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ON IRREDUCIBLE SIGN–TRIPOTENT MATRICES

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I. Introduction

A matrix whose entries consist of the symbols $+, -, 0$ is called a *sign-pattern matrix.* For a real matrix B, by sgn$B$ we mean the sign pattern matrix in which each positive (respectively, negative, zero) entry is replaced by $+$ (respectively, $-, 0$). For each $n \times n$ sign-pattern matrix A, there is a natural class of real matrices whose entries have the sign indicated by A. If $A = [a_{ij}]$ is an $n \times n$ sign-pattern matrix, then the *sign-pattern class* of A is defined by

$$(1.1) \quad Q(A) = \{B = [b_{ij}] |\ \mathrm{sgn} b_{ij} = a_{ij} \text{ for all } i \text{ and } j \text{ in } \{1, 2, \cdots, n\}\}.$$

If A and B are $n \times n$ sign-pattern matrices, then $A + B$ exists, that is, $A + B$ is qualitatively defined if $a_{ij}b_{ij} \neq -$ for all i and j in $\{1, 2, \cdots, n\}$. If $a_{ij}b_{ij} = -$, then $a_{ij} + b_{ij}$ is $-$ or $+$. We cannot determine the sign of the entry $a_{ij} + b_{ij}$. That is, $A + B$ is not defined. Similary, the product AB exists if no two terms in the sum

$$(1.2) \qquad \sum_{k=1}^{n} a_{ik}b_{kj}$$

are oppositely signed, for all i and j in $\{1, 2, \cdots, n\}$. For example, let

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then we can have

$$\mathrm{sgn}A + \mathrm{sgn}B = \begin{bmatrix} - & + \\ + & - \end{bmatrix} \quad \text{and} \quad \mathrm{sgn}A\mathrm{sgn}B = \begin{bmatrix} + & - \\ - & + \end{bmatrix}.$$

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But both sgn$A$ + sgn$C$ and sgn$A$sgn$C$ cannot be defined.

If $A = [a_{ij}]$ is an $n \times n$ sign-pattern matrix, then a product of the form

$$(1.3) \qquad \gamma = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k j},$$

in which the index set $\{i_1, i_2, \cdots, i_k, j\}$ consists of distinct indices, is called a *chain* of length k from i_1 to j. If the index $j = i_1$, then γ is called a *simple cycle* of length k. A cycle or a chain is said to be *negative* (resp. *positive*) if it contains an odd (resp. even) number of negative entries and no entries equal to zero.

Recall that a real $n \times n$ matrix C is said to be *idempotent* if $C = C^2$. Analogously, a square sign-pattern matrix A is said to be *sign idempotent* if $C^2 \in Q(A)$ whenever $C \in Q(A)$; henceforth we write $A = A^2$. Also, a matrix C is said to be *tripotent* if $C = C^3$. So, a square sign-pattern matrix A is said to be *sign tripotent* if $C^3 \in Q(A)$ whenever $C \in Q(A)$; henceforth we write $A = A^3$. One important reason for studying sign idempotent and sign tripotent is that powers of sign idempotent matrices and sign tripotent matrices preserve not only the sign pattern, but also the cycle structure of the matrix.

In [1], Eschenbach characterized the sign idempotent matrices. In this paper, our objective is to characterize $n \times n$ irreducible sign patterns that are sign tripotent.

II. Results

Let $P = [p_{ij}]$ be the product matrix A^3. If P is defined for an $n \times n$ sign-pattern matrix $A = [a_{ij}]$, then

$$(2.1) \qquad p_{ij} = \sum_{l=1}^{n} \sum_{k=1}^{n} a_{ik} a_{kl} a_{lj}.$$

We let the index set $\{1, 2, \cdots, n\}$ be represented by N, and the class of sign tripotent matrices be denoted by ST.

The first lemma is clear, and we state it without proof.

LEMMA 1. *The class ST is closed under the following operations:*

i) *signature similarity;*
ii) *permutation similarity; and*
iii) *transposition.*

If A is a sign idempotent matrix, then A is a sign tripotent matrix. But sign tripotent matrices need not be sign idempotent in general. For example, let

$$A = \begin{bmatrix} - & + \\ + & - \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} + & - \\ - & + \end{bmatrix} \neq A, \quad A^3 = \begin{bmatrix} - & + \\ + & - \end{bmatrix} = A.$$

In [1], Eschenbach characterized the sign idempotent matrices. In this note, we will characterize the $n \times n$ irreducible sign tripotent matrices which are not sign idempotent. That is, A is an irreducible sign tripotent which is not idempotent.

LEMMA 2. *Let* $A = [a_{ij}] \in ST$. *If* $a_{ij} = 0$ *for some* $i, j \in N$, *then* $a_{ik}a_{kl}a_{lj} = 0$ *for all* $k, l \in N$.

Proof. By the definition of addition,

$$\text{sgn} a_{ij} = \text{sgn} \sum_{l=1}^{n} \sum_{k=1}^{n} a_{ik}a_{kl}a_{lj} = \text{sgn} a_{ik}a_{kl}a_{lj} = 0$$

for all $k, l \in N$.

THEOREM 3. *Let* $A = [a_{ij}] \in ST$. *If A is an irreducible, then* $\text{sgn} a_{ij} = \text{sgn} a_{ji}$ *for all* $i, j \in N$.

Proof. Suppose that $a_{ij} \neq 0$. Since $a_{ij} \neq 0$, there exist $k, l \in N$ such that $\text{sgn} a_{ij} = \text{sgn} a_{ik}a_{kl}a_{lj} \neq 0$, i.e., there exist $a_{ik} \neq 0, a_{kl} \neq 0$ and $a_{lj} \neq 0$. For ith column and jth row of A, since A is irreducible, there exist $p, q \in N$ such that $a_{jp} \neq 0$ and $a_{qi} \neq 0$. For qth column and pth row of A, since A is irreducible, there exist $r, s \in N$ such that $a_{pr} \neq 0$ and $a_{sq} \neq 0$. Continuous process, the matrix A appears row and column that are overlap. Without loss of generality, let $s = p$ and $r = q$. Since $a_{pq} \neq 0$, $a_{jp}a_{pq}a_{qi} \neq 0$. That is, $\text{sgn} a_{ji} = \text{sgn} a_{jp}a_{pq}a_{qi} \neq 0$. So, if $a_{ij} \neq 0$, then $a_{ji} \neq 0$. Thus,

$$\begin{aligned} \text{sgn} a_{ij}a_{ji} &= \text{sgn} a_{ij}a_{ji}a_{ik}a_{ik} \\ &= \text{sgn}(a_{ij}a_{ji}a_{ik})\text{sgn} a_{ik} = (\text{sgn} a_{ik})^2 = +. \end{aligned}$$

Therefore, $\text{sgn} a_{ij} = \text{sgn} a_{ji}$ Now, suppose that $a_{ij} = 0$. Assume that $a_{ji} \neq 0$. Then, we have a contradiction, by the same as the proof of case $a_{ij} \neq 0$. Therefore, $\text{sgn} a_{ij} = \text{sgn} a_{ji}$ for all $i, j \in N$. The proof is completed.

Now, suppose that $A = [a_{ij}]$ is an irreducible sign tripotent matrix which is not idempotent matrix.

LEMMA 4. *Let $a_{ij} \neq 0$ for all $i, j \in N$. Let $B, C, D \in Q(A)$. Then $F = BCD \in Q(A)$ if and only if*

$$(2.2) \qquad \begin{aligned} &\text{i)} & a_{ik}a_{kl}a_{lj}a_{ij} &> 0 \\ &\text{ii)} & a_{ij}a_{ji} &> 0 \\ &\text{iii)} & a_{ii} &< 0 \\ &\text{iv)} & a_{ik}a_{kl}a_{li} &< 0. \end{aligned}$$

Proof. Suppose that $F \in Q(A)$. Since $a_{ij} = a_{ik}a_{kl}a_{lj}$,

$$a_{ij}a_{ij} = a_{ik}a_{kl}a_{lj}a_{ij} > 0.$$

Thus, i) holds. Since $a_{ij} = a_{ji}$, $a_{ij}a_{ji} > 0$. Thus, ii) holds. Now, we will prove iii). By [4, Theorem 4.1], A is an idempotent if an only if $a_{ii} > 0$, $i = 1, 2, \cdots$, $a_{ij}a_{ji} > 0$, $i \neq j$ and $a_{ij}a_{jk}a_{ik} > 0$, i, j, k distinct. For $k, p \in N$, $a_{kk} = a_{kp}a_{pp}a_{pk} = a_{kp}a_{pk}a_{pp} = a_{pp}$. So, $a_{kk} = a_{pp}$ for all $k, p \in N$. Assume that $a_{ii} > 0$ for $i \in N$. By ii), $a_{ij}a_{ji} > 0$, $i \neq j$. For any $k, l, p \in N$, $a_{kl}a_{lp}a_{kp} = a_{kp}a_{pl}a_{lk} = a_{kk} > 0$. So, if $a_{ii} > 0$, then A is idempotent. This is a contradiction. Thus, $a_{ii} < 0$ for all $i \in N$. Since $a_{ii} = a_{ik}a_{kl}a_{li} < 0$, iv) holds.

Conversely, let $F = BCD$. Then, for any $i, j \in N$,

$$\begin{aligned} a_{ij} = a_{ik}a_{kl}a_{lj} &= \sum_{l=1}^{n}\sum_{k=1}^{n} a_{ik}a_{kl}a_{lj} \\ &= \text{sgn} \sum_{l=1}^{n}\sum_{k=1}^{n} b_{ik}c_{kl}d_{lj} = \text{sgn} f_{ij}. \end{aligned}$$

Therefore, $F = BCD \in Q(A)$. The proof is completed.

Now, we define the *tripath* following as;

$$t(i_{k-3}, i_k) = a_{i_{k-3}i_{k-2}}a_{i_{k-2}i_{k-1}}a_{i_{k-1}i_k}. \tag{2.3}$$

Then for any tripath $t(i_{k-3}, i_k)$, $\text{sgn}t(i_{k-3}, i_k)a_{i_k i_{k-3}} = +$.

LEMMA 5. *Let $a_{ij} \neq 0$ for all $i, j \in N$. Then $A \in ST$ if and only if $\gamma = a_{i_1 i_2} \cdots a_{i_k i_1}$ is positive cycle for even number k and negative cycle for odd number k.*

Proof. Suppose that γ is positive cycle for even number k and negative cycle for odd number k. Then, by lemma 4, the proof is trivial.

Suppose that $A \in$ST. If $k = 1$, then $a_{ii} < 0$. If $k = 2$, then $\gamma = a_{ij}a_{ji} > 0$. By induction, assume that $k \geq 3$. First, suppose that k is an odd number. If $k = 3$, then $\gamma = a_{i_1 i_2}a_{i_2 i_3}a_{i_3 i_1} < 0$ $(k \equiv 0 \ (mod\ 3))$. If $k = 5$, then $\gamma = a_{i_1 i_2} \cdots a_{i_5 i_1}$. $\text{sgn}\gamma = \text{sgn}t(i_1, i_4)a_{i_1 i_4}a_{i_1 i_4}a_{i_4 i_5}a_{i_5 i_1} = -$ $(k \equiv 2 \ (mod\ 3))$. If $k = 7$, then $\gamma = a_{i_1 i_2} \cdots a_{i_7 i_1}$. $\text{sgn}\gamma = -$ $(k \equiv 1 \ (mod\ 3))$. Now we consider three cases; If $k \equiv 0 \ (mod\ 3)$, then $\gamma = t(i_1, i_4) \cdots t(i_{k-2}, i_1)$. The number of tripath of γ is $\frac{k}{3}$. Since k is an odd number, $\frac{k}{3}$ is also. So,

$$\begin{aligned}\text{sgn}\gamma &= \text{sgn}t(i_1, i_4)a_{i_1 i_4} \cdots t(i_{k-2}i_1)a_{i_{k-2}i_1}a_{i_1 i_4} \cdots a_{i_{k-2}i_1} \\ &= \text{sgn}a_{i_1 i_4} \cdots a_{i_{k-2}i_1}.\end{aligned}$$

Let $\gamma_1 = a_{i_1 i_4} \cdots a_{i_{k-2}i_1}$.The number of terms of γ_1 is $\frac{k}{3}$. Thus, by induction, $\text{sgn}\gamma_1 = -$. Therefore, $\text{sgn}\gamma = -$. Second case, if $k \equiv 1 \ (mod\ 3)$, then $\gamma = t(i_1, i_4) \cdots t(i_{k-3}, i_k)a_{i_k i_1}$. The number of tripath of γ is $\frac{k-1}{3}$. Since k is an odd, $k - 1$ is even. So, $\frac{k-1}{3}$ is an even number. Thus,

$$\begin{aligned}\text{sgn}\gamma &= \text{sgn}t(i_1, i_4)a_{i_1 i_4} \cdots t(i_{k-3}, i_k)a_{i_{k-3}i_k}a_{i_1 i_4} \cdots a_{i_k i_1} \\ &= \text{sgn}a_{i_1 i_4} \cdots a_{i_k i_1}.\end{aligned}$$

Since the number of terms of $a_{i_1 i_4} \cdots a_{i_k i_1}$ is $\frac{k-1}{3} + 1$, by induction, $\text{sgn}\gamma = 1$. Finally, if $k \equiv 2 \ (mod\ 3)$, then $\gamma = t(i_1, i_4) \cdots t(i_{k-4}, i_{k-1})$ $a_{i_{k-1}i_k}a_{i_k i_1}$. Since $k - 2$ is an odd number, $\frac{k-2}{3}$ is also. The number of tripath of γ is $\frac{k-2}{3}$. Thus, by induction, $sgn\gamma = -$.
Next, if k is an even number, then the proof is similar to odd cases. The proof is completed.

By above statements, we obtain the following:

THEOREM 6. *Let $A = [a_{ij}]$ be irreducible sign pattern matrix such that $a_{ij} \neq 0$ for all $i, j \in N$. If A is not idempotent, then $A \in ST$ if and only if every odd cycle of A is negative and every even cycle is positive.*

Let $I_+ = [i_{ij}]$ be $n \times n$ identity sign-pattern matrix. That is, $i_{kk} = +$ for all $k \in N$ and $i_{kj} = 0$ for all $k \neq j$.

THEOREM 7. *Let A be an irreducible sign tripotent matrix. $A^2 = I_+$ if and only if A is a permutation matrix pattern.*

Proof. Suppose that $A^2 = I_+$. Assume that A is not permutation pattern. Since A is irreducible, there exists $j \in N$ such that $a_{ij} \neq 0$ and $a_{kj} \neq 0$ for some $i \neq k$. By theorem 3, $a_{ji} \neq 0$ and $a_{jk} \neq 0$. Since $A^2 = I_+$, for $i \neq k$,

$$\begin{aligned} 0 = i_{ik} &= \sum_{l=1}^{n} a_{il}a_{lk} \\ &= a_{i1}a_{1k} + \cdots + a_{ij}a_{jk} + \cdots + a_{in}a_{nk}. \end{aligned}$$

but $a_{ij}a_{jk} \neq 0$. This is a contradiction. Therefore, A is a permutation matrix pattern.

Conversely, suppose that A is a permutation matrix pattern. Since A is a permutation pattern, there exists exactly one nonzero entry at each row i. Without loss of generality, let $a_{ik} \neq 0$. Let $A^2 = B = [b_{ij}]$. Then

$$b_{ii} = \sum_{k=1}^{n} a_{ik}a_{ki} = \sum_{k=1}^{n} a_{ik}^2 = +.$$

For $i \neq j$, $b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$. Since A is a permutation, there exists exactly one nonzero entry at each row i, i.e., $a_{ik} \neq 0$, for some $k \in N$. If $a_{kj} \neq 0$ for some $j \in N$, then $a_{jk} \neq 0$. This is a contradiction because A is a permutation matrix pattern. That is, there exists exactly one nonzero entry at each column k. So, $a_{kj} = 0$ for all $j \in N$. Thus, $b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = 0$ for all $i \neq j$. Therefore, $A^2 = I_+$.

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