in k, and from the relation (7), we may assume that

$$C_{j,\alpha}^k \to d_{j,\alpha}^0, \quad X_\alpha^k \to X_\alpha^0, \quad U^k \to U^0 \quad and \quad h^k \to h_0.$$

So the equation (10) converges to T'' linearly independent equations of the form

$$\sum_{0 \leq |\alpha| \leq T''} d_{j,\alpha}^0 X_\alpha^0 = 0, \qquad j = 1, 2, ..., T''. \tag{11}$$

Then (11) shows that $(I(F^0 - U^0 G^0))$ (as defined in (4)) has codimension at least $T'' + 1$. (Recall that $f(0) = 0$ if f is in this ideal by the construction of F and G). Recall also that for each k, $h_k \circ g^k$ vanishes to order $2T''+1$. Thus T'' conditions apply only to the values of $F^k - U^k G^k$ on the manifold $\{z; h_k(z) = 0\}$. Therefore if we let $I(F^0 - U^0 G^0, h_0)$ denote the ideal obtained by adjoining the function h_0, then we still have $D(I(F^0 - U^0 G^0, h_0)) \geq T'' + 1$. Since we can consider h_0 is a coordinate function, Theorem 2.3 tells us that

$$D(I) \leq (\tau^*(I))^{n-1}$$

where $I = I(F^0 - U^0 G^0, h_0)$. Therefore,

$$(\Delta_1(z_0)/2)^{n-1} < T'' + 1 \leq D(I) \leq (\tau^*(I))^{n-1}. \tag{12}$$

If we combine (12) and Theorem 2.7, we will get

$$\Delta_1(z_0) < 2\tau^*(I(F^0 - U^0 G^0, h_0)) \leq 2 \sup_{U \in \mathcal{U}(N)} \tau^*(I(U, z_0)) = \Delta_1(z_0).$$

This contradiction proves that the assumption (6) is false and therefore that Theorem 1 holds. □

REMARK. If we take $\widetilde{r} = r$ then (2) holds for r.

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