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# SOME FIXED POINT THEOREMS FOR COMMUTING AND COMPATIBLE MAPPINGS IN NONARCHIMEDEAN MENGER PROBABILISTIC METRIC SPACES

NAN-JING HUANG, BYUNG-SOO LEE AND YONG KAB CHOI

## 1. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers , $\mathbb{R}_+$ the non-negative real numbers and $\mathcal{D}$ the set of all distribution functions.

DEFINITION 1.1. Let $X$ be a nonempty set and $\mathcal{F} : X \times X \to \mathcal{D}$ . $(X, \mathcal{F})$ is called a nonarchimedean probabilistic metric space if the following conditions are satisfied (for $x, y \in X$, the distribution $\mathcal{F}(x, y)$ is denoted by $F_{x,y}$) :

(i) $F_{x,y}(t) = 1$ for $t > 0$ if and only if $x = y$,
(ii) $F_{x,y} = F_{y,x}$ for any $x, y \in X$,
(iii) $F_{x,y}(0) = 0$ for any $x, y \in X$,
(iv) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(\max\{t, s\}) = 1$ for any $x, y, z \in X$.

DEFINITION 1.2. A triplet $(X, \mathcal{F}, T)$ is called a nonarchimedean Menger probabilistic metric space if $(X, \mathcal{F})$ is a nonarchimedean probabilistic metric space and $T$ is a $T$-norm ([5]) with the following condition ;
(v) $F_{x,z}(\max\{t, s\}) \geq T(F_{x,y}(t), F_{y,z}(s))$ for $t, s \in \mathbb{R}_+$ and $x, y, z \in X$.
Let

$$\Omega = \{g | g : [0,1] \to \mathbb{R}_+ \text{ is continuous, strictly decreasing, } g(1) = 0 \text{ and } g(0) < \infty\}.$$

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DEFINITION 1.3. A nonarchimedean Menger probabilistic metric space $(X, \mathcal{F}, T)$ is called a $(C)_g$ type nonarchimedean Menger probabilistic metric space, if there exist a $g \in \Omega$ such that

$$g(T(s,t)) \le g(s) + g(t) \quad \text{for} \quad s,t \in [0,1].$$

PROPOSITION 1.1. ([2]) (1) *If $(X, \mathcal{F}, T)$ is a $(C)_g$ type nonarchimedean Menger probabilistic metric space, then*

$$gF_{x,y}(t) \le gF_{x,z}(t) + gF_{z,y}(t) \quad \text{for any} \quad x,y,z \in X \quad \text{and} \quad t \ge 0.$$

(2) *If $(X, \mathcal{F}, T)$ is a nonarchimedean Menger probabilistic metric space and $T \ge T_1$, where $T_1(a,b) = \max\{a+b-1, 0\}$, then $(X, \mathcal{F}, T)$ is a $(C)_g$ type nonarchimedean Menger probabilistic metric space, in which $g(t) = 1 - t$.*

*In the sequel we will suppose that $X$ is a complete $(C)_g$ type nonarchimedean Menger probabilistic metric space.*

DEFINITION 1.4. Let $A$ and $S$ be mappings from $X$ into itself. $A$ and $S$ are said to be compatible if

$$\lim_{n \to \infty} gF_{ASx_n, SAx_n}(t) = 0 \quad \text{for} \quad t > 0,$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} AX_n = \lim_{n \to \infty} SX_n = u \in X$.

Obviously, commuting and weakly commuting mappings are compatible, but the converse is not true (see [3,4]).

PROPOSITION 1.2. ([2]) *Let $A$ and $S$ be mappings from $X$ into itself. If $A$ and $S$ are compatible, and $Au = Su$ for some $u$ in $X$, then $ASu = SAu$.*

LEMMA 1.1. ([1]) *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing, upper semicontinuous and $\varphi(t) < t$ for all $t > 0$, then*

(1) *for any sequence $\{t_n\}$ of nonnegative real numbers satisfying the following condition:*

$$t_{n+1} \le \varphi(t_n), \quad n = 1, 2, \cdots,$$

*we have $\lim_{n \to \infty} t_n = 0$.*

(2) *especially, for any $t$ in $\mathbb{R}_+$ satisfying $t \le \varphi(t)$, we have $t = 0$.*

LEMMA 1.2. ([2]) *Let* $\{y_n\} \subset X$ *be a sequence satisfying* $\lim_{n\to\infty} gF_{y_n,y_n+1}(t) = 0$ *for all* $t > 0$. *Suppose* $\{y_n\}$ *is not a Cauchy sequence in* $X$, *then there exist* $\varepsilon$, $t_0 > 0$, *and two sequences of positive integers* $\{m(i)\}$ *and* $\{n(i)\}$ *such that*

(i) $m(i) > n(i) + 1$ *and* $n(i) \to \infty$ $(i \to \infty)$.

(ii) $gF_{y_{m(i)},y_{n(i)}}(t_0) \geq \varepsilon$ *and* $gF_{y_{m(i)-1},y_{n(i)}}(t_0) < \varepsilon$, $i = 1, 2, \cdots$.

## 2. Common fixed point theorems

THEOREM 2.1. *Let* $A, B, P$ *and* $Q$ *be mappings from* $X$ *into itself such that* $AP = PA, BQ = QB$ *and for any* $x, y$ *in* $X$, *any* $t > 0$,

$$\begin{aligned}(gF_{APx,BQy}(t))^2 \leq \Phi((gF_{x,y}(t))^2, &gF_{x,APx}(t)gF_{y,BQy}(t),\\ &gF_{x,BQy}(t)gF_{y,APx}(t), gf_{y,APx}(t)gF_{y,BQy}(t),\\ &gF_{x,y}(t)gF_{x,APx}(t), gF_{x,APx}(t)gF_{y,APx}(t)),\end{aligned} \tag{2.1}$$

*where* $\Phi : \mathbb{R}_+^6 \to \mathbb{R}_+$ *is upper semi-continuous, nondecreasing in each coordinate variable and satisfies*

$$\max\{\Phi(t, t, 0, 2t, t, 2t), \Phi(t, 0, t, 0, 0, 0)\} = \varphi(t) < t \text{ for } t > 0.$$

*Then there exists a unique* $u$ *in* $X$ *such that*

$$Au = Bu = Pu = Qu.$$

*Proof*. For $x_0$ in $X$, let

$$x_{2n+1} = APx_{2n} \text{ and } x_{2n+2} = BQx_{2n+1}, \quad n = 0, 1, 2, \cdots. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}&(gF_{x_{2n+1},x_{2n}}(t))^2\\ &= (gF_{APx_{2n},BQx_{2n-1}}(t))^2\\ &\leq \Phi((gF_{x_{2n},x_{2n-1}}(t))^2, gF_{x_{2n},x_{2n+1}}(t)gF_{x_{2n-1},x_{2n}}(t),\\ &\quad gF_{x_{2n},x_{2n}}(t)gF_{x_{2n-1},x_{2n+1}}(t), gF_{x_{2n-1},x_{2n+1}}(t)gF_{x_{2n-1},x_{2n}}(t),\\ &\quad gF_{x_{2n},x_{2n-1}}(t)gF_{x_{2n},x_{2n+1}}(t), gF_{x_{2n},x_{2n+1}}(t)gF_{x_{2n-1},x_{2n+1}}(t)).\end{aligned}$$

If $gF_{x_{2n},x_{2n+1}}(t) > gF_{x_{2n-1},x_{2n}}$, from Proposition 1.1 and above inequality, we have

$$\begin{aligned}
&(gF_{x_{2n+1},x_{2n}}(t))^2 \\
&\leq \Phi((gF_{x_{2n},x_{2n+1}}(t))^2, (gF_{x_{2n},x_{2n+1}}(t))^2, 0, \\
&\quad 2(gF_{x_{2n},x_{2n+1}}(t))^2, (gF_{x_{2n},x_{2n+1}}(t))^2, 2(gF_{x_{2n},x_{2n+1}}(t))^2) \\
&= \varphi((gF_{x_{2n},x_{2n+1}}(t))^2) \\
&< (gF_{x_{2n},x_{2n+1}}(t))^2,
\end{aligned}$$

which is a contradiction. It shows that $gF_{x_{2n},x_{2n+1}}(t) \leq gF_{x_{2n-1},x_{2n}}(t)$ for $t > 0$, so

$$\begin{aligned}
&(gF_{x_{2n+1},x_{2n}}(t))^2 \\
&\leq \Phi((gF_{x_{2n},x_{2n-1}}(t))^2, (gF_{x_{2n},x_{2n-1}}(t))^2, 0, \\
&\quad 2(gF_{x_{2n},x_{2n+1}}(t))^2, (gF_{x_{2n},x_{2n-1}}(t))^2, 2(gF_{x_{2n},x_{2n-1}}(t))^2) \\
&= \varphi((gF_{x_{2n},x_{2n-1}}(t))^2), \text{ for all } t > 0.
\end{aligned}$$

Similarly, we can prove that

$$(gF_{x_{2n+2},x_{2n+1}}(t))^2 \leq \varphi((gF_{x_{2n+1},x_{2n}}(t))^2) \text{ for } t > 0.$$

In view of Lemma 1.1 and above inequalities, we have

$$\lim_{n\to\infty} gF_{x_n,x_{n+1}}(t) = 0 \text{ for } t > 0. \tag{2.3}$$

Now we show that $\{x_n\}$ is a Cauchy sequence in $X$. By (2.3), it is equivalent to show that $\{x_{2n}\}$ is a Cauchy sequence in $X$. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. By Lemma 1.2 and (2.3), there exist $\varepsilon > 0$, $t_0 > 0$, and two sequences of positive integers $\{m(i)\}$ and $\{n(i)\}$ such that

(i) $m(i) > n(i) + 1$ and $n(i) \to \infty$ $(i \to \infty)$;

(ii) $gF_{x_{2m(i)},x_{2n(i)}}(t_0) \geq \varepsilon$ and $gF_{x_{2m(i)-1},x_{2n(i)}}(t_0) < \varepsilon$, $i = 1, 2, \cdots$.

This leads to

$$\varepsilon \leq gF_{x_{2m(i)},x_{2n(i)}}(t_0) \leq \varepsilon + gF_{x_{2m(i)-2},x_{2m(i)-1}}(t_0) + gF_{x_{2m(i)-1},x_{2m(i)}}(t_0),$$

and so

$$\lim_{i\to\infty} gF_{x_{2m(i)},x_{2n(i)}}(t_0) = \varepsilon. \tag{2.4}$$

On the other hand, we have

$$\begin{aligned}
&(gF_{x_{2m(i)},x_{2n(i)+1}}(t_0))^2 \\
&= (gF_{APx_{2n(i)},BQx_{2m(i)-1}}(t_0))^2 \\
&\leq \Phi((\varepsilon + gF_{x_{2m(i)-1},x_{2m(i)-2}}(t_0))^2, gF_{x_{2n(i)},x_{2n(i)+1}}(t_0)gF_{x_{2m(i)-1},x_{2m(i)}}(t_0), \\
&\quad gF_{x_{2n(i)},x_{2m(i)}}(t_0)(gF_{x_{2m(i)-1},x_{2m(i)}}(t_0) + gF_{x_{2m(i)},x_{2n(i)}}(t_0) \\
&\quad + gF_{x_{2n(i)},x_{2n(i)+1}}(t_0)), (gF_{x_{2m(i)-1},x_{2m(i)}}(t_0) + gF_{x_{2m(i)},x_{2n(i)}}(t_0) \\
&\quad + gF_{x_{2n(i)},x_{2n(i)+1}}(t_0))gF_{x_{2m(i)-1},x_{2m(i)}}(t_0), (gF_{x_{2m(i)-1},x_{2m(i)-2}}(t_0) \\
&\quad + \varepsilon)gF_{x_{2n(i)},x_{2n(i)+1}}(t_0), gF_{x_{2n(i)},x_{2n(i)+1}}(t_0)(gF_{x_{2m(i)-1},x_{2m(i)}}(t_0) \\
&\quad + gF_{x_{2m(i)},x_{2n(i)}}(t_0) + gF_{x_{2n(i)},x_{2n(i)+1}}(t_0))).
\end{aligned}$$

Letting $i \to \infty$ and taking an upper limit, using (2.4) and the upper semi-continuity, we obtain

$$\overline{\lim_{i\to\infty}} (gF_{x_{2m(i)},x_{2n(i)+1}}(t_0))^2 \leq \Phi(\varepsilon^2, 0, \varepsilon^2, 0, 0, 0) < \varepsilon^2 \tag{2.5}$$

Since

$$\varepsilon \leq gF_{x_{2m(i)},x_{2n(i)}}(t_0) \leq gF_{x_{2m(i)},x_{2n(i)+1}}(t_0) + gF_{x_{2n(i)+1},x_{2n(i)}}(t_0),$$

it follows from (2.5) that

$$\varepsilon \leq \overline{\lim_{i\to\infty}} gF_{x_{2m(i)},x_{2n(i)+1}}(t_0) < \varepsilon,$$

which is a contradiction. Therefore, $\{x_{2n}\}$ is a Cauchy sequence in $X$ and so $\{x_n\}$ is also a Cauchy sequence in $X$. Let $x_n \to u \in X$ $(n \to \infty)$.

By (2.1) and (2.2), we have

$$\begin{aligned}
&(gF_{APu,X_{2n}}(t))^2 \\
&= (gF_{APu,BQx_{2n-1}}(t))^2 \\
&\leq \Phi((gF_{u,x_{2n-1}}(t))^2, gF_{u,APu}(t)gF_{x_{2n-1},x_{2n}}(t), \\
&\quad gF_{u,x_{2n}}(t)gF_{x_{2n-1},APu}(t), gF_{x_{2n-1},APu}(t)gF_{x_{2n-1},x_{2n}}(t), \\
&\quad gF_{x_{2n-1},u}(t)gF_{u,APu}(t), gF_{u,APu}(t)gF_{x_{2n-1},APu}(t)) \text{ for } t > 0.
\end{aligned}$$

Letting $n \to \infty$ and taking an upper limit, we obtain

$$\begin{aligned}(gF_{APu,u}(t))^2 &\le \Phi(0,0,0,0,0,(gF_{u,APu}(t))^2)\\ &\le \varphi((gF_{u,APu}(t))^2) \text{ for } t > 0.\end{aligned}$$

It follows from Lemma 1.1 that

$$gF_{APu,u}(t) = 0 \text{ for } t > 0.$$

This implies $u = APu$. Similarly, we can prove that $u = BQu$. Since $AP = PA$, from (2.1) we have

$$\begin{aligned}(gF_{Pu,u}(t))^2 &= (gF_{APPu,BQu}(t))^2\\ &\le \Phi((gF_{Pu,u}(t))^2, 0, (gF_{Pu,u}(t))^2, 0, 0, 0)\\ &\le \varphi((gF_{Pu,u}(t))^2) \text{ for } t > 0.\end{aligned}$$

This shows $gF_{Pu,u}(t) = 0$ for $t > 0$. So $Pu = u$. Similarly, $Qu = u$. Therefore,

$$Au = APu = u = BQu = Bu.$$

From (2.1), it is easy to prove that $u$ is a unique common fixed point of $A, B, P$ and $Q$.

This completes the proof of Theorem 2.1.

THEOREM 2.2. *Let $A$, $B$ and $S$ be self mappings of $X$ which satisfy $A(X) \cup B(X) \subset S(X)$, $A$ and $S$ are compatitle and $B$ and $S$ are compatitle. Suppose further that*

$$\begin{aligned}(gF_{Ax,By}(t))^2 \le &\Phi(gF_{Sx,Ax}(t)gF_{Sy,By}(t), gF_{Sx,By}(t)gF_{Sy,Ax}(t),\\ &gF_{Sx,Ax}(t)gF_{Sx,By}(t), gF_{Sy,Ax}(t)gF_{Sy,By}(t))\end{aligned} \tag{2.6}$$

*for $t > 0$ and $x, y$ in $X$, where $\Phi : \mathbb{R}_+^4 \to \mathbb{R}_+$ is upper semi-continuous and nondecreasing in each coordinate variable, and satisfies*

$$\Phi(t, t, a_1 t, a_2 t) < t \textit{ for } t > 0,$$

*where* $a_1, a_2 \in \{0,1,2\}$ *with* $a_1 + a_2 = 2$. *If* $S$ *is continuous, then* $A, B$ *and* $S$ *have a unique common fixed point.*

*Proof.* Since $A(X) \cup B(X) \subset S(X)$, for any given $x_0$ in $X$ we can choose $\{x_n\}$ in $X$ such that

$$Ax_{2n} = Sx_{2n+1}, \quad Bx_{2n+1} = Sx_{2n+2}, \quad n = 0, 1, 2, \cdots.$$

As in Theorem 2.1, we can prove that $\{Sx_n\}$ is a Cauchy sequence in $X$. Letting $Sx_n \to u$ $(n \to \infty)$, we know that

$$Ax_{2n} \to u, \quad Bx_{2n+1} \to u \quad (n \to \infty).$$

Since $S$ is continuous, $SSx_{2n} \to Su$, $SAx_{2n} \to Su$, and since $A$ and $S$ are also compatible, $ASx_{2n} \to Su$ by Definition 1.4. Similarly, $BSx_{2n+1} \to Su$ and $SBx_{2n+1} \to Su$.

From (2.6) we have

$$\begin{aligned}
&(gF_{ASx_{2n},Bu}(t))^2 \\
&\leq \Phi(gF_{SSx_{2n},ASx_{2n}}(t)gF_{Su,Bu}(t), gF_{SSx_{2n},Bu}(t)gF_{Su,ASx_{2n}}(t), \\
&\quad gF_{SSx_{2n},ASx_{2n}}(t)gF_{SSx_{2n},Bu}(t), gF_{Su,ASx_{2n}}(t)gF_{Su,Bu}(t))
\end{aligned}$$

for $t > 0$. Taking the limit as $n \to \infty$ yields

$$(gF_{Su,Bu}(t))^2 \leq \Phi(0,0,0,0) = 0 \text{ for } t > 0.$$

Therefore, $Su = Bu$. Similarly, $Su = Au$. Since $A$ and $S$ are compatible, and $B$ and $S$ are also compatible, it follows from $Su = Au = Bu$ and Proposition 1.2 that

$$SAu = ASu \text{ and } SBu = BSu.$$

Now (2.6) implies

$$\begin{aligned}
(gF_{ASu,Su}(t))^2 &= (gF_{ASu,Bu}(t))^2 \\
&\leq \Phi(gF_{SSu,ASu}(t)gF_{Su,Bu}(t), gF_{SSu,Bu}(t)gF_{Su,ASu}(t), \\
&\quad gF_{SSu,ASu}(t)gF_{SSu,Bu}(t), gF_{Su,ASu}(t)gF_{Su,Bu}(t)) \\
&\leq \Phi(0, (gF_{ASu,Su}(t))^2, 0, 0) \text{ for } t > 0.
\end{aligned}$$

Hence $gF_{ASu,Su}(t) = 0$ for $t > 0$. This implies that $ASu = Su$. Similarly, $ASu = Su$. So we have $SSu = SAu = ASu = Su$. These show that $Su$ is a common fixed point of $A, B$ and $S$. From (2.6), it is easy to prove that $Su$ is a unique common fixed point of $A, B$ and $S$. This completes the proof of Theorem 2.2.

The following is an immediate consequence of Theorem 2.1.

THEOREM 2.3. *Let $A_i$ and $P_i$ be mappings from $X$ into itself such that $A_iP_i = P_iA_i, i = 1, 2, \cdots$ and for any $x, y$ in $X$ and any $t > 0$,*

$$\begin{aligned}&(gF_{A_iP_ix,A_{i+1}P_{i+1}y}(t))^2\\ \leq&\Phi((gF_{x,y}(t))^2, gF_{x,A_iP_ix}(t)gF_{y,A_{i+1}P_{i+1}y}(t),\\ &gF_{x,A_{i+1}P_{i+1}y}(t)gF_{y,A_iP_ix}(t), gF_{y,A_iP_ix}(t)gF_{y,A_{i+1}P_{i+1}y}(t),\\ &gF_{x,y}(t)gF_{x,A_iP_ix}(t), gF_{x,A_iP_ix}(t)gF_{y,A_iP_ix}(t)),\end{aligned}$$

*where $\Phi : \mathbb{R}_+^6 \to \mathbb{R}_+$ is the same as in Theorem 2.1. Then there exists a uique $u$ in $X$ such that*

$$u = A_iu = P_iu, i = 1, 2, \cdots.$$

## References

1. S. S. Chang, *Fixed Point Theory with Applications*, Chongqing Fublishing House Chongqing, 1984.
2. S. S. Chang, N.J. Huang and D.P. Wu, *Common fixed point theorems for compatible mappings in nonarchimedean Menger probabilistic metric spaces with applications*, J. Sichuan Univ. **28** (1991), 19-25.
3. G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. & Math. Sci. **9** (1986), 771-779.
4. G. Jungck, B. E. Rhoades, *Some fixed point theorems for compatible maps*, Internat. J. Math. & Math. Sci. **16** (1993), 417-428.
5. B. Schweizer, A. Sklar, *Probabilistic metric spaces*, North-Holland, 1983.

Department of Mathematics
Sichuan University Chengdu
P. R. China

Department of Mathematics
Kyungsung University
Pusan 608-736, Korea

Department of Mathematics
Gyeongsang National University
Jinju 660-701, Korea