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ON UNIFORMITIES OF BCK–ALGEBRAS

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In [1], Alo and Deeba introduced the uniformity of a BCK-algebra by using ideals. Meng [5] introduced the concept of dual ideals in BCK-algebras. We note that the concept of dual ideals is not a dual concept of ideals. In this paper, by using dual ideals, we consider the uniformity of a BCK-algebra.

By a BCK-algebra we mean an algebra $(X; *, 0)$ of type (2,0) satisfying the following axioms:

(I) $(x * y) * (x * z) \leq (z * y)$,
(II) $x * (x * y) \leq y$,
(III) $x \leq x$,
(IV) $x \leq y$ and $y \leq x$ implies $x = y$,
(V) $0 \leq x$,

where $x \leq y$ is defined by $x * y = 0$.

A BCK-algebra X is said to be bounded if there exists $1 \in X$ such that $x \leq 1$ for all $x \in X$. In a bounded BCK-algebra, we denote $1 * x$ by Nx. In what follows, X denotes a bounded BCK-algebra.

DEFINITION 1. ([5, 6]) A nonempty subset D of X is called a dual ideal if it satisfies:

(D_1) $1 \in D$,
(D_2) $N(Nx * Ny) \in D$ and $y \in D$ implies $x \in D$.

We note that the intersection of dual ideals is also a dual ideal.

LEMMA 2. ([5, 6]) *If A is a nonempty subset of X, then the set*

$$\{x \in X \mid \text{there exist } a_i \in A, i = 1, \cdots, n, \text{ such that } (\cdots(Nx * Na_1) * \cdots) * Na_n = 0\}$$

is the least dual ideal containing A, which is called the dual ideal generated by A.

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LEMMA 3. ([5, 6]) *Let D be a dual ideal in X. Define, for $x, y \in X$, $x \sim y$ if and only if $N(x * y) \in D$ and $N(y * x) \in D$. Then for any $x, y, u, v \in X$*

(1) $x \sim x$,
(2) $x \sim y \Rightarrow y \sim x$,
(3) $x \sim y$ and $y \sim u \Rightarrow x \sim u$,
(4) $x \sim y$ and $u \sim v \Rightarrow x * u \sim y * v$.

DEFINITION 4. ([7]) Let M be any nonempty set and let U and V be any subsets of $M \times M$. Define

$U \circ V = \{(x, y) \in M \times M |$ for some $z \in M, (x, z) \in U$ and $(z, y) \in V\}$,
$U^{-1} = \{(x, y) \in M \times M | (y, x) \in U\}$,
$\triangle = \{(x, x) \in M \times M | x \in M\}$.

By a uniformity on M we mean a nonempty collection K of subsets of $M \times M$ which satisfies the following conditions:

(U_1) $\triangle \subset U$ for any $U \in K$,
(U_2) if $U \in K$, then $U^{-1} \in K$,
(U_3) if $U \in K$, then there exists a $V \in K$ such that $V \circ V \subset U$,
(U_4) if $U, V \in K$, then $U \cap V \in K$,
(U_5) if $U \in K$ and $U \subset V \subset M \times M$, then $V \in K$.

The pair (M, K) is called a uniform structure.

THEOREM 5. *For each dual ideal D of X, define*

$$U_D = \{(x, y) \in X \times X | N(x * y) \in D \text{ and } N(y * x) \in D\}$$

and let

$$K^* = \{U_D | D \text{ is a dual ideal of } X\}.$$

Then K^* satisfies the conditions $(U_1) - (U_4)$.

Proof. Let $(x, x) \in \triangle$. Since $N(x * x) = N0 = 1 \in D$ for any dual ideal D, it follows that $(x, x) \in U_D$ for every $U_D \in K^*$, which proves that (U_1) holds.

Note from Lemma 3 that for any $U_D \in K^*$, $(x, y) \in U_D$ if and only if $N(x * y) \in D$ and $N(y * x) \in D$ if and only if $(y, x) \in U_D$ if and only if $(x, y) \in {U_D}^{-1}$. Hence ${U_D}^{-1} = U_D \in K^*$, which is (U_2).

Assume that $U_D \in K^*$. Let $A = \{D_\alpha | D_\alpha \subset D\}$ be the collection of dual ideals contained in D. Clearly, A is not empty. Let I be the dual ideal generated by $\bigcup_\alpha D_\alpha$. Then $U_I \in K^*$. It is sufficient to show that $U_I \circ U_I \subset U_D$. If $(x,y) \in U_I \circ U_I$, then there exists $z \in X$ such that $(x,z) \in U_I$ and $(z,y) \in U_I$. It follows from Lemma 3 that $(x,y) \in U_I$, that is,

$$N(x * y) \in I \quad \text{and} \quad N(y * x) \in I.$$

Since I is the minimal dual ideal containing $\bigcup_\alpha D_\alpha$ and since $\bigcup_\alpha D_\alpha \subset D$, it follows that $I \subset D$. Hence $N(x*y), N(y*x) \in D$. Thus $(x,y) \in U_D$ and so $U_I \circ U_I \subset U_D$, which is (U_3).

Finally we prove (U_4). This will follow from the observation that $U_C \cap U_D = U_{C\cap D}$ for all $U_C, U_D \in K^*$. Let $(x,y) \in U_C \cap U_D$. Then $(x,y) \in U_C$ and $(x,y) \in U_D$, which imply that

$$N(x*y), N(y*x) \in C \quad \text{and} \quad N(x*y), N(y*x) \in D.$$

Hence $N(x*y), N(y*x) \in C \cap D$, that is, $(x,y) \in U_{C\cap D}$. So $U_C \cap U_D \subset U_{C\cap D}$. Likewise we can show that $U_{C\cap D} \subset U_C \cap U_D$. Thus $U_C \cap U_D = U_{C\cap D}$ and this proves requirement (U_4).

THEOREM 6. *Let $K = \{U \subset X \times X | U \supset U_D$ for some $U_D \in K^*\}$. Then K satisfies a uniformity on X and hence the pair (X, K) is a uniform structure.*

Proof. Using Theorem 5, we can show that K satisfies the conditions $(U_1)-(U_4)$. To prove (U_5), let $U \in K$ and $U \subset V \subset X \times X$. Then there exists a $U_D \in K^*$ such that $U_D \subset U \subset V$, which implies that $V \in K$. This completes the proof.

DEFINITION 7. For $x \in X$ and $U \in K$, we define

$$U[x] = \{y \in X | (x,y) \in U\}.$$

THEOREM 8. *For each $x \in X$, the collection $\mathcal{U}_x = \{U[x] | U \in K\}$ forms a neighborhood base at x, making X a topological space.*

Proof. First note that $x \in U[x]$ for each x. Second,

$$U_1[x] \cap U_2[x] = (U_1 \cap U_2)[x],$$

which means that the intersection of neighborhoods is a neighborhood. Finally, if $U[x] \in \mathcal{U}_x$ then by (U_3) there exists a $E \in K$ such that $E \circ E \subset U$. Then for any $y \in E[x], E[y] \subset U[x]$, so this property of neighborhoods is satisfied.

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