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ON PRINCIPAL IDEALS IN POLYNOMIAL RINGS

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Throughout this paper R will always denote an integral domain with the quotient field K. Let A denote the polynomial ring $R[x]$, I be an ideal of $A, I_K = I \otimes_R K$ and $J = I_K \cap A$.

Kanemitsu and Yoshida([2]) proved as a main result that if I is an ideal in $R[x]$ such that $I \cap R = (0)$, then the following conditions are equivalent :

(1) There exists a Sharma polynomial of degree d in I where d is the least degree of polynomials in I.

(2) I is a principal ideal and $I = I_K \cap A$.
If, moreover, R is noetherian, the above conditions are equivalent to :

(3) $I_{\emptyset(x)}$ is a principal ideal and $I = I_K \cap A$.

In this paper, we prove (1), (2) and (3) are equivalent in the same integral domain R(not necessarily noetherian).

DEFINITION 1. Let $f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d (a_0 \neq 0)$ be a polynomial in $R[x]$. $f(x)$ is called a Sharma polynomial if there does not exist $t \notin a_0R$ such that $ta_i \in a_0R$ for $0 \leq i \leq d$.

For example, a monic polynomial in $R[x]$ is a Sharma polynomial. Also, the Sharma polynomials in $Z[x]$ are precisely the primitive polynomials where Z is the ring of integers.

In the following let $f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d (a_0 \neq 0)$ be a polynomial in $A = R[x]$ and $\emptyset(x) = x^d + \alpha_1x^{d-1} + \cdots + \alpha_d$ be the polynomial in $K[x]$ such that $\alpha_i = a_i/a_0$ for $1 \leq i \leq d$.

Also we put $I_{h(x)} = \{a \in R \mid ah(x) \in R[x]\}$ if $h(x) \in K[x]$.

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PROPOSITION 2. ([2, proposition 4]) *Let R be an integral domain with the quotient field K and $f(x), \emptyset(x), a_0, \cdots, a_d$ be as in the above. Then the following conditions are equivalent:*

(1) *$f(x)$ is a Sharma polynomial, that is, there does not exist $t \notin a_0R$ such that $ta_i \in a_0R$ for $0 \leq i \leq d$.*
(2) *$c(f(x))^{-1} = R$ where $c(f(x)) = (a_0, a_1, \cdots, a_d)R$ and $c(f(x))^{-1} = R :_K c(f(x))$.*
(3) *$I_{\emptyset(X)} = a_0R$.*

REMARK 1. For an ideal I of $A = R[x]$, I_K is a principal ideal of $K[x]$ and so $I_K = h(x)K[x]$ where $h(x)$ is a monic polynomial in $K[x]$. Put $d = deg\ h(x)$. Then d is the least degree of polynomials in I. Clearly, $h(x)$ is uniquely determined by I. We note that $I \cap R = (0)$ if and only if $I_K \subset K[x]$.

REMARK 2. Let $J = I_K \cap A$ and $J_K = J \otimes_R K$. Then we have $I_K = J_K$ and $min\{deg\ g(x) \mid g(x) \in I\} = min\{deg\ g(x) \mid g(x) \in J\} = min\{deg\ g(x) \mid g(x) \in I_K\} = min\{deg\ g(x) \mid g(x) \in J_K\}$.

LEMMA 3. *Let I be a principal ideal in $A = R[x]$ such that $I = f(x)A$, where $f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d$. Let $I = I_K \cap A$ and $I_K = h(x)K[x]$, where $h(x)$ is a monic polynomial in $K[x]$ Then $h(x) = \emptyset(x)$, where $\emptyset(x) = x^d + \alpha_1x^{d-1} + \cdots + \alpha_d$ such that $\alpha_i = a_i/a_0$ for $1 \leq i \leq d$.*

Proof. Since $\emptyset(x) \in I_K, \emptyset(x)$ is minimal degree in I_K by Remark 2 and $h(x)$ and $\emptyset(x)$ are monic polynomials of degree d in I_K. So, $h(x) - \emptyset(x) \in I_K$ and $deg(h(x) - \emptyset(x)) < d$. Therefore $h(x) - \emptyset(x) = 0$, that is, $h(x) = \emptyset(x)$.

Now we will prove the main result which is the extension of [2,Theorem 5].

THEOREM 4. *Let $A = R[x]$ be a polynomial ring over an integral domain with the quotient field K and let I be an ideal in A such that $I \cap R = (0)$. Let $I_K = I \otimes_R K$ and $h(x)$ be the monic polynomial over K such that $I_K = h(x)K[x]$. Then the following conditions are equivalent :*

(1) *There exists a Sharma polynomial of degree d in I where d is the least degree of polynomials in I.*

(2) *I is a principal ideal and $I = I_K \cap A$.*
(3) *$I_{h(x)}$ is a principal ideal and $I = I_K \cap A$.*

Proof.
(1) $\Longleftrightarrow$ (2) See[2, Theorem 5].
(3) $\Longrightarrow$ (1) Suppose that $I_h(x) = aR$. Put $ah(x) = f(x)$. Then $f(x) \in I_K \cap A = I$. Since the leading coefficient of $f(x)$ is a and $I_h(x) = aR$, it follows that $f(x)$ is a Sharma polynomial by Proposition 2. By Remark 2, $f(x)$ is a polynomial of least degree d in I. Consequently, we proved that (3) implies (1).
(2) $\Longrightarrow$ (3) Since I is a principal ideal, let $I = f(x)A$. Then $\emptyset(x) = h(x)$ by Lemma 3.
On the other hand, let $f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d$. Then d is the smallest degree of polynomials in I. Suppose there exist $t \notin a_0R$ such that $ta_i \in a_0R$ for $0 \leq i \leq d$. Put $\alpha = t/a_0$. Then $\alpha \notin R$ and $\alpha a_i \in R$ for $1 \leq i \leq d$. Let $g(x) = f(x)(x + \alpha)(= xf(x) + \alpha f(x))$. Since $xf(x) \in A$ and every coefficient αa_i of $\alpha f(x)$ is contained in R, we have $g(x) \in R[x] = A$. Thus $g(x) \in I_K \cap A = I = f(x)A$. Hence $g(x)/f(x) = x + \alpha \in A$. So $\alpha \in R$. This is a contradiction. This means $f(x)$ is a Sharma polynomial of degree d in I. So by Proposition 2, $I_{\emptyset(x)}$ is a principal ideal.
Therefore, $I_{h(x)}$ is a principal ideal.

The following proposition is also the extension of [2,Proposition 6].

PROPOSITION 5. *Let R be a unique factorization domain with the quotient field K and I be an ideal of $A = R[x]$ such that $I \cap R = (0)$. Put $I_K = I \otimes_R K$ and $J = I_K \cap A$. Then J is a principal ideal of A.*

Proof. Since I_K is principal, We can assume $I_K = h(x)K[x]$ where $h(x) = x^d + c_1x^{d-1} + \cdots + c_d$ for $1 \leq i \leq d$. Then $I_{h(x)} = \cap_{i=1}^{d}\{a \in R \mid ac_i \in R\}$. Since R is a unique factorization domain, by[3, Theorem 8.34], $I_{h(x)}$ is principal.
On the other hand, $I_K = J_K$ by Remark 2. Hence J is a principal ideal of A by Theorem 4.

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