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AN EXTENSION THEOREM FOR THE FOLLAND–STEIN SPACES

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1. Introduction

This paper is the third of a series in which smoothness properties of functions in several variables are discussed. The germ of the whole theory was laid in the works by Folland and Stein[4]. On nilpotent Lie groups, they defined analogues of the classical L^p Sobolev or potential spaces in terms of fractional powers of sub-Laplacian, $\mathcal{L}$ and extended several basic theorems from the Euclidean theory of differentiability to these spaces: interpolation properties, boundedness of singular integrals,..., and imbedding theorems. In this paper we study the analogue to the extension theorem for the Folland-Stein spaces. The analogue to Stein's restriction theorem were studied by M.Mekias[5] and Y-M.Kim[6]. First, we have the space of Bessel potentials on the Heisenberg group introduced by Folland[4].

DEFINITION. The Heisenberg group H^n is the Lie group of real dimension 2n + 1, whose underlying space is $R \times C^n$, and whose group law is given by

$$(t,z)(t',z') = (t + t' + 2Im z\bar{z}', z + z').$$

It's Lie algebra is generated by the left invariant vector fields X_j, Y_j, T, j = 1,...,n, given by

$$X_j = \frac{\partial}{\partial x_j}, Y_j = \frac{\partial}{\partial y_j}, T = \frac{\partial}{\partial t}.$$

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The homogeneous dimension of H^n is Q = 2n+2. We define the norm on H^n by

$$|(t,z)| = (t^2 + |z|^4)^{1/4}.$$

2. Sublaplacian

We define the sublaplacian on the Heisenberg group by the following. Let

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2).$$

The operator $\mathcal{L}$ is homogeneous of degree 2, and $\mathcal{L}^t = \mathcal{L}$. The fundamental solution to $\mathcal{L}$ is given by

$$\phi = C(|z|^4 + t^2)^{-n/2}, C = \frac{2^{2-2n}\pi^{n+1}}{\Gamma(n/2)^2}.$$

Thus $\mathcal{L}$ is locally solvable. The convolution of two functions f and g in H^n is defined by

$$f * g(u) = \int f(v)g(v^{-1}u)dv.$$

3. The Bessel potential and spaces S_α^p

We define S_α^p to be the image of L^p under the operator $(I + \mathcal{L})^{-\alpha/2}$.

If $f \in L^p, 1 < p < \infty$, then $(I + \mathcal{L})^{-\alpha/2} f = f * J_\alpha$, where J_α is the Bessel potential defined by

$$J_\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-t} h(x,t) dt.$$

Here $h(x,t)$ is the fundamental solution to $\mathcal{L} + \frac{\partial}{\partial t}$. For more properties of $h(x,t)$ see Folland[4]. We have the following properties of J_α.

(1) J_α is defined for all $x \neq 0$ and even for $x = 0$ when $\alpha > Q$. And J_α is C^∞ away from 0.

(2) As $x \to 0$,

$$|J_\alpha(x)| = O(|x|^{\alpha-Q}) \;\; if \; \alpha < Q$$
$$= O(log\frac{1}{|x|}) \;\; if \; \alpha = Q$$

(3) As $x \to \infty$, $|J_\alpha(x)| = O(|x|^{-N})$ for all N. Hence, $J_\alpha \in L^1$ for all $\alpha > 0$.

The spaces $\Lambda_\alpha^{p,q}(H^n)$ is defined to be the space of those functions in $L^p(H^n)$ for which the following quantity is finite.

$$\int_{H^n} \frac{1}{|u|^{Q+\alpha q}} \left[\int_{H^n} |f(uv) - f(v)|^p dv \right]^{q/p} du < \infty, \quad 0 < \alpha < 1.$$

$$\int_{H^n} \frac{1}{|u|^{Q+\alpha q}} \left[\int_{H^n} |f(uv) + f(uv^{-1}) - 2f(v)|^p dv \right]^{q/p} du < \infty, \quad \alpha \geq 1.$$

For convenience's sake we will notate the convolution operator with J_α as $\mathcal{J}^\alpha$, i.e.

$$(I + \mathcal{L})^{\alpha/2}\phi = \phi * J_\alpha = \mathcal{J}^\alpha \phi$$

There is a theorem of Folland[4] which will be used freely in the course of the proof.

THEOREM 1. (Folland [4]) *$f \in S^p_{\alpha+1}$ if and only if $Xf \in S^p_\alpha$ for all X such that $\mathcal{L} = \sum X_j^2$. Furthermore $||f||_{p,\alpha+1}$ and $||f||_{p,\alpha} + \sum ||X_j f||_{p,\alpha}$ are equivalent.*

For the motivation we will introduce a theorem of Stein[2,3].

THEOREM 2.

(a) *The restriction map $R : L^p_\alpha(R^n) \to \Lambda^{p,p}_\beta(R^m)$ is a bounded map as long as $\beta = \alpha - \frac{n-m}{p} > 0$, $1 < p < \infty$.*

(b) *Conversely, there exists an extension map E*

$$E : \Lambda^{p,p}_\beta(R^m) \to L^p_\alpha(R^n) \quad such \; that$$

$$R(E(g)) = g \; for \; all \; g \in \Lambda^{p,p}_\beta(R^m), \quad \beta > 0, \; 1 < p < \infty.$$

For the proof of the theorem, the reader may consult [5]. Now we will state an analogous theorem for the Folland-Stein spaces. This kind of general Lipschitz spaces are first studied by Folland[4], and Stein[7].

THEOREM 3. *There exists an extension map* $E : \Lambda_{\beta}^{p,p}(R_t \times R^{2n-1}) \to S_{\alpha}^{p}(H^n)$, *where* $R_t \times R^{2n-1}$ *is one of hyperplanes of* $R_t \times R^{2n}$ *such that* $R(E(g)) = g$ *for all* $g \in \Lambda_{\beta}^{p,p}(R_t \times R^{2n-1}$, *if* $\beta = \alpha - \frac{1}{\alpha} > 0, \frac{1}{p} < \alpha < 1, 1 < p < \infty$.

The proof of the theorem consists of three parts. First, we define the extension map E and show it's well defined. Second, we show the boundedness of the extended map compared with the original map. Lastly, we need to show the smoothness of the extended map in terms of the L^p norm.

Proof. We define the extension map as follows. Let $g \in \Lambda_{\beta}^{p,p}(R \times R^{2n-1})$, choose $\psi \in C_0^{\infty}(R \times R^{2n-1})$ such that $\int \psi(y)dy = 1, \quad \psi(y) = 0 \; for \; |y| > 1$. Also choose $\lambda \in C_0^{\infty}(R)$ such that $\lambda(0) = 1$. Define the extension map as follows.

$$E(g) = f(x, \xi) = \lambda(\xi)|\xi|^{-Q+1} \int g(xy^{-1})\psi\Big(\frac{y}{|\xi|}\Big)dy$$

Then

$$\begin{aligned} f(x, 0) &= \lim_{\xi \to 0} \lambda(\xi)|\xi|^{-Q+1} \int g(xy^{-1})\psi\Big(\frac{y}{|\xi|}\Big)dy \\ &= \lim_{\xi \to 0} \lambda(\xi)|\xi|^{-Q+1} \int g(x(|\xi|y)^{-1})\psi(y)|\xi|^{Q-1}dy \\ &= \int g(x)\psi(y)dy = g(x). \end{aligned}$$

Note that $||f(,)||_p \leq ||g||_p$. Indeed,

$$\begin{aligned} f(x, \xi) &= \lambda(\xi)|\xi|^{-Q+1} \int g(xy^{-1})\psi\Big(\frac{y}{|\xi|}\Big)dy \\ &= g * K_{\xi}(y), \end{aligned}$$

Where $K_{\xi}(y) = \lambda(\xi)|\xi|^{-Q+1}\psi(\frac{y}{|\xi|})$. Note that $\int K_{\xi}(y)dy \leq A < \infty$ independent of ξ and $g \in L^p$. Hence

$$||f(,)||_p \leq ||g||_p.$$

Now we have to show that $f \in S^p_\alpha(H^n)$. Put $F = \mathcal{J}^{1-\alpha} f$, for $0 < \alpha < 1$. Since $\mathcal{J}^{1-\alpha}$ is an isomorphism between S^p_α and S^p_1, $f \in S^p_\alpha(H^n)$ if and only if $F = \mathcal{J}^{1-\alpha} f \in S^p_1(H^n)$. So it suffices to show that $F \in S^p_1(H^n)$, i.e.,

$$F \in L^p(H^n), \quad and \ \frac{\partial F}{\partial x_k}, \ \frac{\partial F}{\partial \xi} \in L^p(H^n).$$

First, $F \in L^p(H^n)$ is clear because $f \in L^p(H^n)$ and $F = \mathcal{J}^{1-\alpha} f = f * J_{1-\alpha}$,

$$||F||_{L^p(H^n)} \leq ||J_{1-\alpha}||_{L^1(H^n)} ||f||_{L^p(H^n)} \leq C||f||_{L^p(H^n)}$$

since $||J_\beta||_{L^1} \leq C$ for any $\beta > 0$. Now we will consider $\frac{\partial F}{\partial x_k}$.

$$\begin{aligned} F(x,\xi) &= \int_{R^{2n}} \int_R J_{1-\alpha}(z,\eta) f(xz^{-1}, \xi\eta^{-1}) dz d\eta \\ &= \int_{R^{2n}} \int_R J_{1-\alpha}(z,\eta) f(xz^{-1}, \xi - \eta) dz d\eta. \end{aligned}$$

Note that $\int \frac{\partial}{\partial x_k} \psi(x) dx = 0$, and because ψ vanishes outside the unit sphere, we have

$$\begin{aligned} \left|\frac{\partial f}{\partial x_k}(x,\xi)\right| &= |\lambda(\xi)|\xi|^{-Q}| \int_{|y|\leq|\xi|} g(xy^{-1}) \frac{\partial \psi}{\partial y_k}\left(\frac{y}{|\xi|}\right) dy| \\ &\leq A|\xi|^{-Q+1} \int_{|y|\leq|\xi|} \left|g(xy^{-1}) - g(x)\right| dy \end{aligned}$$

From the size of $J_{1-\alpha}(z,\xi)$, we can say that there exists a constant C such that

$$|J_{1-\alpha}(z,\xi)| \leq C(|z|^2 + |\xi|^2)^{\frac{-Q+1-\alpha}{2}}.$$

Then the following estimate holds.

$$\begin{aligned} &\left|\frac{\partial F}{\partial x_k}(x,\xi)\right| \leq \int_{R^{2n}} \int_R |J_{1-\alpha}(z,\eta)| \ \left|\frac{\partial f}{\partial x_k}(xz^{-1}, \xi - \eta)\right| dz d\eta \\ &\leq A \int_{R^{2n}} \int_R |J_{1-\alpha}(z,\eta)| |\xi - \eta|^{-Q} \int_{|y|\leq|\xi-\eta|} \left|g(xy^{-1}z^{-1}) - g(xz^{-1})\right| dy dz d\eta \\ &\leq A \int_{R^{2n}} \int_R (|z|^2 + |\eta|^2)^{\frac{1-\alpha-Q}{2}} |\xi - \eta|^{-Q} \int_{|y|\leq|\xi-\eta|} \left|g(xy^{-1}z^{-1}) - g(xz^{-1})\right| dy dz d\eta \end{aligned}$$

Consider the function

$$\Phi((x,\xi),(z,\eta))$$
$$=(|z|^2+|\eta|^2)^{\frac{1-\alpha-Q}{2}}|\xi-\eta|^{-Q}\int_{|y|\leq|\xi-\eta|}|g(xy^{-1}z^{-1})-g(xz^{-1})|dy.$$

We will apply to it Minkowski's inequality for integrals. First, we will estimate the following integral.

$$\int_{R\times R^{2n-1}}|\Phi((x,\xi),(z,\eta))|^p dx$$
$$=\int\left((|z|^2+|\eta|^2)^{\frac{(1-\alpha-Q)p}{2}}|\xi-\eta|^{-Qp}\int_{|y|\leq|\xi-\eta|}\left|g(xy^{-1}z^{-1})-g(xz^{-1})\right|dy\right)^p dx$$
$$=(|z|^2+|\eta|^2)^{\frac{(1-\alpha-Q)p}{2}}|\xi-\eta|^{-Qp}\int\left(\int_{|y|\leq|\xi-\eta|}|g(xy^{-1}z^{-1})-g(xz^{-1})|dy\right)^p dx.$$

Then, by Minkowski's inequality, we obtain the following inequality.

$$\int_{R\times R^{2n-1}}|\Phi((x,\xi),(z,\eta))|^p dx)^{1/p}$$
$$\leq(|z|^2+|\eta|^2)^{\frac{1-\alpha-Q}{2}}|\xi-\eta|^{-Q}\int_{|y|\leq|\xi-\eta|}||g(*y^{-1})-g(*)||_p dy.$$

Set $w(y)=||g(*y^{-1})-g(*)||_p$ and

$$\Omega(\rho)=(\rho)^{1-Q-\alpha}\int_{|y|<\rho}w(y)dy.$$

With these notations, we find the following estimate holds.

$$\left\|\frac{\partial F}{\partial x_k}(*,\xi)\right\|_p\leq C\int_R\int_{R\times R^{2n-1}}(|z|^2+|\eta|^2)^{\frac{1-\alpha-Q}{2}}|\xi-\eta|^{-Q}$$
$$\int_{|y|\leq|\xi-\eta|}||g(*y^{-1})-g(*)||_p dydzd\eta$$
$$\leq C\int_R\left(\int_{R\times R^{2n-1}}(|z|^2+|\eta|^2)^{\frac{1-\alpha-Q}{2}}dz\right)|\xi-\eta|^{-1+\alpha}\Omega(|\xi-\eta|)d\eta$$
$$\leq C\int_R|\xi-\eta|^{-1+\alpha}\Omega(|\xi-\eta|)|\eta|^{-\alpha}d\eta,$$

and therefore

$$\begin{aligned}\int_R \left\| \frac{\partial F}{\partial x_k}(*,\xi) \right\|_p^p d\xi &\le C \int_0^\infty |\Omega(\rho)|^p d\rho \\ &= C \int_0^\infty \left(\rho^{-Q+1-\alpha} \chi(r<\rho) r^{Q-2+\alpha} \frac{\tilde{w}(r)}{r^\alpha} dr \right)^p d\rho \\ &\le C \int_0^\infty \frac{|\tilde{w}(r)|^p}{r^{\alpha p}} dr \\ &= C \int_{R\times R^{2n-1}} \frac{\|g(\circ y^{-1}) - g(\circ)\|_p^p}{|y|^{Q-1+\beta p}} dy < \infty\end{aligned}$$

where $\tilde{w}(r) = r^{2-Q} \int_{|y|=r} \|g(*y^{-1}) - g(y)\|_p dy$.

The first, third inequality holds by Hardy inequality (see Appendix A in[8]) and the last inequality holds by the assumption that $g \in \Lambda_\beta^{p,p}(R \times R^{2n-1})$.

Here, constantd C may differ in each occurence. From this we have showd that $\frac{\partial F}{\partial x_k} \in L^p(H^n)$. The fact that $\frac{\partial F}{\partial \xi} \in L^p(H^n)$ can be proved in a similar way.

References

1. P. I. Lizorkin, *Characteristics of the boundary values of functions from $L_\alpha^p(E^n)$ on hyperplanes*, Dok. Akad. Nauk. SSSR. **150** (1963), 986-989.
2. E. M. Stein, *The Characterization of functions arising as potentials*, Bull. Amer. Math. Soc. **76** (1961), 102-104 **68** (1962), 577-582.
3. G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13** (1974), 160-208.
4. G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complexes and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 420-522.
5. M. Mekias, *Restriction theorem for Sobolev spaces*, M. I. T. Thesis, 1987.
6. Y. M. Kim, *Restriction theorem on the Folland-Stein spaces*, J. Korean Math. Soc. **30** (1993), 25-39.
7. N. Aronszajn and K. T. Smith, *Theory of Bessel potentials I*, Ann. Inst. Fourier. **11** (1961), 385-475.
8. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton U. Press, Princeton, N.J., 1970.

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