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HEIGHTS ON SINGULAR PROJECTIVE CURVES

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Introduction

In this paper we show that for each divisor class c of degree zero on a projective curve C (not necessarily smooth), there exists a unique function $\widehat{h}_c$ on C up to bounded functions. Section 1 contain basic definitions and a brief summary of classical results on Jacobians and heights. In section 2, we prove the existence of "canonical height" on a singular curves and in section 3 we prove the analogouse results on Néron functions for singular curves. This is a part of the author's doctorial thesis at Ewha Womens University under the guidence of professor Sung Sik Woo.

1. Heights on projective curves

Throughout Chapter 1, F is a field with a proper set of absolute values M_F satisfying the product formula. We denote by K some finite extension of F.

DEFINITION. Let $P \in P^n(K)$ be a point in projective space, rational over K. Let $(x_0, x_1, ..., x_n)$ be the coordinates for P with $x_i \in K$. The height of P (relative to K) is defined by

$$H_k(P) = \prod_{\nu \in M_K} sup_i \|x_i\|_\nu.$$

We define the logarithmic height to be

$$h_K(P) = log H_K(P).$$

The Height of P is well defined by the product formula and for any finite extension L of F, $H_L(P) = H_K(P)^{[L:K]}$. Thus we define;

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DEFINITION. Let $P \in P^n(F^a)$ be a point in a projective space, where F^a is the algebraic closure of F. We define an absolute height of P by

$$H(P) = H_K(P)^{1/[K:F]}$$

and

$$h(P) = log H(P)$$

for any finite extension K of F over which P is rational.

DEFINITION. Let V be a Projective variety defined over a field K. Let $\varphi\colon V \to P^m$ be a morphism defined over K. Then for each $P \in V(K)$, $\varphi(P) \in P^m(K)$. Thus we define the height on V relative to φ by

$$H_{K,\varphi}(P) = H_K(\varphi(P)), \quad h_{K,\varphi}(P) = log H_{K,\varphi}(P)$$

and

$$H_\varphi(P) = H(\varphi(P)), \quad h_\varphi(P) = log H_\varphi(P).$$

Let V be a projective variety, and X_0 be a Cartier divisor on V. We define

$$L(X_0) = \{f \in K(V) \mid (f) \geq -X_0\}$$

and

$$\mathcal{L} = \mathcal{L}(X_0) = \{X \in Div(V) \mid X \geq 0, X - X_0 = (f)\}$$

then $L(X_0)$ is finite dimensional vector space over K and $\mathcal{L} = \mathcal{L}(X_0)$ is an invertible sheaf on V, called a linear system on V.

Let $(f_0, f_1, ..., f_m)$ be any set of generators of $L(X_0)$. Then this defines a rational map of V into P^m

$$\varphi = (f_0, f_1, ..., f_m)\colon V \to P^m.$$

If two morphisms $\varphi\colon V \to {P_K}^n, and\ \psi\colon V \to {P_K}^m$ are obtained by the same linear system $L(X_0), then\ h_\varphi - h_\psi$ is bounded, denoted by $h_\varphi = h_\psi + O(1)$.

If X is a divisor in a linear system $\mathcal{L}$ on V, then we denoted by h_X the height function h_φ associated with any one of the maps φ derived from this linear system $\mathcal{L}$. Then h_X is well defined up to bounded function O(1).

Let $c \in Pic(V)$ be a divisor class. Then for any divisors X and Y, $h_X = h_Y + O(1)$. We denote by h_c the height h_X for any divisor X in c. Thus h_c is uniquely determined up to bounded function O(1).

Let A be an abelian variety defined over a field K. The principal relation between divisor classes is that given $c \in Pic(B)$, the association

$$\alpha \mapsto \alpha^* \quad for \quad \alpha \in Hom(A, B)$$

is quadratic in α. In other word, if we let

$$D_c(\alpha, \beta) = (\alpha + \beta)^* c - \alpha^* c - \beta^* c$$

then $D_c(\alpha, \beta)$ is bilinear in (α, β). From this fundamental relation, we obtains; for $c \in Pic(A)$, there exists a unique quadratic form q_c and a linear form l_c such that

$$h_c = q_c + l_c + O(1).$$

If c is even, that is $(-1)^* c = c$, then $l_c = 0$.

The sum $q_c + l_c$ will be denoted by $\widehat{h}_c$, and is called the *Néron-Tate height*, or *canonical height* (see [5] Chap.4,Chap.5).

Let C be a complete non-singular curve of genus $g \geq 2$ and J be its Jacobian. We fix an embedding, and assume $C \subset J$. Let

$$\Theta = \overbrace{C + C + \cdots + C}^{\text{(g-1)-times}}$$

be the divisor on J, and let θ be the divisor class of Θ in Pic(J). Let $\delta_J = \delta$ be the class

$$\delta = -s^*{}_2(\theta) + p^*{}_1(\theta) + p^*{}_2(\theta)$$

in $Pic(J \times J)$, and

$$\psi_\theta \colon J \to Pic(J)$$

be the homomorphism denoted by $a \mapsto \theta_a - \theta$ where s_2 is the sum map from $J \times J$ to J, and p_1, p_2 are projections on the first and second factor of $J \times J$ respectively. Then (J, δ) is a dual variety of J and ψ_θ gives an isomorphism of J with $Pic_0(J)$. Thus we have

$$^t\delta(a) = \psi_\theta(a)$$

where for any point a in J, the intersection

$$^t\delta(a) = \delta.(J \times a)$$

is defined as a divisor class on J and Pic_0 means divisor classes which are algebraically equivalent to zero. Considering $C \times J \subset J \times J$, let $\delta_{C \times J}$ be the restriction of δ to $C \times J$ then

$$y \mapsto \ ^t\delta_{C \times J}(y)$$

gives an isomorphism of $J(K)$ with $Pic_0(C)_K$. Thus $(J(K), \delta_{C \times J})$ is a dual variety of C.

Let $c \in Pic_0(C)$ and let $S(c)$ be the corresponding point in J. Then by functoriality of height gives

$$h_\delta(x, S(c)) = h_c(x) + O(1)$$

for $x \in C$ where h_δ is the canonical height on $J \times J$ associated with the divisor class δ (see[5] Chap.5 §5).

LEMMA 1.1. *Let $f: U \to V$ be a rational map of one variety into another, and assume that V is complete non-singular. Let Y be a divisor in $D_a(V)$ such that $f^*(Y)$ is defined. Then $f^*(Y) \in D_a(U)$.*

Proof. See ([4] Chap.V,§1).

THEOREM 1.2. *Let C be a complete non-singular curve of genus $g \geq 2$. Then there exists a natural homomorphism*

$$Pic_0(C) \longrightarrow \{\text{real valued functions on } C\}/O(1)$$

given by $c \mapsto \widehat{h}_c$ *where* $Pic_0(C)$ *be the group of divisor classes of degree zero. Furthmore if* $f : C_1 \to C_2$ *is a morphism of complete non-singular curves, then for* $c \in Pic_0(C_2)$ *the divisor class on* C_2*, we have*

$$\widehat{h}_{f*c} = \widehat{h}_c \circ f + O(1).$$

Proof. Take

$$\widehat{h}_c = h_\delta(, S(c))$$

then this is independent of the class c up to bounded functions. Now

$$\widehat{h}_{f*c} = \widehat{h}_c \circ f + O(1)$$

is followed by Lemma 1.1 and the functoriality of height.

2. Heights on singular projective curves

Let C be a projective curve and S be the set of singular points on C. Note that S is a finite subset of C. Let $x \in C$ be a closed point and $f \in O_{C,x}$ be a regular element in the local ring $O_{C,x}$. We define

$$ord_x(f) = l(O_{C,x}/(f))$$

where l denote the length of $O_{C,x}$-modules. Since

$$ord_x(fg) = ord_x(f) + ord_x(g)$$

for regular elements f and g in $O_{C,x}$, we can define

$$ord_x(f/g) = ord_x(f) - ord_x(g)$$

for any element f/g of the total ring of fractions of $O_{C,x}$. Let D be a Cartier divisor on C represented by f_x/g_x in a neighborhood of x. Then we associate the Weil divisor

$$\sum_{x \in C} ord_x(D)x$$

corresponding to the Cartier divisor D. The degree of a Cartier divisor D is defined by

$$deg(D) = \sum_{x \in C} ord_x(D)[K(x) : K].$$

We denote $Pic_0(C)$ to be the group of divisor classes of degree zero. Given Cartier divisor D on C, there exists an element f in $K(C)$ such that

$$S \cap supp(D - (f)) = \emptyset.$$

Thus for each Cartier divisor D on C, we can associate to a Weil divisor on $C - S$.

LEMMA 2.1. *Let $f\colon C' \to C$ be the normalization of C, and D be a Cartier divisor on C. Then*

$$deg(D) = deg(f^* D)$$

Proof. See ([1] Chap.9 §1).

Let $f\colon C' \to C$ be the normalization. Then we have (see [2],P.282)

$$0 \to \bigoplus_{P \in C} \overline{O^*}_P / O^*_P \to Pic(C) \xrightarrow{f^*} Pic(C') \to 0$$

where $\overline{O}_P$ be the integral closure of the local ring O_P at P. In particular, f^* maps $Pic_0(C)$ into $Pic_0(C')$.

LEMMA 2.2. *Let $f\colon V \to W$ be a morphism of varieties, and let $\mu\colon V' \to V$ and $\lambda\colon W' \to W$ be the normalizations of V and W respectively. Then there exists a morphism $f'\colon V' \to W'$ such that $\lambda \circ f' = f \circ \mu$.*

Proof. See [3] (Chap.2 , 2.14).

THEOREM 2.3. *Let C be a projective curve over a field K. Then there exists a natural homomorphism*

$$Pic_0(C) \longrightarrow \{\textit{real valued functions on } C\}/O(1)$$

given by $c \mapsto \widehat{h}_c$. *Furthermore if* $\varphi\colon C_1 \to C_2$ *is a morphism on projective curves, and* $c \in Pic_0(C_2)$ *such that* $\varphi^* c \in Pic_0(C_1)$. *Then we have*

$$\widehat{h}_{\varphi^* c} = \widehat{h}_c \circ \varphi + O(1)$$

Proof. Let $f\colon C' \to C$ be the normalization. Then for $c \in Pic_0(C)$, $f^* c \in Pic_0(C')$ by Lemma 2.1. Let P_1, P_2 be two distinct points in $f^{-1}(Q)$. Then

$$h_c \circ f(P_1) = h_c \circ f(P_2) = h_c(Q)$$

but

$$\begin{aligned} h_c \circ f &= h_{f^* c} \mod O(1) \\ &= h_\delta(\ , S(f^* c)) \mod O(1) \end{aligned}$$

where S is the isomorphism of $Pic_0(C')$ and the Jacobian J of C'. Thus

$$\begin{aligned} h_c(Q) &= h_c \circ f(P_1) \mod O(1) \\ &= h_\delta(P_1, S(f^* c)) \mod O(1) \end{aligned}$$

and

$$\begin{aligned} h_c(Q) &= h_c \circ f(P_2) \mod O(1) \\ &= h_\delta(P_2, S(f^* c)) \mod O(1) \end{aligned}$$

Thus $h_\delta(P, S(f^* c))$ is independent up to bounded functions for any choice $P \in f^{-1}(Q)$. Therefore we take

$$\widehat{h}_c(Q) = h_\delta(P, S(f^* c))$$

for any point P in $f^{-1}(Q)$. For the functoriality, we let $f_1 : C_1' \to C_1$ and $f_2\colon C_2' \to C_2$ be the normalizations of C_1 *and* C_2 respectively. Let (J_1, δ_1), (J_2, δ_2) be the Jacobian varieties of C_1', C_2' respectively. Then by Lemma 2.2 there exists a morphism $\varphi'\colon C_1' \to C_2'$ such that $\varphi \circ f_1(P) = f_2 \circ \varphi'(P)$. Note that

$$h_{\varphi^* c} = h_c \circ \varphi + O(1)$$

and for each $Q \in C_1$, we have

$$\begin{aligned} h_{\varphi^* c}(Q) &= h_{f^*{}_1(\varphi^* c)} \circ f_1(P) + O(1) \quad for\ P \in f^{-1}{}_1(Q) \\ &= h_{\delta_1}(f_1(P), S_1(f^*{}_1(\varphi^* c)) + O(1) \\ &= \widehat{h}_{\varphi^* c}(Q) + O(1) \end{aligned}$$

where S_1 is the isomorphism of $Pic_0(C_1')$ and J_1. Also we have

$$\begin{aligned} h_c \circ \varphi(Q) &= h_c(\varphi(Q)) \\ &= h_{f^*{}_2 c}(\varphi'(P)) + O(1) \quad since\ \varphi'(P) \in f^{-1}{}_2(\varphi(Q)) \\ &= h_{\delta_2}(\varphi'(P), S_2(f^*{}_2 c)) + O(1) \\ &= \widehat{h}_c(\varphi(Q)) + O(1) \end{aligned}$$

where S_2 is the isomomorphism of $Pic_0(C_2')$ and J_2. Therefore we have

$$\widehat{h}_{\varphi^* c} = \widehat{h}_c \circ \varphi + O(1).$$

3. Néron functions on singular projective curves

On a projective variety, a Weil function associated with a divisor is defined only up to a bounded function. However on an abelian variety, this is defined up to a constant functions Γ satisfying some properties. Functions normalized as such are called *Néron functions*. The Weil functions on arbitrary complete non-singular varieties associated with divisors are obtained by pull back from abelian varieties. However, the divisors are restricted those which are algebraically equivalent to zero. Again such Weil functions are called Néron functions. Having normalized the Néron functions up to additive constants, we can get rid of these constants if we evaluate these functions by additivity on 0-cycles of degree zero on an abelian variety or non-singular projective variety. We then obtain a bilinear pairing between divisors (algebraically equivalent to zero on an arbitrary variety) and 0-cycles of degree zero. This pairing is called the Néron pairing (see [5] Chap.10,Chap 11).

For a singular projective curve, we have:

THEOREM 3.1. *Let C be a projective curve defined over a field K and S be the set of singular points on C. Let $Z_0'(C)_K$ be the set of zero cycles of degree zero on C rational over K whose supports are disjoint from S. Then for each $D \in Z_0'(C)_K$, λ_D is uniquely determined up to constant functions Γ satisfying*

(1) *The association $D \mapsto \lambda_D$ is a homomorphism mod Γ*
(2) *If $D = (f)$ is principal, then $\lambda_D = \lambda_f \mod \Gamma$*
(3) *If $\psi: V \to W$ be a morphism of projective curves defined over K, and let $Y \in Z_0'(W)$ such that $\psi^{-1}(Y) \in Z_0'(V)$, than*

$$\lambda_{\psi^{-1}(Y)} = \lambda_Y \circ \psi \mod \Gamma$$

Proof. Let $f: C' \to C$ be the normalization, and let λ, λ' be two Weil functions corresponding to D. Then $\lambda \circ f$ *and* $\lambda' \circ f$ are Weil functions on C' associated with the Cartier divisor $f^*D \in Z_0(C')$. Since C' is nonsingular and f^*D is algebraically equivalent to zero, we have

$$\lambda \circ f = \lambda' \circ f \mod \Gamma.$$

For $Q \in C$, let $P_1, \ldots, P_t$ be distinct points in $f^{-1}(Q)$, then

$$\lambda(Q) = \lambda \circ f(P_1) = \cdots = \lambda \circ f(P_t)$$

and

$$\lambda'(Q) = \lambda' \circ f(P_1) = \cdots = \lambda' \circ f(P_t)$$

Therefore, $\lambda = \lambda' \mod \Gamma$.

COROLLARY 3.2. *Let C be a projective curve defined over K such that C(K) is zariski dense in C. Let $\alpha \in Z_0'(C)$, $\beta \in Z_0(C)$ be two zero cycles with disjoint supports. Then there exists unique pairing $< \alpha, \beta >_\nu$ satisfying the following properties.*

(1) *The pairing is bilinear.*
(2) *If $\alpha = (f)$ is principal, then $< \alpha, \beta >_\nu = \nu \circ f(\beta)$.*
(3) *If $\beta \in Z_0'(C)$, then $< \alpha, \beta >_\nu = < \beta, \alpha >_\nu$.*
(4) *The function $x \mapsto < \alpha, (x) - (x_0) >_\nu$ from $C(K) - supp(\alpha)$ to R is continuous and locally bounded.*

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